A PROPERTY OF CERTAIN SQUARES

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This note arises from the serendipitous observation that both $144 = 12^2$ and $1444 = 38^2$ are perfect squares. A further examination of the first few perfect squares and the numbers obtained by appending a digit to them yields the following:

TABLE I

$1^2 = 1,$	$16 = 4^2,$	$10^2 = 100,$	—
$2^2 = 4,$	$49 = 7^2,$	$11^2 = 121,$	—
$3^2 = 9,$	—	$12^2 = 144,$	$1444 = 38^2,$
$4^2 = 16,$	$169 = 13^2,$	$13^2 = 169,$	—
$5^2 = 25,$	$256 = 16^2,$	$14^2 = 196,$	—
$6^2 = 36,$	$361 = 19^2,$	$15^2 = 225$	—
$7^2 = 49,$	—	$16^2 = 256$	—
$8^2 = 64,$	—	$17^2 = 289$	—
$9^2 = 81,$	—	$18^2 = 324$	$3249 = 57^2$

The question now arises, whether there are any more such pairs of perfect squares, and what regularities apply. Consider the general case. The Diophantine equation (with integer x, y, and b) is

$$10x^2 + b = y^2, \quad 0 \le b \le 9, \quad x \ge 0, \quad y \ge 0. \tag{1}$$

(i) If $x = 0$, the only solutions are

$$x = 0, \quad \text{with} \quad (b, y) = (0,0),\ (1,1),\ (4,2),\ \text{or}\ (9,3). \tag{2}$$

(ii) If $x = 1$, the only solution clearly is

$$x = 1, \quad y = 4, \quad b = 6. \tag{3}$$

(iii) If $x \ge 2$, since 10 is not a perfect square, $b \ne 0$. Furthermore, let us write (with integer u and v)

$$y = 10u + v, \quad -4 \le v \le 5, \tag{4}$$

then
$$10x^2 + b = 100u^2 + 20uv + v^2,$$

or
$$10(x^2 - 10u^2 - 2uv) = v^2 - b. \tag{5}$$

Since the left-hand side is divisible by 10, if $b = 5$, then necessarily $v = 5$, and then

$$x^2 = 10u(u-1) + 2,$$

which is impossible; because the last digit of a perfect square cannot be 2. Thus, $b \neq 5$. For $v^2 - b$ to be divisible by 10, the only remaining possibilities are

$$(b, v) = (1, \pm 1), (4, \pm 2), (6, \pm 4), (9, \pm 3). \tag{6}$$

Let us put

$$z = \lfloor x\sqrt{10} \rfloor; \tag{7a}$$

that is (with integer z and real ζ),

$$x\sqrt{10} = z + \zeta, \quad 0 < \zeta < 1, \tag{7b}$$

so that

$$3x \leq z < 4x; \tag{7c}$$

and (with integer c)

$$y = z + c, \tag{8a}$$

so that

$$c > \zeta, \tag{8b}$$

and so

$$c \geq 1. \tag{8c}$$

Then, by (1), (7b), and (8a),

$$(z + \zeta)^2 + b = (z + c)^2, \tag{9}$$

whence, by (6), (7c), and (8b),

$$6x(c - \zeta) \leq 2z(c - \zeta) = b - c^2 + \zeta^2 < b \leq 9. \tag{10}$$

If $x \geq 2$ and $c \geq 2$, then $c - \zeta > c - 1 \geq 1$, and so (10) becomes $12 < 9$, a contradiction. Thus, with (8c), and by (3),

$$\text{if} \quad x \geq 1, \quad \text{then} \quad c = 1. \tag{11}$$

TABLE II

x	b	z	ζ	c	y
1	6	3	0.1623	1	4
2	9	6	0.3246	1	7
4	9	12	0.6491	1	13
5	6	15	0.8114	1	16
6	1	18	0.9737	1	19
12	4	37	0.9473	1	38
18	9	56	0.9210	1	57

Table II above shows the parameters for the solutions given in Table I.

If we put $$\zeta = 1 - \eta, \tag{12}$$

and, by (8a) with (11), $$z = y - 1, \tag{13}$$

then, by (7b), $$x\sqrt{10} = y - \eta, \quad 0 < \eta < 1. \tag{14}$$

i.e., $$y = \lceil x\sqrt{10} \rceil. \tag{15}$$

This establishes the following result.

Theorem 1. *For every integer $x \geq 1$, the fraction $\frac{y}{x}$ is the rational least upper bound of $\sqrt{10}$ having x as denominator.*

It follows from (15) that there is a simple algorithm for finding possible solutions of (1). All that is necessary is, for each possible integer value of x, to compute y by applying (15), and b from (1), in the form

$$b = y^2 - 10x^2 > 0. \tag{16}$$

If the resulting value of b is less than 10, we have an acceptable solution to our problem. For example, consider the three consecutive cases:

TABLE III

x	y	x^2	y^2	b
79	250	6241	62500	90
80	253	6400	64009	9
81	257	6561	66049	439

Only $x = 80$ yields an acceptable solution.

Using this algorithm, all the acceptable solutions of the Diophantine equation (1) for $0 \leq x \leq 500{,}000$ have been computed and are tabulated below.

TABLE IV

n	x	y	x^2	y^2	b
1	0	1	0	1	1
2	0	2	0	4	4
3	0	3	0	9	9
4	1	4	1	16	6
5	2	7	4	49	9
6	4	13	16	169	9
7	5	16	25	256	6
8	6	19	36	361	1
9	12	38	144	1444	4
10	18	57	324	3249	9
11	43	136	1849	18496	6
12	80	253	6400	64009	9
13	154	487	23716	237169	9
14	191	604	36481	364816	6
15	228	721	51984	519841	1
16	456	1442	207936	2079364	4
17	684	2163	467856	4678569	9
18	1633	5164	2666689	26666896	6
19	3038	9607	9229444	92294449	9
20	5848	18493	34199104	341991049	9
21	7253	22936	52606009	526060096	6
22	8658	27379	74960964	749609641	1
23	17316	54758	299843856	2998438564	4
24	25974	82137	674648676	6746486769	9
25	62011	196096	3845364121	38453641216	6
26	115364	364813	13308852496	133088524969	9
27	222070	702247	49315084900	493150849009	9
28	275423	870964	75857828929	758578289296	6
29	328776	1039681	108093658176	1080936581761	1

We immediately observe that there appears to be a periodicity in the values of the incremental digit b: the period takes the form "1, 4, 9, 6, 9, 9, 6," in the data we have. This feature has been emphasized in the table by drawing a line above the data with $b = 1$. In each period, the values 1 and 4 occur only once, while 6 occurs twice and 9 occurs thrice.

At this point, bearing in mind the result embodied in Theorem 1, it seemed appropriate to examine the literature of this problem.[1] The search began with the discussion of *continued fractions* in Hardy & Wright (**HAR62**), thence moved on to Chrystal (**CHR64**), and so to Dickson (**DIC52**) and Gelfond (**GEL61**). It is found in the literature that the equation[2]

$$y^2 - 10x^2 = 1 \tag{17}$$

will have infinitely many solutions; and that, if the solution (x_1, y_1) is *minimal*, in the sense that

$$y_1 + x_1\sqrt{10} = \min\{\, y + x\sqrt{10} : x > 0,\ y > 0,\ y^2 - 10x^2 = 1 \,\}, \tag{18}$$

then *all* the solutions of (17) take the form

$$x_n = \frac{\alpha^n - \beta^n}{2\sqrt{10}}, \quad y_n = \frac{\alpha^n + \beta^n}{2}, \tag{19}$$

where

$$\alpha = y_1 + x_1\sqrt{10}, \quad \beta = y_1 - x_1\sqrt{10}. \tag{20}$$

We can readily derive from this that, in our case, $x_1 = 6$, $y_1 = 19$, and so

$$\alpha = 19 + 6\sqrt{10} \approx 37.973666, \tag{21a}$$

$$\beta = 19 - 6\sqrt{10} \approx 0.026334. \tag{21b}$$

Now consider a recurrence relation of the form

$$s_{n+2} = As_{n+1} + Bs_n. \tag{22}$$

This will apply to both the x_n and the y_n given in (19), if we can find coefficients A and B, such that, for *all* n, both α and β satisfy

$$\lambda^{n+2} = A\lambda^{n+1} + B\lambda^n,$$

i.e.,

$$\lambda^2 = A\lambda + B. \tag{23}$$

Substituting (21), we see that this is indeed possible, when

$$721 \pm 228\sqrt{10} = A(19 \pm 6\sqrt{10}) + B,$$

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1 See Chrystal (**CHR64**), Chaps. 32 and 33; Dickson—where a further, very extensive, historical bibliography is supplied—(**DIC52**), Chap. 12; Gelfond (**GEL61**), Chaps. 4 and 5; and Hardy & Wright (**HAR62**), Chap. 10.

2 A very careful analysis of $y^2 - ax^2 = 1$, a generalized form of (17), is given by Gelfond. This equation is often referred to as *Pell's equation;* but appears to have been first considered by *Fermat*.

and these two equations have the unique solution

$$A = 38, \quad B = -1. \tag{24}$$

Thus, both the x_n and the y_n satisfy the recurrence

$$s_{n+2} = 38s_{n+1} - s_n. \tag{25}$$

We can verify that these relations are borne out by the solutions tabulated on page 4 above, for the $b = 1$ entries only. Thus, we have the following result.

Theorem 2. *All the integer solutions of* (1) *when* $b = 1$ *are given by the recurrence* (25), *with initial values* $x_0 = 0, x_1 = 6, y_0 = 1, y_1 = 19$.

It is interesting to note that *all* the x and y entries in Table IV appear to satisfy the relation (25), when one skips from each entry to the *seventh* entry down the table as successor.

Now suppose that (x, y) is any Diophantine (integer) solution of (1) with $b = 1$ and that (p, q) is any Diophantine solution of (1) with $b \neq 1$ (that is, by (6), with b chosen arbitrarily to be 4, 6, or 9). Let us write

$$\left.\begin{aligned} u &= qx + py, \\ v &= 10px + qy. \end{aligned}\right\} \tag{26}$$

Then

$$\begin{aligned} v^2 - 10u^2 &= (10px + qy)^2 - 10(qx + py)^2 \\ &= (100p^2 - 10q^2)x^2 + (q^2 - 10p^2)y^2 \\ &= (q^2 - 10p^2)(y^2 - 10x^2) = b \times 1 = b; \end{aligned} \tag{27}$$

so that (u, v) is a solution of (1) with the given value of b; and, clearly, u and v will be integers, since p, q, x, and y are integers. Furthermore, if we consider

$$(x, y) = (x_n, y_n), \quad (x_{n+1}, y_{n+1}), \quad \text{and} \quad (x_{n+2}, y_{n+2}), \tag{28}$$

related by (25), then, by (26), the corresponding pairs

$$(u, v) = (u_n, v_n), \quad (u_{n+1}, v_{n+1}), \quad \text{and} \quad (u_{n+2}, v_{n+2}), \tag{29}$$

will be related by

$$\begin{aligned} u_{n+2} &= qx_{n+2} + py_{n+2} = q(38x_{n+1} - x_n) + p(38y_{n+1} - y_n) \\ &= 38(qx_{n+1} + py_{n+1}) - (qx_n + py_n) = 38u_{n+1} - u_n, \end{aligned} \tag{30a}$$

and

$$\begin{aligned} v_{n+2} &= 10px_{n+2} + qy_{n+2} = 10p(38x_{n+1} - x_n) + q(38y_{n+1} - y_n) \\ &= 38(10px_{n+1} + qy_{n+1}) - (10px_n + qy_n) = 38v_{n+1} - v_n. \end{aligned} \tag{30b}$$

That is to say, starting with any integer solution (p, q) of (1) with given integer b, we can generate an infinite sequence of such solutions (u_n, v_n), in one-to-one correspondence with the solutions (x_n, y_n) of (1) with $b = 1$. All these sequences satisfy the same recurrence, (25). Note that this is equivalent to

$$s_n = 38s_{n+1} - s_{n+2}, \tag{31}$$

so that the recurrence proceeds without end, both forward and backward (i.e., both as n increases and as it decreases), yielding integer solutions. Also, taking $(x, y) = (x_0, y_0) = (0, 1)$, we see that

$$(u_0, v_0) = (p, q). \tag{32}$$

Substituting (19) in (26), we get that

$$u_n = qx_n + py_n = \left(\frac{q + p\sqrt{10}}{2\sqrt{10}}\right)\alpha^n - \left(\frac{q - p\sqrt{10}}{2\sqrt{10}}\right)\beta^n \tag{33a}$$

and $$v_n = 10px_n + qy_n = \left(\frac{q + p\sqrt{10}}{2}\right)\alpha^n + \left(\frac{q - p\sqrt{10}}{2}\right)\beta^n, \tag{33b}$$

Thus we get the following result.

Theorem 3. *Given any integer solution (p, q) of (1) with given b, we can generate an infinite sequence of such integer solutions (u_n, v_n), given explicitly by (33). These solutions are in one-to-one correspondence, given by (26), with the solutions (x_n, y_n) of (1) with $b = 1$.*

It is easily verified from (19) that

$$x_{m+n} = x_m y_n + x_n y_m, \quad y_{m+n} = y_m y_n + 10x_m x_n. \tag{34}$$

Hence, by (33), $$u_{m+n} = q(x_m y_n + x_n y_m) + p(y_m y_n + 10x_m x_n),$$

i.e., $$u_{m+n} = v_m x_n + u_m y_n, \tag{35a}$$

and $$v_{m+n} = 10p(x_m y_n + x_n y_m) + q(y_m y_n + 10x_m x_n),$$

i.e., $$v_{m+n} = 10u_m x_n + v_m y_n. \tag{35b}$$

From this we obtain the following result.

Theorem 4. *We can replace (p, q) by any member of the sequence of pairs (u_m, v_m) in (26) and (33) without changing the sequence of solutions defined in Theorem 3.*

Now, since, by (1), $q^2 - 10p^2 = b > 0$, and we assume $p \geq 0$ and $q \geq 0$, it follows that

$$q + p\sqrt{10} \geq q - p\sqrt{10} > 0. \tag{36}$$

We can write (33) as

$$u_n = \mu\alpha^n - \nu\beta^n, \quad v_n = \sqrt{10}\,(\mu\alpha^n + \nu\beta^n), \tag{37}$$

with

$$\mu = \frac{q + p\sqrt{10}}{2\sqrt{10}} \geq \nu = \frac{q - p\sqrt{10}}{2\sqrt{10}} > 0, \quad \frac{\mu}{\nu} \geq 1. \tag{38}$$

(Equality occurs in (38), if and only if $p = 0$.) Therefore, by (21), and since

$$\alpha\beta = 1, \tag{39}$$

there will be a unique integer $m \geq 0$, such that

$$\alpha^{2m+2} > \frac{\mu}{\nu} \geq \alpha^{2m}; \tag{40}$$

and then

$$\mu\alpha^{-m} \geq \nu\beta^{-m},$$

i.e.,

$$u_{-m} \geq 0; \tag{41a}$$

while

$$\mu\alpha^{-m-1} < \nu\beta^{-m-1},$$

i.e.,

$$u_{-m-1} < 0. \tag{41b}$$

Furthermore, by (21), (37), and (38),

$$u_{n+1} - u_n = \mu\alpha^n(\alpha - 1) + \nu\beta^n(1 - \beta) > 0, \tag{42}$$

so that the sequence of u_n is *strictly monotone-increasing*, and so

$$u_n \geq 0 \quad \text{if and only if} \quad n \geq -m. \tag{43}$$

We have thus shown that there will be a solution (u_{-m}, v_{-m}) with minimal positive first component (the second component is always positive).

Also, by (1) and (38),

$$\sqrt{\mu\nu} = \frac{\sqrt{q^2 - 10p^2}}{2\sqrt{10}} = \frac{\sqrt{b}}{2\sqrt{10}}. \tag{44}$$

Thus, by (40),

$$\alpha^{-m} \geq \sqrt{\frac{\nu}{\mu}}, \tag{45a}$$

and $\alpha^{-2m-2} < \frac{\nu}{\mu}$, so that $\alpha^{-m-1} < \sqrt{\frac{\nu}{\mu}}$, whence

$$\alpha^{-m} < \alpha\sqrt{\frac{\nu}{\mu}}. \tag{45b}$$

Thus, by (37), (45b), (39), (21), and (44),

$$u_{-m} = \mu\alpha^{-m} - \nu\beta^{-m} < \alpha\sqrt{\mu\nu} - \beta\sqrt{\mu\nu} = (\alpha - \beta)\sqrt{\mu\nu}$$

$$= 12\sqrt{10} \times \frac{\sqrt{b}}{2\sqrt{10}} = 6\sqrt{b} \le 6;$$

i.e.,

$$u_{-m} < 6. \tag{46}$$

This yields the following theorem.

Theorem 5. *The sequence of solutions specified in Theorem 3 contains a member between* $x_0 = 0$ *and* $x_1 = 6$.

It follows that we can always adopt (p, q) as the member (u_{-m}, v_{-m}) defined above, which lies in $(0, 6)$. Let us do this, from now on; so that the member of the sequence (u_n, v_n) with minimal non-negative first component will be identified as $(u_0, v_0) = (p, q)$. This proves the following theorem.

Theorem 6. *All the integer solutions of* (1) *with* $y > 0$ *are identified as the sequences defined in Theorem 3, with*

$$(p, q) = \left\{ \begin{array}{l} (0, 1) \ \ \textit{for} \ \ b = 1, \\ (0, 2) \ \ \textit{for} \ \ b = 4, \\ (0, 3) \ \textit{or} \ (2, 7) \ \textit{or} \ (4, 13) \ \ \textit{for} \ \ b = 9, \\ (1, 4) \ \textit{or} \ (5, 16) \ \ \textit{for} \ \ b = 6 \end{array} \right\}. \tag{47}$$

Proof. Theorem 2 restates the known result, that all integer solutions of (1) when $b = 1$ are given by (19) with (20), or, alternatively, by the sequence satisfying (25), with initial values $(0, 1)$ and $(6, 19)$. Theorem 3 tells us that, if (p, q) is any integer solution of (1) with some other allowable value of b (i.e., by (6), one of 4, 6, and 9), then we can generate a sequence of solutions (u_n, v_n) which correspond one-to-one to the (x_n, y_n), *via* (26) or (33). Theorem 4 tells us that there will always be a member of such a sequence lying strictly between $x_0 = 0$ and $x_1 = 6$. Therefore, by inspection of Table IV, we see that the only such sequences possible are those enumerated in (47).[3]

3 There is the additional, isolated case of $x = 0, y = 0, b = 0$—see (2)—but this leads to no other solutions; because the general solution (37) has $\mu = \nu = 0$.

Now, let us suppose that, for given n, (x_n, y_n), (u_n, v_n), and (r_n, t_n) are all integer solutions of (1), belonging to sequences defined as in (19), (20), and (33), and that

$$0 = x_0 \leq u_0 \leq r_0 < x_1 = 6 \tag{48a}$$

and

$$1 = y_0 < v_0 < t_0 < y_1 = 19. \tag{48b}$$

Then, by (33),

$$y_0x_n + x_0y_n < v_0x_n + u_0y_n < t_0x_n + r_0y_n < y_1x_n + x_1y_n,$$

or, by (34),

$$x_n < u_n < r_n < x_{n+1}; \tag{49a}$$

and

$$10x_0x_n + y_0y_n < 10u_0x_n + v_0y_n < 10r_0x_n + t_0y_n < 10x_1x_n + y_1y_n,$$

or

$$y_n < v_n < t_n < y_{n+1}. \tag{49b}$$

This last result yields the following.

Theorem 7. *The seven sequences of integer solutions of* (1) *for various b, developed in Theorems 2–6 are interlaced, in the sense that each sequence retains its member-by-member order, relative to the others. In other words, if we list solutions* (as in Table IV) *in order of increasing y-values, every seventh entry belongs to the same sequence.*

We have thus proved that the conjectures generated by examination of Table IV are universally true for the Diophantine problem (1). Now let us generalize this to

$$ax^2 + b = y^2, \quad 0 \leq b \leq a-1, \quad x \geq 0, \quad y \geq 0. \tag{50}$$

As was mentioned in Footnote 2,[4] so long as a is not a perfect square, the equation with $b = 1$,

$$y^2 - ax^2 = 1, \tag{51}$$

is known to have infinitely many integer solution (x, y), all of them taking the form[5]

$$x_n = \frac{\alpha^n - \beta^n}{2\sqrt{a}}, \quad y_n = \frac{\alpha^n + \beta^n}{2}, \tag{52}$$

where[6]

$$\alpha = y_1 + x_1\sqrt{a}, \quad \beta = y_1 - x_1\sqrt{a}, \tag{53}$$

4 Compare (17).

5 Compare (19).

6 Compare (20).

and (x_1, y_1) is the (unique) solution which minimizes $y + x\sqrt{a}$. The entire argument presented above for $a = 10$ clearly generalizes to any a which is not a perfect square.[7] Thus, Theorems 2–7 hold for any such a, not just for $a = 10$, *mutatis mutandis*, as to the particular solution (x_1, y_1).

For instance, if $a = 7$, then $0 \le b \le 6$. If $x = 0$, then the only acceptable solutions have $(b, y) = (0, 0)$, $(1, 1)$, and $(4, 2)$; if $x = 1$, then $(b, y) = (2, 3)$ is the only solution; and if $x \ge 2$, then, putting[8]

$$y = 7u + v, \quad -3 \le v \le 3, \tag{54}$$

we get[9]

$$7(x^2 - 7u^2 - 2uv) = v^2 - b. \tag{55}$$

Since the left-hand side is divisible by 7, it is easy to verify by enumeration that the only possibilities are $(b, v) = (1, \pm 1)$, $(2, \pm 3)$, and $(4, \pm 2)$. The argument in (7)–(15), now with[10]

$$x\sqrt{7} = z + \zeta, \quad 0 < \zeta < 1, \tag{56a}$$

so that

$$2x \le z < 3x; \tag{56b}$$

and (with integer c)

$$y = z + c, \tag{8a}$$

yields

$$4x(c - \zeta) \le 2z(c - \zeta) = b - c^2 + \zeta^2 < b \le 4, \tag{57}$$

which recovers that,

$$\text{if} \quad x \ge 1, \quad \text{then} \quad c = 1. \tag{11}$$

Hence,[11]

$$y = \lceil x\sqrt{7} \rceil, \tag{58}$$

yielding the appropriate version of Theorem 1.

Table IV is replaced by Table V, shown below.

We again observe a periodicity in the values of b, of the form "1, 4, 2." This feature has been emphasized in the table by drawing a line above the data with $b = 1$.

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7 The irrationality of $\sqrt{a}$ is used in proving the result embodieed in (51)–(53).

8 Compare (4).

9 Compare (5).

10 Compare (7b), (7c).

11 Compare (15).

TABLE V

n	x	y	x^2	y^2	b
1	0	1	0	1	1
2	0	2	0	4	4
3	1	3	1	9	2
4	3	8	9	64	1
5	6	16	36	256	4
6	17	45	289	2025	2
7	48	127	2304	16129	1
8	96	254	9216	64516	4
9	271	717	73441	514089	2
10	765	2024	585225	4096576	1
11	1530	4048	2340900	16386304	4
12	4319	11427	18653761	130576329	2
13	12192	32257	148644864	1040514049	1
14	24384	64514	594579456	4162056196	4
15	68833	182115	4737981889	33165873225	2
16	194307	514088	37755210249	264286471744	1
17	388614	1028176	151020840996	1057145886976	4

This shows that, for $a = 7$, we have

$$x_1 = 3, \quad y_1 = 8, \tag{59}$$

and so, by (53),

$$\alpha = 8 + 3\sqrt{7} \approx 15.937254 \tag{60a}$$

and

$$\beta = 8 - 3\sqrt{7} \approx 0.062746. \tag{60b}$$

Theorems 2–7 now follow, just as before, with appropriate modifications.

REFERENCES

CHR64 G. CHRYSTAL, *Algebra—An Elementary Text-book for the Higher Classes of Secondary Schools and for Colleges*. Volume 2, Seventh Edition, Chelsea Publishing Co., New York, 1964, 628 pp.

DIC52 L. E. DICKSON, *History of the Theory of Numbers*. Volume 2, *Diophantine Analysis*. Seventh Edition, Chelsea Publishing Co., New York, 1952, 803 pp.

GEL61 A. O. GELFOND, *The Solution of Equations in Integers*. W. H. Freeman & Co., San Francisco, 1961, 63 pp.

HAR62 G. H. HARDY, E. M. WRIGHT, *An Introduction to the Theory of Numbers*. Fourth Edition, Corrected, Clarendon Press, Oxford, 1962, 421 pp.