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Random Sequences in Generalized Cantor Sets

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This article presents a fast algorithm for generating random points in the finitely and infinitely defined generalized Cantor sets in the unit real interval.

KEY WORDS: Random sequences; generalized Cantor sets.

1. THE CANTOR SETS

Consider the closed unit interval

$$D_0^{(0)} = C_0 = U = [0, 1] = \{x: 0 \leqslant x < 1\} \tag{1.1}$$

Call C_0 the *generalized discrete Cantor set of order 0* and denote its length by

$$\delta_0 = 1 \tag{1.2}$$

Then we define C_n, the *generalized discrete Cantor set*[2] *of order n*, to be the union of 2^n equal closed intervals,

$$D_j^{(n)} = [\lambda_j^{(n)}, \mu_j^{(n)}] = \{x: \lambda_j^{(n)} \leqslant x \leqslant \mu_j^{(n)}\} = [\lambda_j^{(n)}, \lambda_j^{(n)} + \delta_n] \tag{1.3}$$

each of length

$$\delta_n = \mu_j^{(n)} - \lambda_j^{(n)} \tag{1.4}$$

and we obtain C_{n+1} by removing an open interval of length σ_{n+1} from the center of every interval $D_j^{(n)}$ ($j = 0, 1, 2,..., 2^n - 1$) making up C_n. Thus, we have

$$D_{2j}^{(n+1)} = [\lambda_{2j}^{(n+1)}, \mu_{2j}^{(n+1)}] = [\lambda_j^{(n)}, \lambda_j^{(n)} + \delta_{n+1}] \tag{1.5a}$$

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[2] See Mathematical Society of Japan (1977), p. 277.

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and

$$D_{2j+1}^{(n+1)} = [\lambda_{2j+1}^{(n+1)}, \mu_{2j+1}^{(n+1)}] = [\lambda_j^{(n)} + \delta_{n+1} + \sigma_{n+1}, \mu_j^{(n)}]$$
$$= [\mu_j^{(n)} - \delta_{n+1}, \mu_j^{(n)}] \tag{1.5b}$$

It follows that

$$\delta_n = 2\delta_{n+1} + \sigma_{n+1} \tag{1.6}$$

This sequence of sets clearly depends on the choice of the sequence

$$\Sigma = [\sigma_n]_{n=1}^{\infty} \qquad \text{with} \quad \sigma_1 > \sigma_2 > \cdots \sigma_n > \cdots > 0 \tag{1.7}$$

with

$$\delta_n > \sigma_{n+1} > 0 \tag{1.8}$$

Theorem 1. The relation (1.6), with (1.2), has the unique solution

$$\delta_n = 2^{-n} \left\{ 1 - \sum_{m=1}^{n} 2^{m-1} \sigma_m \right\} \tag{1.9}$$

and, further, if (1.7) holds, then (1.8) will hold if and only if

$$\sum_{m=1}^{\infty} 2^{m-1} \sigma_m \leqslant 1 \tag{1.10}$$

Proof. When $n=0$, the sum in (1.9) vanishes and the equation reduces to (1.2). Let us put

$$\gamma_n = 2^n \delta_n \tag{1.11}$$

so that, by (1.2),

$$\gamma_0 = 2^0 \delta_0 = 1 \tag{1.12}$$

Then (1.6) becomes

$$\gamma_m - \gamma_{m-1} = -2^{m-1} \sigma_m \tag{1.13}$$

and, by "telescoping" [summing (1.12) from $m=1$ to $m=n$ and canceling intermediate terms] we get that

$$\gamma_n - \gamma_0 = -\sum_{m=1}^{n} 2^{m-1} \sigma_m \tag{1.14}$$

which, by (1.11) and (1.12), yields that

$$2^n\delta_n = 1 - \sum_{m=1}^{n} 2^{m-1}\sigma_m \tag{1.15}$$

This is the (necessarily unique) solution (1.9).

Now, let (1.7) hold. If (1.10) is true, then

$$\sum_{m=1}^{n} 2^{m-1}\sigma_m + 2^n\sigma_{n+1} = \sum_{m=1}^{n+1} 2^{m-1}\sigma_m < \sum_{m=1}^{\infty} 2^{m-1}\sigma_m \leqslant 1 \tag{1.16}$$

whence

$$1 - \sum_{m=1}^{n} 2^{m-1}\sigma_m > 2^n\sigma_{n+1} \tag{1.17}$$

and therefore (1.8) follows, by (1.9). Conversely, if (1.8) is true, then, by (1.9),

$$1 - \sum_{m=1}^{n} 2^{m-1}\sigma_m = 2^n\delta_n > 2^n\sigma_{n+1} \tag{1.18}$$

whence

$$\sum_{m=1}^{n+1} 2^{m-1}\sigma_m < 1 \tag{1.19}$$

By letting $n \to \infty$, and noting that every $\sigma_m > 0$ by (1.7), we obtain (1.10). □

The limit set of the C_n is the *generalized Cantor set* (*of infinite order*), the intersection of all the C_n:

$$C = \bigcap_{n=0}^{\infty} C_n \tag{1.20}$$

The *original* "Cantor sets,[3]" are recovered from the generalized sets by putting

$$(\forall n \geqslant 1)\ \sigma_n = 3^{-n} \tag{1.21}$$

We then observe that (1.7) holds, and that

$$\sum_{m=1}^{n} 2^{m-1}\sigma_m = \tfrac{1}{3}\sum_{m=0}^{n-1} (\tfrac{2}{3})^m = \tfrac{1}{3}\left[\frac{1-(\frac{2}{3})^n}{1-\frac{2}{3}}\right] = 1 - (\tfrac{2}{3})^n \tag{1.22}$$

[3] See Georg Cantor's epoch-making series of papers (Cantor, 1879–1884).

whence, by (1.9),

$$\delta_n = 2^{-n}\left\{1 - \sum_{m=1}^{n} 2^{m-1}\sigma_m\right\} = 2^{-n}\{1 - [1 - (\tfrac{2}{3})^n]\}$$
$$= 2^{-n}(\tfrac{2}{3})^n = (\tfrac{1}{3})^n = 3^{-n} = \sigma_n \tag{1.23}$$

[Of course, the result (1.23) is obvious from the geometric construction—at each stage, every closed interval becomes two closed intervals, of one-third the length, by the removal of the open "middle third."] From (1.23) we see at once that (1.8) holds; and [compare (1.22)]

$$\sum_{m=1}^{\infty} 2^{m-1}\sigma_m = \tfrac{1}{3}\sum_{m=0}^{\infty} (\tfrac{2}{3})^m = 1 \tag{1.24}$$

which agrees with (1.10).

The Cantor set has a number of interesting topological properties. For example,[4] it is closed, compact, nowhere dense (i.e., it has no interior), dense-in-itself (i.e., every point is a limit point), perfect (i.e., closed and dense-in-itself), and totally disconnected (i.e., it contains no intervals). It contains an uncountable infinity of points (i.e., it has the same cardinality as the unit interval), but has Lebesgue measure zero. It is often used to yield counter-intuitive examples in topology and analysis; for example, a continuous, strictly increasing function that has derivative zero almost everywhere.

Theorem 2. If, in binary notation, we have

$$j = \alpha_n + 2\alpha_{n-1} + 2^2\alpha_{n-2} + \cdots + 2^{n-1}\alpha_1 = (\alpha_1\alpha_2\cdots\alpha_n)_2 \tag{1.25}$$

where

$$(\forall i \mid 0 \leqslant i < n)\ \alpha_i = 0 \text{ or } 1 \tag{1.26}$$

and we write

$$\varepsilon_n = \delta_n + \sigma_n \tag{1.27}$$

then

$$\lambda_j^{(n)} = \alpha_n\varepsilon_n + \alpha_{n-1}\varepsilon_{n-1} + \alpha_{n-2}\varepsilon_{n-2} + \cdots + \alpha_1\varepsilon_1 \tag{1.28}$$

Proof. When $n=0$, the sums in (1.25) and (1.28) disappear and (since $j < 2^n = 1$) only $j=0$ is allowed; so $\lambda_0^{(0)} = 0$, as required by (1.1). Suppose that (1.25)–(1.27) hold for all $n \geqslant 0$, and that (1.28) holds for all $n \leqslant k$, for some k. Then, by (1.25), for $n = k$, we have that

$$2j = 2\alpha_k + 2^2\alpha_{k-1} + 2^3\alpha_{k-2} + \cdots + 2^k\alpha_1 \tag{1.29a}$$

[4] See the general literature of *point-set topology* (e.g., Bourbaki, 1966, p. 338; Dugundji, 1966, pp. 22, 104–105, 112; Kelley, 1955, pp. 165–166; Mathematical Society of Japan, 1977, p. 277; Nanzetta and Strickland, 1971, p. 72; Pervin, 1964, p. 136; Sierpiński, 1956, pp. 143–145; Steen and Seebach, 1970, pp. 57–58).

and

$$2j+1=1+2\alpha_k+2^2\alpha_{k-1}+2^3\alpha_{k-2}+\cdots+2^k\alpha_1 \tag{1.29b}$$

i.e., both indices again take the form (1.25) with $n=k+1$. By (1.5), with (1.28) for $n=k$,

$$\begin{aligned}\lambda_{2j}^{(k+1)}&=\lambda_j^{(k)}\\ &=\alpha_k\varepsilon_k+\alpha_{k-1}\varepsilon_{k-1}+\alpha_{k-2}\varepsilon_{k-2}+\cdots+\alpha_1\varepsilon_1\end{aligned} \tag{1.30a}$$

and

$$\begin{aligned}\lambda_{2j+1}^{(k+1)}&=\lambda_j^{(k)}+\varepsilon_{k+1}\\ &=\varepsilon_{k+1}+\alpha_k\varepsilon_k+\alpha_{k-1}\varepsilon_{k-1}+\alpha_{k-2}\varepsilon_{k-2}+\cdots+\alpha_1\varepsilon_1\end{aligned} \tag{1.30b}$$

Both these results agree with (1.28) for $n=k+1$. Therefore, by induction, we have (1.28) for all $n\geqslant 0$. □

We note from (1.2), (1.6), and (1.27) that

$$\delta_0=1, \qquad \delta_1=\tfrac{1}{2}(1-\sigma_1), \qquad (\forall n\geqslant 0) \quad \delta_{n+1}=\tfrac{1}{2}(\delta_n-\sigma_{n+1}) \tag{1.31}$$

and

$$\varepsilon_1=\tfrac{1}{2}(1+\sigma_1), \qquad (\forall n\geqslant 1) \quad \varepsilon_{n+1}=\tfrac{1}{2}(\varepsilon_n-\sigma_n+\sigma_{n+1}) \tag{1.32}$$

Corollary 1. If (1.21) holds, with (1.1)–(1.5), then we have

$$\begin{aligned}\lambda_j^{(n)}&=2\times 3^{-n}(\alpha_n+3\alpha_{n-1}+3^2\alpha_{n-2}+\cdots+3^{n-1}\alpha_1)\\ &=(0.(2\alpha_1)(2\alpha_2)\cdots(2\alpha_n))_3\end{aligned} \tag{1.33}$$

Proof. We return to the original Cantor sets. Here, by (1.21), (1.23), and (1.27), we have

$$\varepsilon_n=\delta_n+\sigma_n=2\times 3^{-n} \tag{1.34}$$

Since Theorem 2 still holds in this particular case, (1.28) becomes (1.33), by (1.34), and the corollary follows at once from (1.20). □

This particular result is, of course, well known.

In recent years, the Cantor set and its generalizations have played a major role as part of the basis of the theory of *fractals*.[5] As such, it has become of interest to applied mathematicians and computer scientists specializing in *computer graphics*. Since many calculations in computer graphics, as elsewhere in general computing, require prohibitively large

[5] See the literature of *fractals* (e.g., Barnsley, 1988, pp. 44–45, 75, 83, 130, 134, 147, 150–151, 156, 165, 179, 182, 187, 192, 265, 275, 302; Mandelbrot, 1983, pp. 3, 4, 14, 21, 76, 80–81, 82, 181, 313, 357, 406, 407, 409).

amounts of time and effort to carry out by the classical methods, the *Monte Carlo method* has been evolved to perform them by means of *random sampling*.[6] We are thus led to the problem of generating a sequence of random variables, independently uniformly distributed in C_n.

2. RANDOM POINTS IN C_n

The purpose of this article is to find a way to generate a sequence

$$\zeta = [\zeta_r]_{r=1}^{\infty} \tag{2.1}$$

of random variables (r.v.) ζ_r, independently uniformly distributed (i.u.d.) in the generalized Cantor set of order n, C_n, defined by (1.1)–(1.5), with respect to a given sequence Σ, as in (1.7). We have available to us so-called *canonical random generators*,[7] which yield sequences of "canonical random variables" (c.r.v.),

$$\xi = [\xi_r]_{r=1}^{\infty} \tag{2.2}$$

i.u.d. in the unit interval [0, 1]. (Such are the "random generators" we use in Monte Carlo computations, usually in the form of *pseudorandom* routines.) Therefore, if we can, for each $r = 1, 2, 3,\dots$, use ξ_r alone to generate ζ_r, the given statistical independence of the ξ_r will guarantee that of the ζ_r, and we have merely to solve the problem of making the ζ_r uniform in C_n. Henceforth, therefore, we shall omit the subscript r and seek to generate a r.v. ζ, uniform in C_n, from a r.v. ξ, uniform in [0. 1].

Since C_n is the disjoint union of 2^n equal intervals $D_j^{(n)}$, each of length δ_n [given by (1.9) or (1.31) in terms of Σ], the remaining problem divides into two parts: (i) to select an index κ, uniformly distributed in the set $\{0, 1, 2,\dots, 2^n - 1\}$, and thereby an interval $D_\kappa^{(n)} = [\lambda_\kappa^{(n)}, \mu_\kappa^{(n)}]$; (ii) to select a point ζ, uniformly distributed in the interval $D_\kappa^{(n)}$. It is clearly possible, given a c.r.v. η, to perform the second task by putting

$$\zeta = \lambda_\kappa^{(n)} + \delta_n \eta \tag{2.3}$$

3. THE ALGORITHMS

Consider a *canonical random variable* (c.r.v.) given in binary notation as

$$\xi = (0.\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n \cdots)_2 \tag{3.1}$$

[6] See the literature of *Monte Carlo methods*, e.g., Buslenko *et al.*, 1962; Carter and Cashwell, 1965; Ermakov, 1971; Halton, 1970; Hammersley and Handscomb, 1964; Kleijnen, 1975; Rubinstein, 1981; Sobol', 1973; Spanier and Gelbard, 1969; Yakowitz, 1977; Zaremba, 1968.

[7] See Halton, 1991, p. 66.

It is known[8] that all its binary digits α_i are independently uniformly distributed in $\{0, 1\}$ (i.e., independently take the values 0 and 1 with equal probability 1/2), and therefore that the two numbers,

$$\kappa = (\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n)_2 \quad \text{and} \quad \eta = (0.\alpha_{n+1}\alpha_{n+2}\alpha_{n+3}\cdots)_2 \tag{3.2}$$

are themselves uniformly distributed, η (like ξ) in the closed real interval $[0, 1]$, and κ in the integer set $\{0, 1, 2,..., 2^n - 1\}$.

Before computing the r.v. ζ (or a sequence of such r.v.), we must compute δ_n (as the computer variable `D`) and $\varepsilon_1, \varepsilon_2,..., \varepsilon_n$ (as the computer array `E[1, ..., n]`, where the computer constant `n` denotes n). To do this, we use (1.27), (1.32), the computer variable `r` as working space, and the given computer array `S[1, ..., n]`, in the following pseudocode algorithm.

Algorithm 1.

```
E[1] ← (1+S[1])/2;
for r=2 to n do E[r] ← (E[r−1]−S[r−1]+S[r])/2;
D ← E[n]−S[n];
```

Clearly, this algorithm takes time $O(n)$ to perform.

Now, to complete the tasks (i) and (ii) stated earlier, we need only execute the following pseudocode algorithm. Here, the computer variables `K`, `L`, `X`, `Y`, and `Z`, respectively, denote κ, $\lambda_\kappa^{(n)}$, ξ, η, and ζ; while `r`, `V`, and `W` are working space.

Algorithm 2.

	NOTES
`L ← 0;`	
`V ← X×2^n;`	$V = (\alpha_1\alpha_2\cdots\alpha_n.\alpha_{n+1}\alpha_{n+2}\cdots)_2$
`Y ← FRACT(V);`	$\eta = (0.\alpha_{n+1}\alpha_{n+2}\cdots\alpha_r)_2$
`K ← INTPT(V);`	$\kappa = (\alpha_1\alpha_2\cdots\alpha_n)_2$
`W ← K/2^n;`	$W = (0.\alpha_1\alpha_2\cdots\alpha_n)_2$ initially
`for r=1 to n do`	$W = (0.\alpha_k\alpha_{k+1}\cdots\alpha_n)_2$ in pass k
`{L ← L+E[r]×INTPT(2×W);`	$\lambda = \alpha_1\varepsilon_1 + \cdots + \alpha_k\varepsilon_k$
`W ← FRACT(2×W);`	$W = (0.\alpha_{k+1}\alpha_{k+2}\cdots\alpha_n)_2$
`}`	$\lambda = \alpha_1\varepsilon_1 + \cdots + \alpha_n\varepsilon_n$ finally
`Z ← L+D×Y;`	$\zeta = \lambda + \delta_n\eta$

Clearly, the algorithm takes time $O(n)$ to generate each of the required r.v. ζ.

Finally, we should consider the particular case of the original Cantor sets. In this case, (1.28) is replaced by (1.33), and the following algorithm applies.

[8] See Halton, 1991, Theorem A, pp. 68–69.

Algorithm 3.

```
L ← 0;
V ← X×2^n;
Y ← FRACT(V);
K ← INTPT(V);
W ← K/2^n;
while W>0 do
    {L ← 3×L+INTPT(2×W);
     W ← FRACT(2×W);
    }
Z ← 2×L/3^n+D×Y;
```

NOTES

$V = (\alpha_1 \alpha_2 \cdots \alpha_n . \alpha_{n+1} \alpha_{n+2} \cdots)_2$

$\eta = (0.\alpha_{n+1} \alpha_{n+2} \cdots \alpha_r)_2$

$\kappa = (\alpha_1 \alpha_2 \cdots \alpha_n)_2$

$W = (0.\alpha_1 \alpha_2 \cdots \alpha_n)_2$ initially

$W = (0.\alpha_k \alpha_{k+1} \cdots \alpha_n)_2$ in pass k

$L = 3^{k-1}\alpha_1 + \cdots + 3\alpha_{k-1} + \alpha_k$

$W = (0.\alpha_{k+1} \alpha_{k+2} \cdots \alpha_n)_2$

$L = 3^{n-1}\alpha_1 + \cdots + 3\alpha_{n-1} + \alpha_n$ finally

$\lambda = 2 \times 3^{-n} \times L; \quad \zeta = \lambda + \delta_n \eta$

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