A Denotational Semantics Approach to Functional and Logic Programming

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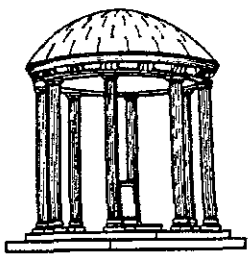

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A Denotational Semantics Approach to Functional and Logic Programming

by

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ABSTRACT

This dissertation addresses the problem of incorporating into lazy higher-order functional programming the relational programming capability of Horn logic. The language design is based on set abstraction, a feature whose denotational semantics has until now not been rigorously defined. A novel approach is taken in constructing an operational semantics directly from the denotational description.

The main results of this dissertation are:

(i) *Relative set abstraction* can combine lazy higher-order functional programming with not only first-order Horn logic, but also with a useful subset of higher-order Horn logic. Sets, as well as functions, can be treated as first-class objects.

(ii) *Angelic powerdomains* provide the semantic foundation for relative set abstraction.

(iii) The computation rule appropriate for this language is a modified parallel-outermost, rather than the more familiar left-most rule.

(iv) Optimizations incorporating ideas from narrowing and resolution greatly improve the efficiency of the interpreter, while maintaining correctness.

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1 INTRODUCTION

This chapter places functional and Horn logic programming within the general context of declarative programming languages. After reviewing previous functional/relational combinations, we provide an overview of our approach.

1.1 Declarative vs. Imperative Languages

Programming languages can be divided into two broad categories: *imperative* languages, which includes languages such as Fortran, Pascal and Ada [M83], and *declarative* languages, which includes (declarative subsets of) languages such as Prolog [CM81] and Lisp [M65]. One way to appreciate their difference is via Kowalski's celebrated equation "Algorithm = Logic + Control," [K79] meaning that an algorithm may be described as a combination of logical relationships and execution control. Imperative programming languages, having evolved from Von Neumann machine languages, express the program control explicitly, and leave the program logic implicit in the form of assertions that are invariant at various control points. Declarative programming languages reverse the relative emphasis of logic and control; they express the program logic explicitly, leaving much of the control implicit. In contrast to the machine orientation of imperative languages, declarative languages are programmer-oriented, and their syntax and semantics are based on mathematical theories predating the electronic computer.

Two important declarative subgroups are **functional** and **logic**. The most expressive functional programming languages (those which treat functions as computational objects) are based on *lambda calculus*; these include pure Scheme [AS85] and pure ML [M84], Miranda [T85] and Haskell [HW88]. Simpler (first-order) functional languages have been based on an *algebra of programs* [B78] and *recursion equation systems* [O85]. Most logic programming languages have likewise been based on the *first-order predicate calculus* [VK76, GM84], higher-order predicate

calculus [MN86], and also *equational logic* [O85, YS86, F84].

The benefits of declarative languages are, firstly that algorithms expressed in a declarative language are often easier to understand than when expressed in a more procedural language, since the parts of a program combine in a more predictable way. Often the declarative programs are shorter. Because of their absence of side-effects and explicit sequencing, they have great potential for parallel execution. Even on purely sequential machines they execute efficiently enough to be useful in many applications [P87]. As the ratio of programmer costs to hardware costs rises, and with programs becoming longer and more complex, declarative languages are becoming ever more attractive. Furthermore, declarative languages show great potential for implementation on massively parallel hardware [CK81, D74, GJ89, LWH88, M80, M82]. Declarative languages do not over specify the order of operations, and for many subcalculations execution order may be chosen arbitrarily. When execution order is known to be irrelevant, sub-calculations may be freely executed in parallel.

Another advantage of declarative over imperative languages, and indeed, a central focus of this dissertation, is that the *semantics* of declarative languages can be more easily given a mathematically rigorous treatment. Programming language semantics is a study of the association between programs and the mathematical objects which are their meaning. Different methods have been proposed to describe this association, e.g., denotational, operational, axiomatic, etc. [P81]. To specify the semantics of a language denotationally means to specify functions that assign mathematical objects to the programs *and* to parts of the programs in such a way that the semantics of a program expression depends only on the semantics (i.e. not the form) of its subexpressions. This kind of semantics seems most useful for describing the language constructs, i.e. for encapsulating the essence of the language design. Since the constructs of a declarative language are patterned after mathematical ideas, denotational semantics would seem seem to be the easiest way to describe a declarative language. Operational semantics specifies an abstract machine which would compute the output of a program. That is, an operational semantics can be viewed as a high-level description of a possible implementation. Since the constructs of an imperative language are designed with conventional hardware capabilities in mind, operational semantics would seem to be the easiest way to describe an im-

perative language. Axiomatic semantics seems most useful for proving properties of specific programs in a language. Since we are interested in language design and implementation, we will concentrate on denotational and operational semantics. As we are designing a declarative language, the defining semantics is the denotational; the operational semantics will be considered correct only to the extent it agrees with the denotational.

1.2 Paradigms of Declarative Programming

Various declarative paradigms have evolved independently of each other. We discuss some of the more popular ones, emphasizing their unique features.

1.2.1 Functional Programming

The functional programming paradigm, based on function definition and application, offers powerful tools for program modularization and abstraction. Typically, a computation may be decomposed into a hierarchy of smaller components. For instance, one operation might produce a data structure used by the next. In a purely imperative style, rather than defining each operation independently, references to their common data structures make the definitions of these operations mutually dependent. This makes it difficult to understand one part of the program independently of the rest, and hinders the reuse of code. In a more functional style of programming, an operation can provide data for the next via function composition. For instance, an expression denoting the result of one function is used as an input argument to another. The location or name of the intermediary result need not be explicitly given, thus allowing the two functions to be defined independently of each other. More specific features common to functional programming are listed below.

1) With a form of outermost, or *lazy evaluation*, the data structure defined by an argument expression is computed only to the extent necessary for its caller. With this strategy, a function can define a data structure that is conceptually infinite, such as an infinite list. With infinite lists, even interactive input-output can be described in a purely functional notation, as shown by Henderson [H80b]. The functions being composed can be modeled as concurrent processes.

2) Functional languages permit *higher-order* objects, i.e., functions may take other functions as arguments and produce other functions as results. With this feature, one can more easily abstract code common to several routines. With the

ability to create general-purpose program fragments, more program parts can be easily reused in new systems.

3) *Static scoping* permits functions to be defined within the context of other function definitions. This provides modularization via hierarchical functional decomposition and supports information hiding.

4) To help the programmer avoid errors, many programming languages provide a type system. An object's type constrains the range of values it may take, so that gross errors are caught early. Many functional programming languages provide *polymorphic* typing, so that typing can be parameterized.

5) To enhance its acceptance for practical programming, powerful compilation techniques have been developed, such as incremental compilation and combinator approaches [P87].

To give a flavor of the functional style of programming, we present a small program, which given a function to insert a binary operator between components of a list, easily defines functions to sum lists of numbers, compute their products, and even to append lists:

```
reduce (func, identity, list) is
      if null(l) then identity else
      func( head(list), reduce( func, identity, tail(list)) )

sumlist is lambda(list). reduce( +, 0, list)

prodlist is lambda(list). reduce( *, 1, list)

append(a, b) is reduce( cons, b, a)
```

1.2.2 (Horn) Logic Programming

In logic programming, computations are specified via logical constraints. Instead of defining the solution as a function from input to output, one merely states, in the form of *relations*, the properties the solution must satisfy. This gives programs a more flexible execution moding; not until execution need it be determined which parameters of a relation will be given values and which parameters are to be computed. Execution is a search procedure to find one or more solutions. In this sense, logic programming can be even more declarative than functional programs. Its chief advantages are:

1) conceptual simplicity, in that a program can be viewed as a set of assertions in first-order predicate logic; and

2) automatic availability of a function's inverse, i.e. when a function is programmed as a relation between its input and its output, its inverse is also automatically defined.

The most popular form of logic programming is based on first-order *Horn* logic, a subset of predicate logic. A Horn logic program defines and reasons with relations. Conceptually speaking, each program clause states that for all instantiations of its variables by first-order terms, if all the righthand side predicates are true, then so is the predicate on the lefthand side. We use the term *subgoal* to refer to one of the righthand side predicates. If a clause has no subgoals, then, for all possible instantiations, the lefthand side predicate is true. Such a clause is called a *unit* clause.

For instance, one can describe the append function as a relation between its input arguments and its output, as in the following Horn logic program:

```
append( [], X, X).
append( [H|T], Y, [H|Z]) :- append( T, Y, Z).
```

The goal is to find a set of bindings for the variables so that all the predicates in the user query are true. In a user query, one can either provide two lists to be appended, or one can request that two lists be found which, when appended, produce a known list:

```
?-    append( [1,2], [3,4], Ans).
?-    append( left, Right, [1,2,3,4]).
```

Since the universe of first-order terms can be enumerated, one could, in principle, generate all possible instantiations for the goal, and check each instantiation for suitability by verifying, if possible, the truth of each instantiated goal predicate via a derivation. A more efficient operational procedure, called *resolution*, tries to derive the truth of the original uninstantiated user query, binding values to the variables in the goal and program clauses only to the extent needed to keep the derivation going. Such variables are called *logical* variables, in current Prolog terminology.

Two basic operations of the resolution procedure are *unification* and *search*. In *unification*, the logical variables in a program and in the goal are given values so that

a predicate in the user query will equal the lefthand side predicate of the partially-instantiated program clause. When this can be done, the instantiated righthand side of the program clause replaces the matched predicate in the user query. In first-order Horn logic, two terms are equal if and only if they are identical, so unification can be efficiently implemented. Higher-order unification is much more difficult; in fact, testing the equality of two higher-order terms is not decidable, in general [R86].

"Search" refers to the fact that, an any stage in the derivation, one may have several applicable program clauses to choose from. *Breadth-first* search, in which one tries all possible choices, will find all solutions. This property is called *completeness*. Usually however, a *depth-first* search implemented via backtracking is used, because of its smaller space requirements. Depth-first search, however, sacrifices completeness.

One disadvantage of first-order Horn logic programming is its lack of higher-order capability, i.e. the inability to use relations themselves as objects. Warren described a way to encode some higher-order Horn logic programs within first-order Prolog [W83a]. The programmer associates a special (first-order) term with each predicate to be passed as an argument (or returned as a result) in place of the predicate itself. To apply the 'predicate', one calls a special 'apply' predicate, which has recorded which predicates are associated with which terms, applying the associated predicate. Though Warren has described a useful Prolog programming technique, higher-order predicates defined via his technique may lack *referential transparency*. Referential transparency requires that when two predicates name the same relation, they may be interchanged anywhere in the program without changing the program's declarative meaning (though execution efficiency may be affected). With Warren's scheme, a "higher-order" predicate testing two input predicates for equality would test the associated terms, instead. The term encodings might differ even if the predicates themselves define the same relation. Though Warren's technique provides the desired linguistic expressiveness, the possibility of nontransparent usage makes reasoning about programs more difficult. It is for this same reason that we consider first-order logic insufficient as the basis for higher-order programming, in contrast to the position taken in [G88]. One would prefer that higher-order capability be directly supported in the language's actual formal semantics.

Seeing the limitations of Warren's encodings, researchers [MN86, R86] are investigating programming languages based on subsets of of higher-order Horn logic. Unfortunately, full higher-order predicate logic is uncomputable, so most approaches restrict the higher-order capability to handle only specific subclasses of higher-order objects, or impose restrictions limiting their use. Miller and Nadathur have described a form of Horn logic incorporating terms from Church's typed lambda calculus [MN86], introducing some higher-order capability. However, their functions (lambda expressions) cannot be recursively defined and were not intended to provide full higher-order capability. Rather, they intended their system to be a useful tool for meta-programming, i.e. for building program transformation systems and theorem provers.

The next section discusses another variation of logic programming based on equational logic.

1.2.3 Equational Logic Programming

Though this dissertation does not deal explicitly with equational programming, we feel that some discussion is warranted, as equational programming is capable of combining many of the features of both functional and Horn logic programming [O85, F84, YS86, DP85]. Like Horn logic programming, certain forms of equational programming compute by solving constraints [YS86, F84, DP85]. As in functional programming, certain forms of equational programming can define functions, executing them with reasonable efficiency [O85]. As with Horn logic programming, it provides no higher-order capability.

In equational logic, rather than defining general predicates or relations, the program is a collection of parameterized equalities between terms, each implying that, for all instantiations of the (logical) variables, the two resulting terms are equal. Alternatively, the equality of a parameterization can be made conditional upon other pairs of terms being proven equal first. The goal of an equational logic program is to instantiate a user query so as to make it a logical consequence of the program.

To test whether two terms are equal, one uses the equations as substitutions, to see whether one can rewrite the two terms to identical forms. This can be very difficult, as one must compare each equivalent form of the left term in the goal

with each equivalent form of the right. Furthermore, for each term, the class of equivalent terms may be infinite.

To avoid the computational problems of unrestricted equational programming, *term-rewriting systems* were developed. In proving equality of two terms, the program equations are used in one direction only, i.e. one instantiates a program equation so that its left side matches a portion of the term being rewritten, and that portion is then replaced by the instantiated right-hand side. In order to guarantee the sufficiency of this mechanism, the equation set must be *confluent*. That is, it must be guaranteed that, if a term can be rewritten using more than one equation, all results must eventually converge to a common result. It is easiest to prove confluence when it can be shown that rewriting is always guaranteed to terminate, in which case, every term has a unique *normal form*, i.e., rewrites to a single term which can no longer be rewritten.

Confluent and terminating term-rewriting systems are called *canonical*. When a canonical term-rewriting system is viewed as an equational program, it has the property that all equal terms will rewrite to a common term, called a *normal form*, which cannot be further rewritten. This greatly increases the efficiency of the equality test, as one need not only compare two normal forms to test whether two terms are equal. However, one disadvantage of using canonical term-rewriting systems for functional programming is that the termination requirement rules out functions and relations operating on infinite data structures.

An operational mechanism known as *narrowing* [H80a], which combines reduction and unification, allows one to solve for logical variables [DP85] [YS86] in a goal equation. This technique reduces the parameterized equation via the rewrite rules as much as possible. When the presence of logical variables in the goal prevents further reduction, the variables are replaced by somewhat more defined values (terms which may contain new logical variables) in order that reduction may continue. When the equality becomes apparent, the accumulated bindings for the logical variables provide the solution. For completeness, each time a logical variable is narrowed, one must compute in a breadth-first manner many alternative narrowings. Unfortunately, the branching on narrowing tends to quickly get out of hand.

Constructor-based equational programming systems [R85] [JS86] [F84b] seem

to ameliorate the above problems. Certain functors are taken as irreducible data constructors, and other functors are assumed to name functions. An equation's left side is restricted to contain only one function name, placed at the outermost level, thus distinguishing between equations to define functions, and equations stating properties of functions, and permitting only the former. A term now no longer stands for itself, but rather denotes the normal form (which is built solely of constructors). With this restriction, the distinction between term-rewriting and first-order functional programming begins to blur, and with narrowing, one gains Horn logic's capability to satisfy constraints, as seen in the following example:

```
append([ ], y) = y
append([h | t], y) = cons(h, append(t, y))
?-append([1,2],[5,6]).

?-append(x, y) = [1, 2, 3].
```

In a constructor-based system, narrowing becomes more efficient (at each step, fewer narrowings need be considered) [F84b] [JS86]. Where reduction without narrowing suffices, non-terminating programs (denoting infinite objects) might possibly be supported. Proofs of confluence are still required, though they are perhaps easier to find. In some ways, reduction, the operational strategy of functional programming, resembles term-rewriting. We should therefore not be surprised if the operational procedure of a language combining functional and logic programming would similarly resemble narrowing.

1.2.4 Functional and Logic Programming Combinations

Sometimes we wish to combine both functions and relations within one program. A number of attempts have been made to combine features of functional and logic programming into a single language (see [BL86] for a recent survey). Either one can add functional programming to a Horn logic base, or one can add relational capability to a functional programming language.

Some have proposed adding function definition capability to Horn logic via an *equational theory* [K83] [GM84]. The equational theory may be provided via a canonical term rewriting system. Syntactic examination of a predicate's arguments no longer suffices when judging whether a program clause is relevant. Instead, the interpreter must ascertain whether a predicate's arguments might rewrite to terms

matching the program clause, or whether patterns in the program clause may be rewritten to match the arguments. The computational complexity of this inference mechanism is a major difficulty.

In describing his language *Fresh* [SP85], Smolka begins with a functional language, written in an equational style to incorporate pattern-matching, and describes an operational semantics reminiscent of narrowing. The resulting language is very expressive, providing a higher-order capability via a technique similar to Warren's (described above). As with Warren's encodings, referential transparency is lost. It is unclear what would be a meaningful purely declarative subset, as Smolka did not provide a denotational description.

Though Horn logic relations are defined in predicate logic, these relations could just as easily be described via set theory. In fact, Horn logic's model-theoretic and fixed-point semantics are described in the language of set theory. One approach to combining functional and logic programming is to add sets as another data type. Robinson and Darlington were the first advocates of adding logic programming capability to functional programming through set abstraction. In describing *SUPERLOGLISP* [BRS82], Robinson suggests that a functional language should have a construct denoting the complete set of solutions to a Horn logic program, and that the user be able to build functions accepting such sets as arguments. Darlington calls his extension *absolute set abstraction* [D83, DFP86] to distinguish it from *relative set abstraction*, discussed later. Absolute set abstraction permits expressions such as

$$\{x : p(x)\},$$

to denote define the set of all x satisfying $p(x)$. In this approach, nondeterminism is replaced by set union, and unification is performed to solve equations between non-ground objects.

Robinson's language, SUPERLOGLISP, combines LISP and Horn logic through absolute set abstraction. He develops many useful implementation ideas, but does not develop a mathematical justification or a formal semantics. Since the base language is LISP, LOGLISP has some higher-order capability, though its use is restricted when accessing the relational features. As in LISP, stream-based programming is not supported, as arguments must be evaluated before being passed

to functions.

Darlington's approach is similar; however, his base functional language is lazy, with polymorphic typing. In his recent paper [DFP86], Darlington sketched a partial and informal operational semantics. In [DG89], he described absolute set abstraction in a strictly first-order equational language as a variation of narrowing in term-rewriting systems.

The work of Darlington and Robinson leaves several important open problems: In what way does this construct interact with other traditional functional language features, such as infinite and higher-order objects? How can the presence of this feature be reflected in the language's denotational semantics? Will all denotable sets be computable? To our knowledge, these semantic issues have never been rigorously worked out.

1.3 Proposed Approach

Our goal is a language incorporating both functional and logic programming, and providing the following features:

1) Simple semantics through *referential transparency.* For instance, functions should be completely described by the mapping defined; two alternative definitions describing the same mapping should be indistinguishable within the language. Analogously, sets should be completely described by the elements contained; we should not be able to distinguish two different expressions of the same set, though orderings of the elements may differ. In other words, we would like to satisfy the *axiom of extensionality* [S77] for functions and sets.

2) Higher-order objects should be *first-class*, i.e., they can be used freely as function or predicate arguments and results.

3) Possibility of efficient execution. Backtracking should not be used where simple rewriting is sufficient, and the interpreter should not rely on computationally explosive primitives, such as higher-order unification or unification relative to an equational theory.

4) Verifiably correct execution mechanism, i.e., the operational and denotational semantics should describe the same language.

We make functional programming with set abstraction the basis for our unified

declarative language because:

1) simple propagation of objects can be managed without equality tests (important for higher-order objects);

2) it should be easier to add the constructs from as small and simple a language as Horn logic into the larger and more complicated functional programming, rather than vice-versa. Functional programming languages have a richer domain of objects, compared with Horn logic's flat domain, and therefore a larger variety of constructs; and

3) ordinary functional computations not making use of set abstraction might be executed without backtracking.

For simplicity, we will not consider typing mechanisms, whether polymorphic or otherwise, though we see no reason why such a feature could not be added.

1.3.1 Relative Set Abstraction

Neither Robinson nor Darlington have been able to implement absolute set abstraction as a first-class object, interacting freely with other functional language features. Both Darlington and Robinson claim that implementation of absolute set abstractions as first-class objects would require higher-order unification, which is not always computable. Even so, some higher-order programs in their languages would merely be unexecutable program specifications. Robinson has criticized existing combinations of higher-order functional programming with first-order relational programming as inelegant [R86]. His goal is to create a purely declarative functional language permitting higher-order relational programming, without arbitrary unorthogonal restrictions on its features. But, this line of work has yet to be fully explored.

Replacing absolute set abstraction with the semantically simpler *relative set abstraction*, the notation avoids the suggestion that full higher-order logic programming capability *ought* to be available. This removes the sense of inelegance Robinson noted. A typical relative set abstraction would be an expression of the form:

$\{f(x) : x \in G \text{ and } C(x)\}$.

Here, the generating set `G` is provided explicitly, and those elements `x` which satisfy the condition `C` are used in computing elements of the new set. Compare this to

the form of a typical absolute set abstraction:

$\{\mathtt{f(x)} : \mathtt{C(x)}\}$.

Here, one "solves" the condition C for suitable values of x, each solution used to compute an element f(x) of the denoted set.

The absolute construct is powerful, but its higher-order extension is problematic. The implementations of the languages of Darlington and Robinson restrict a logical variable to represent only first-order terms. This restricted domain is analogous to the *Herbrand universe* in first-order Horn logic. Thus weakened, absolute set abstraction is no longer more powerful than relative set abstraction. This set of first-order terms, T, can easily be expressed via a recursively-defined relative set abstraction. Thus, any first-order absolute set abstraction can easily be expressed as a relative set abstraction. For instance, the example above would be written as:

$\{\mathtt{f(x)} : \mathtt{x} \in \mathtt{T} \text{ and } \mathtt{C(x)}\}$.

We observe that *relative* set abstraction can *also* provide the needed logic programming capability. We prefer relative set abstraction because it has a more tractable higher-order generalization. Not only is relative set abstraction as expressive as first-order absolute set abstraction (as shown above), but it can mix freely with higher-order constructs, without requiring arbitrary first-order restrictions.

David Turner pioneered the use of relative set abstraction in a functional programming language, KRC [T81]. However, in his language, sets are implemented as lists, and may be accessed as such, thus providing an implicit ordering on the elements. This violates the semantics of sets and does not ensure fairness. If computation with one element of a generator set diverges, the next element is never tried. With this kind of implementation, the construct is no longer described as set abstraction; rather, one speaks of *list comprehensions* [P87]. We, in contrast, advocate *true* relative set abstraction.

In our system, the set of first-order terms is provided as a (semantically unnecessary but operationally convenient) primitive. In computing a relative set abstraction, only if a variable x is recognized as being enumerated from the set of first-order terms is it treated as a logical variable. This special treatment is merely an optimization to the default 'generate and test' mechanism. We show that these set abstractions generated from the Herbrand universe can be identified, and opti-

mized to provide efficiency comparable to Darlington's procedure.

Thus, we propose a lazy, statically-scoped, higher-order functional language with relative set abstraction, to combine higher-order functional and logic programming.

1.3.2 Denotational Semantics

A formal mathematical description of objects computed in this language, objects such as atoms, lists, functions and sets, is given by the specification of the *semantic domain.* The denotational semantics consists of this and a function mapping of the language's syntactic statements to elements of the semantic domain. We choose to define the language through denotational semantics because, through this method, properties of a language can be determined at a glance. Consider for example the denotational equation for a cons structure:

$$\mathcal{E}[\![\texttt{cons}(\textit{expr1},\ \textit{expr2})]\!]\ \rho = <(\mathcal{E}[\![\textit{expr1}]\!]\ \rho)\ ,\ (\mathcal{E}[\![\textit{expr2}]\!]\ \rho)>.$$

This equation describes the meaning of the expression, cons(*expr1*, *expr2*), in the environment ρ, as depending upon the meaning of the subexpressions *expr1* and *expr2*. Note that the environment for one subexpression is unaffected by the presence of the other subexpression. That is, the computation of *expr1* can have no side-effects that might influence the value of *expr2*. Therefore, the two subexpressions can be computed independently, perhaps even in parallel. Through such semantic equations the declarative nature of the language may be seen.

Though others have proposed languages combining functional and logic programming through set abstraction [BRS82] [DFP86], to our knowledge we are the first to give a denotational description. One difficulty was finding a suitable domain to represent set-valued objects. We have found that *angelic powerdomains* suffice, and our reasons for this choice will be explained later in Chapter 3. Powerdomain theory is usually used to describe *nondeterministic* languages, i.e. where a program is said to denote the *set* of objects which *might* be computed in any single execution. Indeed, powerdomains were conceived for that very purpose. Our use of powerdomain theory is unusual, in that, rather than describing the results of a control structure (nondeterminism), it describes sets as an explicit data type.

1.3.3 Correct Operational Semantics

To demonstrate that the language is executable, we provide an operational semantics, and we show its correctness with respect to the denotational semantics. To do so, we show that the denotational description's function to map syntactic statements to the semantic domain can be viewed as a functional program for the interpreter, written in terms of lambda expressions and primitive semantic functions. Each primitive function is defined via a set of equations, and implemented through the use of these equations as simplification rules. This leaves unspecified only the order of evaluation. We choose a variation of the parallel-outermost rule, optimized in that some outermost computations may be delayed. That some degree of parallel evaluation is needed for complete evaluation of sets should not be surprising; as a *complete* implementation of Horn-logic also requires a degree of breadth-first evaluation.

1.3.4 Optimizations

The operational semantics described above is inefficient for two reasons. First, it is pure interpretation; no provision has been made for compile-time pre-computation. In defining the scope of this research, we chose to avoid such issues. Second, the operational semantics is inefficient when the set of first-order terms is the generator of a relative set abstraction (analogous to an absolute set abstraction); the procedure described above would blindly enumerate the infinite set of terms, instantiating a copy of the abstraction for every possible term. To avoid this second cause of inefficiency, optimizations are provided which permit the enumerated variable in such cases to be treated as a logical variable, instantiated only as needed to continue the computation. This optimized operational semantics was inspired by the narrowing technique from term rewriting systems [R85] and the resolution technique from logic programming [L84]. This development expresses our point of view that the logical variable concept is best understood as an operational optimization, and not as part of the language's declarative description.

1.3.5 Scope of the Research

This dissertation describes a programming language combining functional and logic programming, and exhibiting the characteristics we have set forth as desirable. The denotational semantics provides a deep understanding of the meaning of the

language constructs, and serves as a standard for correct implementation. We also develop two variations of operational semantics. The first serves two purposes:

1) to demonstrate that the language *can* in fact be correctly implemented; and

2) to serve as a basis for the second operational semantics.

The second operational semantics contains optimizations essential for efficient logic programming (the relation between the two operational semantics may be thought of as analogous to that between Horn logic proof theory and the resolution method [VK76]), and justifies our view that the logical variable is an operational, not a declarative concept. In this dissertation, we do not discuss the detailed implementation issues necessary for constructing a practical piece of software. Neither operational semantics, if implemented directly, would be very fast. Rather, operational issues are considered only in so far as they give us a deeper theoretical understanding of the language.

This introduction has given an overview of the dissertation, and a summary of related work. The remaining chapters are as follows. Chapter 2 provides the syntax of the new language, along with sample programs. Chapter 3 describes the denotational semantics. Chapter 4 derives a simple operational semantics. Chapter 5 provides a detailed description of the semantic primitives. Chapter 6 improves the operational semantics with optimizations for more efficient logic programming. Chapter 7 discusses and summarizes the results of this research.

2 THE POWERFUL LANGUAGE

This chapter introduces a functional language with relative set abstraction. We call the language **PowerFuL**, because **P**owerdomains provide the semantic basis for the kind of set abstraction needed to unite **F**unctional and **L**ogic programming. We first describe the syntactic features via BNF and informal explanation, and then provide sample programs chosen to illustrate the constructs, and to show that any first-order Horn logic program can be translated into PowerFuL.

Wishing to concentrate of semantic foundations, we emphasize the essential features, leaving out many convenient syntactic niceties. For instance, we do not provide the boolean connectives, as these can be easily programmed via the conditional. We also do not discuss (polymorphic) strong typing and numeric operations, though these features are not incompatible with PowerFuL.

2.1 Syntax of Constructs

A PowerFuL program is an expression to be evaluated. The syntax is:

expr ::= (*expr*) | A_1 | ... | A_n
| `cons(`*expr*`,` *expr*`)` | `car(`*expr*`)` | `cdr(`*expr*`)`
| `TRUE` | `FALSE` | `not(`*expr*`)` | `if(`*expr*`,` *expr*`,` *expr*`)`
| `bool?(`*expr*`)` | `atom?(`*expr*`)` | `pair?(`*expr*`)`
| `func?(`*expr*`)` | `set?(`*expr*`)` | *expr* `=` *expr* | `null?(`*expr*`)`
| *identifier*
| `letrec` *identifier* `be` *expr*`, ...,` *identifier* `be` *expr* `in` *expr*
| λ *identifier* . *expr*
| *expr* *expr*
| `phi` | *set-clause* | `U(`*set-clause*`,` *set-clause*`)`
| `bools` | `atoms` | `terms`

set-clause ::= {*expr* : *qual-list*}
qual-list ::= *qualifier, qual-list* | ϵ
qualifier ::= *enumeration* | *condition*
enumeration ::= *identifier* $\in$ *expr*
condition ::= *expr*

Most of these constructs have close analogs in other functional languages. We provide below a brief explanation of the above syntax, in the order of their appearance in the BNF.

• We can put parentheses around an expression for clarity, or to override the default left-associativity.

• We use A_i to indicate an arbitrary atom. In practice, an initial quote distinguishes an atom from an identifier.

• As in LISP, we use `cons` to construct ordered pairs; `car` and `cdr` select a pair's left and right elements, respectively. Lists may be written in the `[...]` notation, e.g. `['apple, 'orange, 'grape]` and have the usual nested-pair representation using `cons`.

• As in Scheme [AS85], condition predicates end in a '`?`'. The basis of equality testing is `atomeq?`, which compares atoms for equality. The condition `null?` tests whether its argument equals the atom `'nil`. The general equality condition answers false if its arguments are of incompatible types (e.g. an atom and a list), answers true or false if the arguments are two atoms or two booleans, and in the case of two ordered pairs, compares the respective left and right branches, recursively. It is undefined when comparing two functions or two sets. PowerFuL also provides the usual type-checking conditions: `bool?`, `atom?`, etc. As is required for full referential transparency (extensionality), equality between higher-order objects is not defined. The result of equating higher-order objects, such as sets or functions, is $\perp$.

• The conditional, `if(`*condition*`,` *expr2*`,` *expr3*`)`, may also be written as `if` *condition* `then` *expr2* `else` *expr3* `fi`.

• A function parameter is represented by an identifier. Lambda expressions are used to define functions. A lambda expression with more than one parameter, such as

λ `id1 id2.` *body*

is syntactic sugar for a lambda expression with a single parameter x representing a sequence. In this example, an occurrence of `id2` in the body would be replaced by `cdr(`x`)`. Similarly, when applying a "multi-argument" lamba expression, the argument list is converted to the appropriate list via `cons`.

• The syntactic form `letrec` is used to define statically scoped identifiers. A set of identifiers may be defined via mutually recursive definitions. A program is invalid if it contains a reference to an undefined identifier.

• Enumerations are the syntactic basis for relative set abstraction. Each identifier introduced within the set-clause is associated with a set expression to provide possible values. The scope of the enumerated identifier contains the principal expression (left of the ':'), and also all qualifiers to the right of its introduction. In case of name conflict, an identifier takes its value from the latest definition (innermost scope). In any case, the scope of an enumerated identifier never reaches beyond the set-clause of its introduction. When the qualifier is a condition, the expression to the left of the ':' is in the denoted set only if the condition evaluates to true. When there are no qualifiers to satisfy, the set-clause indicates a singleton set, and the ':' is usually omitted. Expressions of the form `U(`set_1`, ..., `set_n`)` are syntactic sugar for a nesting of binary unions.

• The syntax `bools` refers to the set { `TRUE`, `FALSE`}. Similarly, `atoms` is the set containing all the atoms A_i. The set `terms` is a union of `atoms`, `bools`, and any finite object which can be constructed by "cons"-ing together elements of those two sets.

2.2 Program Examples

To illustrate the language constructs and to show their applicability for functional and logic programming, we now provide a series of short programs in PowerFuL.

Functional Programming

```
letrec
   append be λ l1 l2. if null?(l1) then l2
                      else cons(car(l1), append(cdr(l1),l2)) fi
   map be λ f.λ l.if null?(l) then []
                      else cons(f(car(l)), map(f,cdr(l)))fi
   infinite be cons('a, infinite)
in
   ...
```

Higher-order functions and infinite objects can be defined in the usual manner. The `map` example shown above is in curried form.

Set Operations

```
letrec
    crossprod be λ s1 s2. {cons(X,Y) : X∈s1, Y∈s2}
    filter be λ p s. {X : X ∈ s, p(x)}
    intersection be λ s1 s2. {X : X∈s1, Y∈s2, X=Y}
in
   ...
```

The operations `crossprod` and `filter` are similar to those in Miranda [T85]. Note that one *cannot* define an operation to compute the cardinality of a set, nor can one test whether a value is or is not a member. Such an operation would be analogous to Prolog's negation by failure. This work concerns itself solely with Prolog's declarative capabilities, i.e. those based on pure Horn logic.

Logic Programming

```
letrec
   split be λ list. { [X|Y] : X∈terms, Y∈terms, append(X,Y)=list}
   append be λ l1 l2. if null?(l1) then l2
                      else cons(car(l1), append(cdr(l1),l2)) fi
in
   ...
```

The enumerations `X∈terms, Y∈terms` in `split` are needed because the set-abstraction is relative, not absolute.

Higher-order Functional and Horn logic programming

```
letrec
    one be λ v. 'a
    two be λ v. 'b
    three be λ v. 'c
in
   {F :  F ∈ U({one}, {two}, {three}), map(F)(['x,'y,'z]) = ['c,'c,'c]}
```

The result of the above set-abstraction is the set `{three}`. In this example, the generator set for `F, U({one}, {two}, {three})` is first enumerated to obtain a function which is then passed on to `map`. Those functions which satisfy the equality condition are kept in the resulting set, while the others are screened out.

2.3 Translating Horn Logic to PowerFuL

This sections discusses the translation of programs from Horn logic to PowerFuL. First, we reinterpret Horn logic clauses as statements about sets and set-membership, rather than about the truth of predicates. Then we show how such statements can be expressed in PowerFuL.

2.3.1 Converting Horn Logic to Set Logic

The Horn logic domain is the *Herbrand Universe* of first-order terms, i.e., those terms built from the constructs found in the program. Finite sequences of these terms are themselves in the Herbrand Universe (assuming a sequencing constructor is provided). The set of all possible applications of predicates to Herbrand Universe terms is called the Herbrand Base. The meaning of each Horn logic predicate is a subset of the Herbrand Base, that is, those applications of the predicate to terms which the program implies are true. Alternatively, we could say that the predicate is given meaning as the set of arguments on which it is true. For instance, an n-ary predicate is defined by the set of n-tuples for which the predicate is true. Such a set is called a *relation.* In a sense, each predicate names a relation. To say that a predicate applied to a tuple is true, is equivalent to saying that the tuple is an element of the relation. We view Horn logic clauses as statements about

set membership, where each set is a relation representing a predicate. Where a conventional Prolog program asserts *P(tuple)*, we could equivalently assert that *tuple* $\in$ *P*, where *P* now refers to a set.

For example, consider the following program and goal, written in Prolog syntax [CM81].

```
app([], Y, Y).
app([H|T], Y, [H|Z]) :- app(T, Y, Z).
rev([], []).
rev([H|T], Z) :- rev(T, Y), app(Y, [H], Z).
?- rev(L, [a, b, c]).
```

With a syntax more oriented towards sets, we could write:

```
[ [], Y, Y] ∈ app
[ [H|T], Y, [H|Z] ] ∈ app :- [T, Y, Z] ∈ app
[ [], [] ] ∈ rev
[ [H|T], Y] ∈ rev :- [T, Z] ∈ rev, [Z, [H], Y] ∈ app
?- [X, [a, b, c] ] ∈ rev
```

Here, we have used mutually-recursive definite clauses to define sets (instead of predicates). We could call this paradigm *set logic programming*, but it is really just another syntax for Horn logic.

2.3.2 Converting Set Logic to PowerFuL

Any such first order definite-clause set-logic program can be routinely converted into PowerFuL. The `letrec` command provides mutually recursive definitions, and each clause will correspond to a relative set abstraction. Where a predicate/set was defined with several clauses, we use a union of relative set abstractions. Within a relative set abstraction, each logical variable must be formally introduced as representing an element the Herbrand Universe, i.e. `terms`. To indicate that a particular first-order term is a member of a particular set, one lets the set instantiate an enumeration variable, and then one states that the enumeration variable equals the specified term.

Converting the above program to PowerFuL syntax results in:

```
letrec
   app be U( { [ [],L,L ] : L∈terms},
             {[ [H|T], Y, [H|Z] ] : H,T,Y,Z ∈ terms,
                         W∈app, W=[T,Y,Z]})
   rev be U( {[ [], [] ] },
             {[ [H|T], Z] : H,T,Y,Z ∈ terms, V∈rev, W∈app,
                         V = [T, Y], W = [Y, [H], Z]})
in
   { L : L∈terms, V∈rev, V = [L, ['a, 'b, 'c]] }
```

We have taken the liberty of writing `h,t,y,z ∈ terms` instead of four separate enumerations.

2.3.3 Discussion

The PowerFuL program uses sets to express Horn logic predicates, which the Horn logic program used, in turn, to define functions. With so many layers of indirection, it is no wonder the resulting PowerFuL version is ugly. Still, this technique of Horn logic to PowerFuL conversion demonstrates that we have indeed captured the full expressive power of Horn logic.

A better PowerFuL style would be to use Lisp-like functions where functions are intended, and sets only where necessary. One could always provide Prolog notation as a syntactic sugar wherever the relational style is more appropriate, recognizing that its semantics are to be understood in terms of PowerFuL. We have seen that functions such as append can be defined in PowerFuL and used in the usual way, and also can be used within a set abstraction to choose which inputs would yield a desired output. For both uses, only one function definition is required, which is convenient for the programmer. For efficiency, such a function might be compiled differently for use within and outside set abstractions. PowerFuL allows the interpreter to detect where backtracking is needed and where it is not. Actually, a theoretically-complete or *fair* implementation computes elements of sets *in parallel*, not via backtracking. This kind of optimization might permit a program to be much more efficient than would be if all functions had to be defined via set abstraction, or within Horn logic.

Translating programs from PowerFuL to Horn logic is more difficult. To be sure, one might convert PowerFuL programs to Horn logic by *implementing* a PowerFuL

interpreter in Horn logic, *mapping* PowerFuL's higher-order domain into Horn logic's first-order domain. Since PowerFuL's semantic domain is much richer than that of Horn logic, we see no general way to *directly* convert PowerFuL programs to Horn logic. It is easier to restrict oneself to using only PowerFuL's first-order terms, than to arbitrarily expand Horn logic's domain to include higher-order objects.

3 DENOTATIONAL SEMANTICS

This chapter presents a denotational definition of PowerFuL. After motivating some fundamental terminology, we describe PowerFuL's semantic domain, and define the function mapping PowerFuL syntax onto this domain. Especially noteworthy is the use of powerdomains as a semantic basis for a language with set abstraction.

3.1 Semantic Domains

This section reviews some concepts and terminology from the Scott-Strachey theory of denotational semantics. We will not try to provide a rigorous presentation, but only try to motivate some of the basic definitions, which we have taken from [S86]. A more detailed presentation can be found there, as well as in [S77]. Special attention is given to the theory of *powerdomains*, a type of domain construction not usually needed for functional programming.

Intuitively, a *domain* is the set of mathematical entities being manipulated as data objects by a program. Actually, a domain is a partially ordered set with certain technical properties which we will later define. Functions are defined recursively, and it is conceivable that in defining a function, the program might not provide a mapping for every possible input. So as to deal with true functions, rather than *partial* functions, a domain will include a special element to mean *undefined.* This element is usually written as $\perp$ (pronounced "bottom"). The undefined element results when an undefined operation is requested (such as dividing by zero), or when a definition is circular, possibly resulting in an infinite recursion.

Sometimes we wish to create new domains built upon domains already defined. For instance, given domains D_1 and D_2, we can create the domain $D_1 \times D_2$, which consists of all ordered pairs of elements from domains D_1 and D_2, respectively. For instance, domain $I_\perp \times I_\perp$ contains all possible ordered pairs of elements from

$I_\perp$. The elements in this domain are defined to varying degrees. The *least defined* element, $\perp_{I_\perp \times I_\perp}$ is the ordered pair where neither element defined, i.e. $< \perp, \perp >$. Note that, when combining domains like this, it is sometimes ambiguous as to which domain's least element $\perp$ refers. Where necessary to avoid ambiguity, we will subscript $\perp$ by the name of the domain intended. Here, the least defined element, $< \perp, \perp >$, is less defined than $< 2, \perp >$, which is, in turn less defined than $< 2, 3 >$.

Definition 3.1: A binary relation $\sqsubseteq$ over $D \times D$ is a *partial order* if $\sqsubseteq$ is reflexive, antisymmetric and transitive.

The partial ordering of ordered pairs depends upon the partial ordering on the respective elements, as follows.

Definition 3.2: For any two elements $< a_1, b_1 >$ and $< a_2, b_2 >$ of $D_1 \times D_2$, $< a_1, b_1 > \sqsubseteq < a_2, b_2 >$ iff both $a_1 \sqsubseteq_{D_1} a_2$ and $b_1 \sqsubseteq_{D_2} b_2$.

Definition 3.3: For any partial ordering $\sqsubseteq$ on a set D, if there exists an element $c \in D$ such that for all $d \in D$, $c \sqsubseteq d$, then c is the *least* element in D and is denoted by the symbol $\perp$.

Intuitively, one element A is *less defined than* or *approximates* another element B if A can be created by replacing part of B with something undefined.

A list of elements for which each element approximates the next, (e.g. $< \perp, \perp >, < 1, \perp >, < 1, 2 >$) is called a *chain.*

Definition 3.4: For a partially ordered set D, a subset X of D is a *chain* iff X is nonempty and for all $a, b \in X$, either $a \sqsubseteq b$ or $b \sqsubseteq a$.

The integer domain described above is very simple in that, aside from $\perp_I$, all its elements are fully defined. That is, for any $A, B \in I_\perp$, $A \sqsubseteq B$ iff either $A = \perp_I$ or $A = B$. The longest possible chain is of length 2. A domain with this structure is called a *discrete* or *flat* domain. Another example of a discrete domain is the domain of atoms (unlike the domain of integers, this domain has a finite number of elements).

When the information content of two partially-defined elements is consistent, there should exist an element which combines the information of each. We call such an element an *upper bound.* This motivates the following definition.

Definition 3.5: For a set D with a partial ordering $\sqsubseteq$, the expression $a \sqcup b$ denotes the element in D (if it exists) such that:

1) $a \sqsubseteq a \sqcup b$ and $b \sqsubseteq a \sqcup b$; and

2) for all $d \in D$, $a \sqsubseteq d$ and $b \sqsubseteq d$ imply $a \sqcup b \sqsubseteq d$.

Definition 3.6: For a set D partially ordered by $\sqsubseteq$ and a subset X (sometimes written as $\{x_i \mid i \in \mathcal{N}\}$) of D, $\bigsqcup X$ (pronounced the *least upper bound of* X, and sometimes written as $\bigsqcup_i x_i$) denotes the element of D (if it exists) such that:

1) for all $x \in X$, $x \sqsubseteq \bigsqcup X$; and

2) for all $d \in D$, if for all $x_i \in X$ then $x_i \sqsubseteq d$, then $\bigsqcup X \sqsubseteq d$.

It is not always possible to automatically detect (e.g. in an interpreter) that an arbitrary expression denotes $\perp$. When $\perp$ results from an infinite recursion, evaluation will fail to terminate. While waiting for a result, we can never be sure whether or not a result will eventually be forthcoming. Therefore, it is reasonable to demand that, the less defined an argument to a function is (i.e. the more places $\perp$ can be found), the less defined will be the output. This is expressed by a requirement that functions be *monotonic*, as defined below.

Definition 3.7: A function $f : A \mapsto B$ is *monotonic* iff, for every $a, b \in A$, if $a \sqsubseteq b$ (using the the partial order of domain A), then $f(a) \sqsubseteq f(b)$ (using the the partial order of domain B).

A function may be either *strict* or *non-strict*. Informally, a function is strict if the output is undefined whenever the input is undefined.

Definition 3.8: A function $f : A \mapsto B$ is *strict* iff, $f(\perp_A) = \perp_B$.

A constant function might be non-strict, for example the function $f : I_\perp \mapsto I_\perp$ which maps all inputs, including $\perp$, to 1.

In higher-order functional programming, functions are themselves used as data objects. So, domain $D_1 \mapsto D_2$ represents the domain of functions mapping elements of domain D_1 to elements of domain D_2. The partial order on functions is as follows.

Definition 3.9: Given two functions $f, g : D_1 \mapsto D_2$, we say that $f \sqsubseteq g$ iff for all $d \in D_1$, $f(d) \sqsubseteq g(d)$.

If the input domain of a function has infinite size, then the domain of functions will contain chains of arbitrary length, perhaps even chains of infinite length. The

least function is defined as follows.

Definition 3.10: For any domains A and B, the least defined function Ω is the function f such that for all $a \in A$, $f(a) = \perp_B$.

In a functional language, a function is typically defined as the *least fixpoint* of a *functional*. These terms are defined below.

Definition 3.11: For domains A and B, a monotonic function $f : A \mapsto B$ is *continuous* iff for any chain $X \subseteq A$, $f(\bigsqcup X) = \bigsqcup\{f(x) \mid x \in X\}$. A function of more than one variable is continuous iff it is continuous in each variable, individually.

Theorem 3.1: A monotonic function over a domain in which all chains are of finite length (e.g. a discrete domain) is continuous.

Proof: Let A be a domain in which all chains are finite, and let $f : A \mapsto B$ be any monotonic function. Let $X \subseteq A$ be any (finite) chain in A. Because X is finite, there must exist one element of the chain x_n such that for any $x_i \in X$, $x_i \sqsubseteq x_n$ and therefore $\bigsqcup_i x_i = x_n$. Since f is monotonic, the set $Y = \{f(x_i) \mid x_i \in X\}$ is a finite chain in B with $\bigsqcup Y = f(x_n)$. Therefore, $f(\bigsqcup_i x_i) = f(x_n) = \bigsqcup Y = \bigsqcup_i f(x_i)$. **End of Proof**

Definition 3.12: A *functional* is a function $\tau : D \mapsto D$ (usually D is a domain of the form $A \mapsto B$).

Definition 3.13: For a functional $\tau : D \mapsto D$, d is a *fixpoint* of τ iff $d \in D$ and $\tau(d) = d$.

Definition 3.14: For a functional $\tau : D \mapsto D$, d is the *least fixpoint* (lfp) of τ iff d is a fixpoint of τ, and for any other fixpoint e of τ, $d \sqsubseteq e$.

Theorem 3.2 (proved in [S86]): If the domain D is a pointed cpo, then the least fixpoint of a continuous functional $\tau : D \mapsto D$ exists and is defined to be

$$lfp\ \tau = \bigsqcup\{\tau^i(\perp_D) \mid i \geq 0\}, \text{ where } \tau^i = \tau \circ \tau \circ \ldots \circ \tau, \ i \text{ times.}$$

Definition 3.15: The meaning of a recursive specification $f = F(f)$ is taken to be $lfp(F)$, the least fixpoint of the functional denoted by F.

Consider the recursive definition, below, of the factorial function.

$\texttt{fact}(x) \;=\; \texttt{if}((x = 0), 1, x \times \texttt{fact}(x - 1))$.

The factorial is the least fixpoint of the functional

$\lambda f.\ \lambda x.\ \texttt{if}((x = 0), 1, x \times f(x - 1))$.

In essence, the recursive definition defines a chain of non-recursive lambda expressions, each of which we will call $\texttt{fact}_i$ for some integer i, each element of the sequence expanding fact's recursion one step deeper than the previous, and thus defining the factorial for yet another integer. This series of nonrecursive lambda expressions forms a chain in $I_\perp \mapsto I_\perp$, with the full factorial function being the least upper bound of this chain. In general, when an infinite object, such as the factorial function, is the least upper bound of a chain of elements in a domain, we would like the least upper bound itself also to be in the domain. That is, we want each domain to be a *complete partial order.* Since each domain will have a least element, each domain will be a *pointed* cpo.

Definition 3.16: A partially ordered set D is a *complete partial ordering* (*cpo*) iff every chain in D has a least upper bound in D.

Definition 3.17: A complete partial ordering is a *pointed complete partial ordering* (*pointed cpo*) iff it has a least element.

Applied to an argument x, the factorial function is computed by expanding the recursion one step at a time. Mathematically, what we do is compute the chain $\texttt{fact}_i(x)$, until for some i the chain converges. The rationale for doing this will be explained below.

Theorem 3.3: Any functional defined by the composition of monotonic functions and the variable of the functional, is continuous.

The details of the proof of Theorem 3.3 use induction on the structure of the functional. It may be found in [M74].

Domain constructors must also be continuous, if we are to construct new domains from non-trivial base domains. For instance, in constructing the domain $(I_\perp \mapsto I_\perp) \times I_\perp$, we must be certain that $< \bigsqcup_i \texttt{fact}_i,\ 2 > = \bigsqcup_i < \texttt{fact}_i,\ 2 >$. The requirement that domain constructors be continuous plays an important role in defining powerdomains, as seen in the next section. In defining PowerFuL's domain, we only use domain constructors known to be continuous.

Another domain constructor is the *sum*. Given domains $D_1, \ldots, D_n$, the domain $(D_1 + \ldots + D_n)_\perp$ is a domain containing all the elements from the n domains, and a new least element $\perp_{D_1 + \ldots + D_n}$. For any d, $e \in (D_1 + \ldots + D_n)$, $d \sqsubseteq e$ iff *either* $d = \perp_{D_1 + \ldots + D_n}$ *or* both d and e came from the same component domain D_i, and $d \sqsubseteq_{D_i} e$.

Richer domains can sometimes be described as solutions to recursive domain equations [S86]. For instance, the equation

$$D = (A + D \times D)_\perp$$

describes the domain consisting of atoms, and nested ordered pairs whose leaves are atoms, and are nested to arbitrary (and even infinite) depth. The equation is actually read to say that domain D consists of the atoms plus the domain of ordered pairs, the elements of these pairs coming from a domain isomorphic to D.

3.2 Powerdomains

The denotational semantics of set abstraction requires enriching the domain with a new kind of object representing a set of simpler objects. Intuitively, given a domain D, each element of domain D's powerdomain $\mathcal{P}$(D) is to be viewed as a set of elements from D. Powerdomain theory was originally developed to describe the behavior of nondeterministic calculations, for which a program denotes a set of possible results. The original application was operating system modelling, where results depend on the random timing of events, as well as on the values of the inputs. In describing powerdomains, we shall use examples in nondeterminism as motivation for the theory. Be aware, however, that in PowerFuL we use powerdomains to explicitly define sets within a deterministic language.

Suppose an operator accepts an element of domain D, and based on this element produces another element in D, nondeterministically choosing from a number of possibilities. The operator applied to its argument must therefore denote a *set* of objects, i.e. those objects it *might* compute (this set is a subset of D). The set whose elements are (nondeterministically) computed is said to be a member of $\mathcal{P}$(D), the powerdomain of D. The operator is therefore of type D $\mapsto$ $\mathcal{P}$(D). Computation approximates this set by nondeterministically returning a member.

Suppose f and g are nondeterministic computations performed in sequence, first f and then g. For each possible output of operation f, operation g computes

any of a set of possible results. The union of all such sets contains the possible results of the sequence. We express this sequencing of nondeterministic functions by λx. $\mathtt{g}^{+}(\mathtt{f(x)})$. The '+' functional is of type

$$(\mathtt{D} \mapsto \mathcal{P}(\mathtt{D})) \mapsto (\mathcal{P}(\mathtt{D}) \mapsto \mathcal{P}(\mathtt{D})),$$

defined as $\lambda \mathtt{f}.\lambda \mathtt{set}. \cup \{\mathtt{f(x)} : \mathtt{x} \in \mathtt{set}\}$.

The larger the set denoted by f(x) is, the larger the set denoted by $\mathtt{g}^{+}(\mathtt{f(x)})$ will be.

A number of powerdomain constructions have been proposed differing according to the way the partial order is defined (see [S86] for a survey). The Egli-Milner powerdomain was the first powerdomain developed. It is useful for analyzing the operational properties of nondeterministic languages. Using Egli-Milner powerdomains, a nondeterministic program denotes the set of possible results. The least Egli-Milner powerdomain element, $\perp_{\mathcal{P}(D)}$, is the set containing only $\perp$. This is the denotation of a nondeterministic procedure for which no computation path succeeds [S86].

Smyth developed the *demonic* powerdomain, used when the concern is that *all* possible computation paths be successful. (One imagines that, should failure be possible, a demon will guide the nondeterministic calculation toward disaster!) Computation upon Smyth's can be viewed as a process of determining what *cannot* be the result of a nondeterministic program. Adding more possible computation paths *decreases* the likelihood that *all* paths will terminate successfully. So long as computation any branch has not terminated, one assumes that *anything* might result. Therefore, the least element, $\perp_{\mathcal{P}(D)}$, is the set containing all elements of D.

The angelic powerdomain is the dual of the Smyth powerdomain. Computation upon an angelic powerdomain can be viewed as a process of determining what *can* be a successful result of a nondeterministic program. (One imagines that, if a desirable result if possible, an angel will guide the nondeterministic calculation toward success.) One would expect that the larger the set of possible results, the greater the likelihood that *at least one* result will be successful. Both finite failure and nonterminating paths contribute nothing to the set. The least element, $\perp_{\mathcal{P}(D)}$, is the empty set ϕ.

We choose the angelic powerdomain because it is the only one of the three

containing the empty set as an element (the denotational equations which follow make use of ϕ). That the angelic powerdomain is the correct choice for a language combining functional and logic programming can be seen by considering the semantics of Horn logic programming. In an attempt to find values for the goal's logical variables so as to make the goal true, a Horn logic interpreter nondeterministically chooses a logical derivation using the program clauses from among all possible derivations. The set of solutions contains the results of the successful derivations; the derivations which fail or diverge add nothing to the set. If all derivations fail or diverge, the set of solutions is empty. If we view the set of all answer substitutions as a domain D, and the set of correct answer substitutions as an element of $\mathcal{P}(D)$, it is clear that we want $\perp_{\mathcal{P}(D)}$ to be ϕ. Since a description of Horn logic in terms of powerdomains would use the angelic powerdomain, it seems clear that set abstraction for the purpose of incorporating logic programming capabilities should also be described via this powerdomain construction.

We would like to have a partial order on sets which exhibits the property that a set becomes more defined as one adds new elements, *and* that it also becomes more defined as the elements within become more defined (according to the partial order of the base domain). We would like to be able to say that for two sets A and B, if for all $a \in A$ there exists a $b \in B$ such that $a \sqsubseteq_D b$, then $A \sqsubseteq_{\mathcal{P}(D)} B$. However, this is not a partial order, as this would equate $\{d_1, d_2\}$ with $\{d_2\}$ when $d_1 \sqsubseteq d_2$. Yet, though these sets are distinct, they are computationally equivalent (because the angel always chooses the *best* possible result). So theoretically, we are working not with sets, but with equivalence classes of sets. Furthermore, the need for continuity requires that for any chain of elements $t_i \in D$,

$$\{ \bigsqcup_i t_i \} \;=\; \bigsqcup_i \{t_i\},$$

where singleton set $\{t_i\}$ actually represents the equivalence class of sets containing that singleton set. These requirements motivate the following definitions (taken from [S86]).

Definition 3.18: A *Scott-topology* upon a domain D is a collection of subsets of D known as *open sets*. A set $\mathtt{U} \subseteq \mathtt{D}$ is open on the Scott-topology iff:

1) $\mathtt{U}$ is closed upwards, that is, for every $\mathtt{d}_2 \in \mathtt{D}$, if there exists a $\mathtt{d}_1 \in \mathtt{U}$ such that $\mathtt{d}_1 \sqsubseteq \mathtt{d}_2$, then $\mathtt{d}_2 \in \mathtt{U}$; and

2) If $\mathtt{d} \in \mathtt{U}$ is the least upper bound of a chain $\mathtt{C}$ in $\mathtt{D}$, then some $\mathtt{c} \in \mathtt{D}$ is in $\mathtt{U}$.

Definition 3.19: The symbol $\sqsubseteq_\sim$, pronounced 'less defined than or equivalent to', is a relation between sets. For $\mathtt{A}, \mathtt{B} \subseteq \mathtt{D}$, we say that $\mathtt{A} \sqsubseteq_\sim \mathtt{B}$ iff for every $\mathtt{a} \in \mathtt{A}$ and open set $\mathtt{U} \subseteq \mathtt{D}$, if $\mathtt{a} \in \mathtt{U}$ then there exists a $\mathtt{b} \in \mathtt{B}$ such that $\mathtt{b} \in \mathtt{U}$ also.

Definition 3.20: We say $\mathtt{A} \approx \mathtt{B}$ iff both $\mathtt{A} \sqsubseteq_\sim \mathtt{B}$ and $\mathtt{B} \sqsubseteq_\sim \mathtt{A}$. We denote the equivalence class containing $\mathtt{A}$ as $[\mathtt{A}]$. This class contains all sets $\mathtt{B} \subseteq \mathtt{D}$ such that $\mathtt{A} \approx \mathtt{B}$.

We define the partial order on equivalence classes as: $[\mathtt{A}] \sqsubseteq [\mathtt{B}]$ iff $\mathtt{A} \sqsubseteq_\sim \mathtt{B}$. For domain $\mathtt{D}$, the powerdomain of $\mathtt{D}$, written $\mathcal{P}(\mathtt{D})$, is the set of equivalence classes, each member of an equivalence class being a subset of $\mathtt{D}$.

Theorem 3.4 (Schmidt [S86]): The following operations are continuous:

ϕ: $\mathcal{P}(\mathtt{D})$ denotes $[\{\}]$. This is the least element.

$\{\_\}$: $\mathtt{D} \mapsto \mathcal{P}(\mathtt{D})$ maps $\mathtt{d} \in \mathtt{D}$ to $[\{\mathtt{d}\}]$.

$\_\cup\_$: $\mathcal{P}(\mathtt{D}) \times \mathcal{P}(\mathtt{D}) \mapsto \mathcal{P}(\mathtt{D})$ maps $[\mathtt{A}] \cup [\mathtt{B}]$ to $[\mathtt{A} \cup \mathtt{B}]$.

$^+$: $(\mathtt{D} \mapsto \mathcal{P}(\mathtt{D})) \mapsto (\mathcal{P}(\mathtt{D}) \mapsto \mathcal{P}(\mathtt{D}))$ is $\lambda \mathtt{f}.\lambda[\mathtt{A}].[\cup\{\mathtt{f}(\mathtt{a}) : \mathtt{a} \in \mathtt{A}\}]$.

An example will provide intuition about the use of '$^+$'. Suppose we have a set $S = \{1, 2, 3\}$, and we wish to create a new set, each element of which is of the form $f(x)$ where x is in S. Then

$$(\lambda x.\ \{f(x)\})^+(\{1,2,3\}) \ = \ \{f(1), f(2), f(3)\}.$$

In this work we use a noncurried variation of '$^+$', that is, we use it as a primitive function of type

$$((\mathtt{D} \mapsto \mathcal{P}(\mathtt{D})) \times \mathcal{P}(\mathtt{D})) \mapsto \mathcal{P}(\mathtt{D}).$$

This function is strict in its second argument.

3.3 Denotational Semantics of PowerFuL

This section gives the semantics of PowerFuL using the concepts reviewed above. After defining the domain of data objects which can be represented in PowerFuL, we provide a function which shows the way a PowerFuL program can

be mapped onto this domain. The semantic primitives used are, for the most part, quite conventional. However, a few unusual primitives, (called *coercions*) will be discussed in a special section.

PowerFuL's domain is the solution to the following recursive domain equation:

$$\mathtt{D} = (\mathtt{B} + \mathtt{A} + \mathtt{D}\times\mathtt{D} + \mathtt{D}\mapsto\mathtt{D} + \mathcal{P}(\mathtt{D}))_{\perp},$$

where B refers to the booleans, and A to a finite set of atoms. That is, PowerFuL's domain contains booleans, atoms, ordered pairs of smaller elements (to create lists and trees), continuous functions, and powerdomains (sets). Aside from the inclusion of powerdomains, the domain is typical of domains for other lazy, higher-order functional languages. The solution to this recursive domain equation are beyond the scope of this dissertation, but details may be found in [S86, P82, S89].

PowerFuL is a functional programming language, so we present its semantics in the denotational style usual for such languages [S77]. Our convention to differentiate language constructs from semantic primitives is to write the primitives in **boldface**. Language constructs are in `teletype`. Variables in rewrite rules will be *italicized.*

3.3.1 Semantic Equations

The meaning of a syntactic expression is defined in terms of the meaning of its subexpressions. In the definitions below, the semantic function $\mathcal{E}$ maps general expressions to semantic objects (called *denotable values*). The equations for most expressions are the conventional ones for a typical lazy higher-order functional language. The environment, ρ, maps identifiers to denotable values, and belongs to the domain [Id↦D]. The semantic equations for set-abstractions provide the novelty. For simplicity, the semantic equations ignore simple syntactic sugars.

Many of PowerFuL's denotational equations are similar to those of any typical lazy functional language. We present the semantic equations for the various constructs in the order of their appearance in the BNF of Chapter 2.

• Parentheses override the natural left-associativity.

$$\mathcal{E}[\![\mathtt{(expr)}]\!]\,\rho \;=\; \mathcal{E}[\![\mathtt{expr}]\!]\,\rho$$

• For each syntactic atom (represented by $\mathtt{A}_i$) in a program, we assume the existence of an atomic object in the semantic domain (represented by $\mathbf{A}_i$).

$\mathcal{E}[\![\mathtt{A}_i]\!]\ \rho\ =\ \mathbf{A}_i$

• We can group objects into ordered pairs to create lists and binary trees.

$\mathcal{E}[\![\mathtt{cons}(expr1,\ expr2)]\!]\ \rho = <(\mathcal{E}[\![expr1]\!]\ \rho)\ ,\ (\mathcal{E}[\![expr2]\!]\ \rho)>$
$\mathcal{E}[\![\mathtt{car}(expr)]\!]\ \rho\ =\ \mathbf{left}(\ \mathbf{pair!}(\mathcal{E}[\![expr]\!]\ \rho))$
$\mathcal{E}[\![\mathtt{cdr}(expr)]\!]\ \rho\ =\ \mathbf{right}(\mathbf{pair!}(\mathcal{E}[\![expr]\!]\ \rho))$

The primitive functions **left** and **right** select the left and right sides, respectively, of an ordered pair. These primitives are, however, undefined over other types of objects. As PowerFuL is an untyped language, we cannot ensure that the programmer will not try to take the car or cdr of an inappropriate object. Therefore, we protect the primitive function by first applying a *coercion*, **pair!**, to its argument, thus ensuring that its argument is of the appropriate type, handling errors appropriately. Other such coercions are provided for other types of objects, as needed. Later in this section they will be described in more detail. Note that our use of the term 'coercion' differs from the normal usage in that our coercions change an argument's type only as a kind of error-handling.

• We have the booleans, and boolean-valued functions.

$\mathcal{E}[\![\mathtt{TRUE}]\!]\ \rho\ =\ \mathbf{TRUE}$
$\mathcal{E}[\![\mathtt{FALSE}]\!]\ \rho\ =\ \mathbf{FALSE}$
$\mathcal{E}[\![\mathtt{not}(expr)]\!]\ \rho\ =\ \mathbf{not}(\mathbf{bool!}(\mathcal{E}[\![expr]\!]\ \rho))$
$\mathcal{E}[\![\mathtt{if}(expr1,\ expr2,\ expr3\ )]\!]\ \rho\ =\ \mathbf{if}(\mathbf{bool!}(\mathcal{E}[\![expr1]\!]\ \rho),\ (\mathcal{E}[\![expr2]\!]\ \rho),\ (\mathcal{E}[\![expr3]\!]\ \rho))$

• To create boolean values, we can test type (whether an object is a boolean, an atom, an ordered pair, a function or a set), and we can test terms for equality.

$\mathcal{E}[\![\mathtt{bool?}(expr)]\!]\ \rho\ =\ \mathbf{bool?}(\mathcal{E}[\![expr]\!]\ \rho)$
$\mathcal{E}[\![\mathtt{atom?}(expr)]\!]\ \rho\ =\ \mathbf{atom?}(\mathcal{E}[\![expr]\!]\ \rho)$
$\mathcal{E}[\![\mathtt{pair?}(expr)]\!]\ \rho\ =\ \mathbf{pair?}(\mathcal{E}[\![expr]\!]\ \rho)$
$\mathcal{E}[\![\mathtt{func?}(expr)]\!]\ \rho\ =\ \mathbf{func?}(\mathcal{E}[\![expr]\!]\ \rho)$
$\mathcal{E}[\![\mathtt{set?}(expr)]\!]\ \rho\ =\ \mathbf{set?}(\mathcal{E}[\![expr]\!]\ \rho)$
$\mathcal{E}[\![(expr1\ =\ expr2)]\!]\ \rho\ =\ \mathbf{equal?}((\mathcal{E}[\![expr1]\!]\ \rho),(\mathcal{E}[\![expr2]\!]\ \rho))$
$\mathcal{E}[\![\mathtt{null?}(expr)]\!]\ \rho\ =\ \mathbf{if}(\mathbf{atom?}(\mathcal{E}[\![expr]\!]\ \rho)\ \mathbf{then\ is'nil?}([\![expr]\!]\ \rho)\ \mathbf{else\ FALSE\ fi})$

The equality predicate can compare booleans, atoms, and, provided it can compare the respective subtrees, ordered pairs. It does not attempt to compare sets or functions for equality (if you try, it returns $\perp$). Applied to two first-order infinite lists, it returns **FALSE** if they differ, but fails to terminate (returns $\perp$) when they are *identical.* A conventional operation tests whether a "list" is empty (i.e. equals the atom **'nil**).

• We can look up identifiers in the environment, and also create new bindings.

$\mathcal{E}[\![identifier]\!]\ \rho\ =\ \rho(identifier)$
$\mathcal{E}[\![\texttt{letrec}\ defs\ \texttt{in}\ expression]\!]\ \rho\ =\ \mathcal{E}[\![expression]\!]\ (\mathcal{D}[\![defs]\!]\ \rho)$
$\mathcal{D}[\![id\ \texttt{be}\ expr]\!]\ \rho\ =\ \rho[\mathcal{FIX}(\lambda\ X.\ (\mathcal{E}[\![expr]\!]\ \rho[X/id]))/id]$

$\mathcal{D}[\![id\ \texttt{be}\ expr, defs]\!]\ \rho\ =\ (\mathcal{D}[\![defs]\!]\ \rho)\ [\mathcal{FIX}(\lambda\ X.\ (\mathcal{E}[\![expr]\!]\ (\mathcal{D}[\![defs]\!]\ \rho[X/id])))/id]$

In the above equations, $\mathcal{FIX}$ computes the least fixpoint of a functional. Rather than using the fixpoint operator as a primitive, we can define it via:

$\mathcal{FIX}(f)\ =\ f(\mathcal{FIX}(f))$

Taken as an equation, the above is true if $\mathcal{FIX}$ computes *any* fixpoint of its argument. But taken as a recursive definition with Definition 3.17 in mind, $\mathcal{FIX}$ is taken to be the least fixpoint of the functional $\lambda F.\ \lambda f.\ f(F(f))$. Using Theorem 3.2, this can be shown to equal $\lambda f.\ lfp(f)$.

• We can create functions through lambda abstraction, and apply functions to their arguments.

$\mathcal{E}[\![\lambda\ id.\ expr]\!]\ \rho\ =\ \lambda\ x.\ (\mathcal{E}[\![expr]\!]\ \rho[x/id])$
$\mathcal{E}[\![expr1\ expr2]\!]\ \rho\ =\ \textbf{func!}(\mathcal{E}[\![expr1]\!]\ \rho)(\mathcal{E}[\![expr2]\!]\ \rho)$

In the above equations, we considered only functions of one argument. A function of multiple arguments can be considered syntactic sugar either for a curried function, or for a function whose single argument is a list.

• To build sets, the user begins with the empty set, and singleton sets, each constructed from an element of the powerdomain's base domain. The union operation builds larger sets from smaller ones.

$\mathcal{E}[\![\texttt{phi}]\!]\ \rho = \phi$
$\mathcal{E}[\![\{expr\ :\}]\!]\ \rho = \{\mathcal{E}[\![expr]\!]\ \rho\}$
$\mathcal{E}[\![\texttt{U}(expr_1, expr_2)]\!]\ \rho = \textbf{set!}(\mathcal{E}[\![expr_1]\!]\ \rho) \cup \textbf{set!}(\mathcal{E}[\![expr_2]\!]\ \rho)$

To build a new sets out of an old one, we can filter out all but those elements meeting a specified condition, and we can compute with each member of the input set individually, combining the results into a new set.

$\mathcal{E}[\![\{expr\ :\ condition, qualifierlist\}]\!]\ \rho$
$\quad = \textbf{set!}(\textbf{if}\ \mathcal{E}[\![condition]\!]\ \rho\ \textbf{then}\ \mathcal{E}[\![\{expr\ :\ qualifierlist\}]\!]\ \rho\ \textbf{else}\ \phi\ \textbf{fi})$
$\mathcal{E}[\![\{expr\ :\ id \in genrtr, qualifierlist\}]\!]\ \rho$
$\quad = (\lambda\ X.\ \mathcal{E}[\![\{expr\ :\ qualifierlist\}]\!]\ \rho[X/id])^{+}(\textbf{set!}(\mathcal{E}[\![genrtr]\!]\ \rho))$

• The sets denoted by `bools` contains only **TRUE** and **FALSE**, and the set `atoms` contains the atoms. The set of terms includes as subsets not only `bools` and `atoms`, but also any ordered pair which can be constructed from two smaller terms. These sets may be viewed as syntactic sugars, since the user *could* program them using the previously given constructs. In that sense, their presence adds nothing to the expressive power of the language. Nevertheless, providing them in the syntax permits important optimizations through run-time program transformation (discussed later). Thus we have:

$\mathcal{E}[\![\texttt{bools}]\!]\ \rho = \mathcal{F}[\![\texttt{bools}]\!]$
$\mathcal{F}[\![\texttt{bools}]\!] = \{\textbf{TRUE}\} \cup \{\textbf{FALSE}\}$
$\mathcal{E}[\![\texttt{atoms}]\!]\ \rho = \mathcal{F}[\![\texttt{atoms}]\!]$
$\mathcal{F}[\![\texttt{atoms}]\!] = \cup(\{\mathbf{A}_1\}, \ldots, \{\mathbf{A}_n\})$
$\mathcal{E}[\![\texttt{terms}]\!]\ \rho = \mathcal{F}[\![\texttt{terms}]\!]$
$\mathcal{F}[\![\texttt{terms}]\!] = \mathcal{F}[\![\texttt{bools}]\!] \cup \mathcal{F}[\![\texttt{atoms}]\!]$
$\qquad \cup\ (\lambda s.((\lambda t.\{< s, t >\})^{+}(\mathcal{F}[\![\texttt{terms}]\!])))^{+}(\mathcal{F}[\![\texttt{terms}]\!])$

The sets denoted by `bools`, `atoms` and `terms` are semantically superfluous. The user could create these sets with the other constructs. For instance, each reference to the primitive set `terms` could be replaced by:

```
letrec
   bools be U( {TRUE}, {FALSE})
   atoms be U({A1},..., {An})
   terms be U(atoms, bools, {cons(X,Y) : X,Y ∈ terms })
in terms.
```

PowerFuL provides these sets as primitives, so the interpreter can recognize them and treat their enumerated variables as logical variables, for greater efficiency. This will be discussed in greater detail in Chapter 6.

The functions $\mathcal{E}$, $\mathcal{D}$, $\mathcal{F}$ and $\mathcal{FIX}$ are mutually recursive. Their meaning is the least fixed point of the recursive definition. This fixed-point exists because we have combined continuous primitives with continuous combinators. Most of these primitives are fairly standard, and will be described in a later section. Note the use of the primitive '+' (for distributing elements of a powerdomain to a function) in defining the meaning of the set abstraction construct.

3.3.2 Coercions

A few words must be said about some other novel primitives, here called *coercions*. Most primitives are only defined over portions of the domain D. The boolean operators are only defined over $\mathtt{B}_\perp$; the operations **left** and **right** assume the arguments to be ordered pairs; function application (β-reduction) is defined only when the left argument is in fact a lambda expression; and only sets can contribute to a set union.

Five primitives coerce inappropriate arguments to the least-defined object of the appropriate type. The are listed below.

bool!: $\mathtt{D} \mapsto \mathtt{B}_\perp$
atom!: $\mathtt{D} \mapsto \mathtt{A}_\perp$
pair!: $\mathtt{D} \mapsto \mathtt{D}\times\mathtt{D}$
func!: $\mathtt{D} \mapsto [\mathtt{D}\mapsto\mathtt{D}]$
set!: $\mathtt{D} \mapsto \mathcal{P}(\mathtt{D})$

The function **bool!** maps **TRUE** and **FALSE** to themselves, and otherwise maps *arg* to $\perp$. The function **atom!** maps *arg* to itself if *arg* is an atom, and maps *arg* to $\perp$ otherwise. The function **pair!** maps *arg* to itself if *arg* is a member of $\mathtt{D}\times\mathtt{D}$, and to $\perp_{\mathtt{D}\times\mathtt{D}}$ (that is, $< \perp, \perp >$) otherwise. The function **func!** maps *arg* to itself if *arg* is a member of $\mathtt{D}\mapsto\mathtt{D}$ and to $\perp_{\mathtt{D}\times\mathtt{D}}$ (that is, $\lambda \mathtt{x}.\ \perp$) otherwise. The function **set!** maps *arg* to itself if *arg* is a member of $\mathcal{P}(\mathtt{D})$ and to $\perp_{\mathcal{P}(\mathtt{D})}$ (that is, ϕ) otherwise.

The coercions ensure that primitives handles inappropriate input reasonably. For instance, the union constructor is appropriately applied only to sets. If the argument is something other than as set (perhaps $\perp$), then this input is treated as the empty set. This make sense because

1) only sets contain elements — other objects do not;

2) a set is completely defined by the elements it contains; and

3) the empty set is the only set not containing any elements.

Thus, the expression `U('a, expr)` denotes a set containing `'a`, regardless of whether or not `expr` can be computed. This is analogous the the set of solutions to a Horn logic program and goal, the elements of which are determined by successful derivations (or refutations), ignoring derivations which fail or diverge. For uniformity, we define analogous coercions to handle similar questions about primitives of other types.

Theorem 3.5: The coercion **set!** is continuous.

Proof: We will prove the continuity of **set!**. Consider a chain of objects from domain D: t_0, t_1, t_2, ..., such that for $i < j$, $t_i \sqsubseteq t_j$. If there is no i such that t_i is in $\mathcal{P}$(D), then for all i, by definition, $\mathbf{set!}(t_i) = \perp_{\mathcal{P}(D)} = \phi$. Furthermore, $\bigsqcup_i t_i$ will not be in $\mathcal{P}$(D), so by definition,

$$\mathbf{set!}(\bigsqcup_i t_i) = \phi.$$

Thus we have,

$$\bigsqcup_i \mathbf{set!}(t_i) = \phi = \mathbf{set!}(\bigsqcup_i t_i),$$

proving continuity for that case. The only other possibility is that the sequence *does* converge to a powerdomain element. In that case, let t_k be the first member of the chain in $\mathcal{P}$(D). for any $i < k$, t_i can only equal $\perp_D$, so $\mathbf{set!}(t_i)$ equals ϕ, by definition. For $i \geq k$, $\mathbf{set!}(t_i)$ equals t_i, also by definition. Therefore,

$$\begin{aligned}
\bigsqcup_i \mathbf{set!}(t_i) &= \bigsqcup_{i \geq k} \mathbf{set!}(t_i) \\
&= \bigsqcup_{i \geq k} t_i \\
&= \mathbf{set!}(\bigsqcup_{i \geq k} t_i) \\
&= \mathbf{set!}(\bigsqcup_i t_i).
\end{aligned}$$

Hence, **set!** is continuous. **End of Proof**

Proofs of the continuity of the other coercions are similar.

3.4 Summary

In this chapter we described the semantic domain of PowerFuL. It resembles the recursively defined domains of ordinary untyped functional languages, in which

complex objects are built from simpler objects via sequencing constructors (ordered pairs) and function definition. What is new is the use of powerdomain constructors to build sets. We gave the rationale for chosing angelic powerdomains, rather than one of the other types of powerdomains.

The semantic equations for the traditional features are unaffected by the enrichment of the domain. All that was needed to handle set abstraction is the addition of a few new equations. We consider this to be the strongest testimony of the elegance of this approach.

The primitive functions used in the semantic equations are defined in the next chapter. The use of coercions permits simpler definitions of primitives, as we need be concerned only with appropriate subsets of PowerFuL's domain; redundant descriptions of "error handling" is avoided. This will be especially helpful when describing the operational semantics, and its relation to the denotational semantics, in the next chapter.

4 FROM DENOTATIONAL TO OPERATIONAL SEMANTICS

Though the denotational semantics usually provides the clearest and simplest definition of the syntactic constructs, an operational semantics is essential if the language is to be implemented. Therefore, even those defining a language via denotational semantics usually also provide an operational definition. In such cases, one would like to know the extent to which the two semantics agree. A proof of equivalence is rarely given; usually the operational semantics are incomplete when computing infinite objects †. Though Horn logic makes for a very simple language (simple flat domain and only two types of clauses) the proof of correctness is far from trivial [VK76]. The task becomes more difficult as the operational semantics becomes low-level (hardware-oriented). The closer to the denotational definition the operational semantics remains, the easier this proof should be. Therefore, in the remaining chapters, we devise an operational procedure from the denotational equations directly. This ensures that our operational semantics is consistent with respect to the denotational definitions; in this chapter we also speculate upon its completeness.

The semantic equations provided in the last chapter are a recursive definition of the semantic function $\mathcal{E}$, which maps PowerFuL programs into objects from the semantic domain (the value denoted by the program). Assuming that the primitive functions are continuous, the recursive equations define a functional which has a least fixpoint, which can be "computed" by taking successive approximations, as alluded to in Chapter 3. This fixpoint is taken to be the value of $\mathcal{E}$. A procedure to execute $\mathcal{E}$ on its input is, by definition, an interpreter for the PowerFuL language.

In this section we examine ways of executing the function $\mathcal{E}$ when applied to a PowerFuL program and an (initially empty) environment. Given implementations

† *This issue is discussed further in section 4.4.1.*

for the semantic primitives (provided in the next chapter) and the ability to reduce lambda-expressions, a computation rule is all that is needed to convert the semantic equations into an interpreter. If the computation rule is a fixpoint rule, then the computed result will indeed equal the fixpoint value of $\mathcal{E}[\![program]\!]\ \Lambda$ ('Λ' stands for the empty environment). In other words, we interpret the language of denotational semantics as a functional programming language in its own right, not merely reading it as a pseudocode. Since a functional program has both a declarative and an operational reading, the denotational equations will thus provide both the denotational semantics of PowerFuL, and also a simple interpreter consistent with the declarative reading.

To illustrate our approach, suppose that we wish to evaluate the expression:

```
car( cons( cons('a,'b), 'a))
```

The denotational equations for translating syntactic symbols of atoms to real atoms in the semantic domain (differentiated here by the type font), and semantic equations for **cons**, **car** and **cdr** are as follows:

$\mathcal{E}[\![\mathtt{A}_i]\!]\ \rho\ =\ \mathbf{A}_i$
$\mathcal{E}[\![\mathtt{cons}(expr1,\ expr2)]\!]\ \rho = <(\mathcal{E}[\![expr1]\!]\ \rho)\ ,\ (\mathcal{E}[\![expr2]\!]\ \rho)>$
$\mathcal{E}[\![\mathtt{car}(expr)]\!]\ \rho\ =\ \mathbf{left}(\ \mathbf{pair!}(\mathcal{E}[\![expr]\!]\ \rho))$
$\mathcal{E}[\![\mathtt{cdr}(expr)]\!]\ \rho\ =\ \mathbf{right}(\mathbf{pair!}(\mathcal{E}[\![expr]\!]\ \rho))$

The rewrite rules to implement the semantic primitives **left** and **right** are:

left(<*1st*, *2nd*>) = *1st*
right(<*1st*, *2nd*>) = *2nd* .

The denotational equations map syntactic constructs to semantic constructs, and the semantic primitives map semantic objects onto other semantic objects. In this case, both kinds of mappings are defined through rewrite rules. We wish to find the semantic object denoted by the syntactic expression above. That is, we wish to compute:

$\mathcal{E}[\![\mathtt{car(cons(cons('a,'b),'a))}]\!]\ \Lambda.$

Using the semantic equation for **car**, we rewrite the above expression to:

left $\mathcal{E}[\![\mathtt{cons(cons('a,'b),'a)}]\!]\ \Lambda.$

We do not yet have enough information to apply the rewrite rule for **left**, so we must translate more syntax using the semantic equation for **cons**, and will then

obtain:

left $< \mathcal{E}[\![\texttt{cons('a,' b)}]\!]\ \Lambda,\ \mathcal{E}[\![\texttt{'a}]\!]\ \Lambda >$.

We now have enough information to execute the semantic primitive **left**, and will get:

$\mathcal{E}[\![\texttt{cons('a,' b)}]\!]\ \Lambda$.

Further rewriting with the semantic equations produces the final value:

$<$ **'a, 'b** $>$.

Thus, we see that one can sometimes execute a program directly from the denotational semantic equations.

The remainder of this chapter provides a more rigorous development of this technique. The basic ideas come from Vuillemin's pioneering work on correct implementation of recursive programs [V74]. Following Kleene, Vuillemin views the meaning of a recursive definition to be the least fixpoint of an associated functional. As the fixpoint is shown to be the least upper bound of a (possibly infinite) chain of approximations, computation becomes synonymous with the production of better and better approximations. Usually, there are several places in an approximation where one can seek improvement. The computation rule chooses which of these places to work on next. Vuillemin provides conditions under which the successive approximations will converge toward the value implicitly given by the fixpoint definition. We then apply these ideas to PowerFuL's denotational equations, a recursive definition in its own right, to produce our operational semantics.

4.1 Recursion and Least Fixpoints

Consider a recursive definition of the form:

$F(\overline{x}) \equiv \tau[F](\overline{x})$

for function F, where τ is a functional over $D_1 \times \ldots \times D_n \mapsto D$, expressed by composing a term from:

a) the individual variables $\overline{x} =< x_1,\ x_2,\ \ldots,\ x_n >$;

b) known monotonic and continuous functions, called *primitives*; and

c) the function variable, F.

As an example of such a recursive program, consider the following program P for append:

`append(x, y)` $\equiv$ `if(null?(x), y, <car(x),append(cdr(x),y)>).`

Applying the above formalism to this case, the functional variable F is here the recursive function **append**, the parameter list $\overline{x}$ represents the parameters x and y, the functional τ is :

$\lambda F.\ \lambda$ `x.` λ `y. if(null?(x), y, <car(x),F(cdr(x),y)>),`

and the primitive functions are **if**, **null?**, **car?** and **cdr?**.

By Theorem 3.2, there exists for τ as above a *least fixpoint*, and this fixpoint equals

$\bigsqcup_i \tau^i(\Omega)$,

where Ω represents the least-defined object in τ's domain (the function which returns $\perp$ for any arguments). By least fixpoint, we mean the least-defined function for which $F = \tau[F]$. We denote this least fixpoint by f_P, and take it to be the value of the function described in program P.

In our example, $\tau^i(\Omega)$ would approximate the **append** function, in that it would give the correct result provided the first argument is a list of length less than i. The least upper bound of these approximations would handle lists of unbounded length.

The extension of these ideas to a set of mutually-recursive functions is straightforward. From the mutually-recursive functions one would abstract a functional whose fixpoint is defined to be a sequence of functions.

4.2 Computation Rules and Safety

To some input $\overline{d}$, we wish to apply a function defined as above. Let us define a sequence of terms s_0, s_1, s_2, ..., such that the first term s_0 is $F(\overline{d})$, and each term s_{i+1} is computed from s_i by replacing each instance of F in s_i by $\tau(F)$. That is, we expand each occurrence of F in the previous term by the recursive definition. Let us also define a parallel series of terms $s_i[\Omega/F]$, for $i \geq 0$, where $s_i[\Omega/F]$ is computed from s_i by replacing all occurrences of the function variable by the undefined function. Clearly, $s_i[\Omega/F]$ is equal to $\tau^i(\Omega)(\overline{d})$. By definition,

$\bigsqcup_i(s_i[\Omega/F]) = \bigsqcup_i(\tau^i\Omega\overline{d}) = (\bigsqcup_i \tau^i\Omega)\overline{d} = f_p(\overline{d})$

Example: Using the append definition for input lists `list1` and `list2`, s_0 is `append(list1, list2)`, and s_1 is:

if(null?(list1), list2, <car(list1),append(cdr(list1),list2)>).

We can keep expanding the occurrences of append to any depth. The sequence of approximations begins with $s_0[\Omega/F]$, which is **Ω(list1, list2)** (i.e. $\perp$), followed by $s_1[\Omega/F]$, which is:

if(null?(list1), list2, <car(list1),Ω(cdr(list1),list2)>).

In this case, there is only one occurrence of the recursive function to expand at each step. In general there may be more. Let us define a new series t_i similar to s_i, where $t_0 = s_0 = F(\overline{d})$, but where each t_{i+1} is computed from t_i by expanding only *some* of the occurrences of F in t_i, instead of all.

Definition 4.1: A *computation rule* C tells us which occurrences of $F(\overline{e})$ should be replaced by $\tau[F](\overline{e})$ in each step.

For each t_i, we compute $t_i[\Omega/F]$ in the same way we computed $s_i[\Omega/F]$ from s_i.

Theorem 4.1 (Cadiou [V74]): For any computation rule C,

$$\bigsqcup_i(t_i[\Omega/F]) \sqsubseteq f_p(\overline{d}).$$

Proof: For any i, $t_i[\Omega/F] \sqsubseteq s_i[\Omega/F]$, and therefore

$$\bigsqcup_i(t_i[\Omega/F]) \sqsubseteq \bigsqcup_i(s_i[\Omega/F]) = f_p(\overline{d}).$$

End of Proof

Definition 4.2: A computation rule is said to be a *fixpoint computation rule* for program P if for all $\overline{d}$ in the relevant domain,

$$\bigsqcup_i(t_i[\Omega/F]) \equiv f_p(\overline{d}).$$

Vuillemin gives a condition which, if satisfied, means that a computation rule is a fixpoint rule. To explain this condition, we need a bit more notation.

Definition 4.3: A *substitution step* expands some or all occurrences of the recursive function calls in the term. Given a term t_i, let F_y represent the occurrences of F which the computation rule would choose, and let F_n represent those which would not be chosen for expansion in the next step. Let

$$t_i[S/F_y,\ T/F_n]$$

represent the result of replacing all chosen occurrences by S, and the unchosen by T. Then

$$t_i[F/F_y,\ F/F_n]$$

is just t_i, and

$$t_i[\Omega/F_y,\ \Omega/F_n]$$

is the same as

$$t_i[\Omega/F].$$

Let

$$t_i[\Omega/F_y,\ f_p/F_n]$$

represent the result computed if the chosen occurrences were *never* expanded, but all other occurrences expanded arbitrarily far.

Definition 4.4: A *safe substitution step* chooses F_y so that

$$t_i[\Omega/F_y,\ f_p/F_n] \ = \ t_i[\Omega/F_y,\ \Omega/F_n].$$

Intuitively, the computation is safe if the occurrences chosen are so important that, were these never expanded, no other expansions would matter.

Definition 4.5: A computation rule is *safe* if it provides for only safe substitution steps.

Theorem 4.2 (Vuillemin [V74]): If the computation rule used in producing the series v_i is a safe, then

$$f_p(\overline{d}) \ \sqsubseteq \ \bigsqcup_i v_i.$$

To summarize the proof, we note that the fixpoint is equal to the least upper bound of approximations made by the full-substitution rule. We need only show that for each approximation made by the full-substitution rule, a safe rule will produce an approximation that is at least as good. Each approximation using the full-substitution rule expands only a finite number of occurrences. If we perform that many safe substitution steps, then either we will have expanded all of these (guaranteeing a suitably good approximation), or we will have had one or more steps in which none of the remaining occurrences expanded with the full-substitution rule were chosen. When that happens, the safety condition shows that the full-substitution approximation cannot be any better.

Using Theorems 4.1 and 4.2, then for any safe computation rule,

$$f_p(\overline{d}) \ \equiv \ \bigsqcup_i v_i$$

and therefore *all safe computation rules are fixpoint rules.* Intuitively, a safe substitution is one which performs enough essential work. That is, if this work were never done, then all other work would be irrelevant. If enough essential work is performed in each step, then every essential piece of work will eventually be done.

Definition 4.6: The *parallel outermost rule* replaces all outermost occurrences of F simultaneously. Applied to an atom or identifier, there are no occurrences of F to expand. Applied to a term with an occurrence of F at the outermost, it expands only that occurrence. Applied to a term headed by a data constructor (such as an ordered pair), or to a term headed by a primitive function, it expands those occurrences which would be expanded when applying this rule to each argument (if any) individually.

Theorem 4.3 (Vuillemin [V74]): The parallel outermost rule is a safe rule.

Proof: This is proved by structural induction on the term. If the term is an atom or an identifier, then any computation rule trivially produces a safe substitution step. If the term has an occurrence of F at the very outermost, than the parallel-outermost rule expands only that occurrence. If one replaces that occurrence with Ω, the result is $\perp$ no matter what one does with any remaining occurrences, so the parallel-outermost is a safe step. The only alternative is to have a primitive or constructor at the outermost. Applying the parallel-outermost rule to such a term is equivalent to applying it individually to each of the arguments (if any). By the induction hypothesis, parallel-outermost is safe for each argument individually. Since the computation rule is safe for each argument (replacing chosen occurrences by Ω gives the same result as replacing *all* occurrences by Ω), it must be safe for the expression as a whole. **End of Proof**

4.3 Computation of Primitives

To say that a potentially infinite computation is computable, we must be able to describe the result as the least upper bound of a set of finite approximations. By finite, we mean that each approximation must require only a finite number of primitive, mechanical steps. In the preceding sections, we showed how a recursive function, written in terms of primitive functions, could be described as the least upper bound of a chain of computations, each of which expands the recursive definition only a finite number of times. However, to actually compute each approximation,

we must be able to execute the primitive functions. If execution of each primitive function is guaranteed to terminate for any input it may be supplied, then each element of the chain does indeed represent a finite computation, and therefore we have an operational semantics.

On the other hand, if the primitive operations are *not* guaranteed to terminate, then we are faced with the task of approximating the primitives, just as we approximated the recursive function. Rather than taking the least upper bound of a single chain of approximations, we would need to consider an infinite sequence of such chains! We would prefer to compute a single chain of approximations, where each computation step *both* expands some occurrences of the main recursive function *and* computes some of the primitives.

For example, suppose the primitive functions were defined via rewrite rules. The computation rule must specify not only which occurrences of the recursive function to expand, but also which primitive simplification opportunities to take. Those primitives guaranteed to terminate might be simplified as much as possible. In making a finite approximation, one would not only approximate the unexpanded recursive function occurrences by Ω, but applications of (possibly) non-terminating primitives would also be replaced by the appropriate bottom element. Even if primitives *are* guaranteed to terminate, it seems a good idea to simplify them as soon as possible in the hope that some occurrences of the recursive function could be pruned away, perhaps permitting the chain of approximations to terminate.

For example, if we are computing `append([1,2,3], list2)`, then t_1 might be:

if(null?`([1,2,3]), list2, <car([1,2,3]),append(cdr([1,2,3]),list2)>)`,

and $t_1[\Omega/\texttt{append}]$ would be:

if(null?`([1,2,3]), list2, <car([1,2,3]),`Ω`(cdr([1,2,3]),list2)>)`,

which simplifies to `<1,`$\perp$`>`. By performing simplifications as early as possible, however, we have as t_1:

`<1, append([2,3], list2)>`.

It should be possible to generalize the safety condition to deal explicitly with choice of primitive simplification opportunities, just as it now speaks of recursive function occurrences to be expanded. This generalization is not rigorously developed in this dissertation; however, the basic idea is as follows.

The notation for approximating the current term, $t_i[\Omega/F]$ would now mean that we replace not only applications of the recursive function(s) by the appropriately typed undefined element, but we do the same for uncomputed applications of the primitives. The safety condition,

$$t_i[\Omega/F_y,\ f_p/F_n] \ =\ t_i[\Omega/F_y,\ \Omega/F_n],$$

now would state that a computation step is safe if, among those recursive-function occurrences to be expanded and those primitive applications to be simplified, are some so critical that, were these operations never done, the current approximation could never be improved upon no matter how much computation elsewhere were performed. This means, for any part of t_i which is not yet fully computed, the computation rule has chosen vital steps to perform. We may say that such a safe computation step is *fair*, because it simultaneously does work necessary for each incomplete part the result being computed (it may do some *unnecessary* work as well, but this does not concern us).

This is not to say that the occurrences chosen by the safe computation rule *must* be expanded immediately for convergence; the order in which necessary work is performed ought not affect convergence (though speed of convergence might be affected). For any approximation requiring only a finite amount of computation (a finite number of recursive function expansions and a finite number of primitive simplifications), any safe computation rule should eventually produce an approximation that is at least as good.

One might question the claim that, in producing a finite approximation, the order in which necessary work is performed is irrelevant. After all, are not some computation rules more powerful than others? Does not the innermost computation rule sometimes fails to converge, where outermost evaluation succeeds? The key word here is *necessary*. An innermost computation rule may fail to converge if none of the steps necessary to compute some part of the denoted object are ever chosen. Instead, the innermost computation rule might put all its effort into computing an infinite subexpression whose computation is unnecessary to the main result.

Since any computable function can be viewed as the least upper bound of a set of finite approximations, and since a safe computation rule should be able to produce a finite approximation at least as good as any of them, we believe that

a safe computation rule computes the denoted fixpoint value. Furthermore, we believe that, should the attempt be made to do all possible primitive simplifications in each computation step, and should simplifications terminate in each step, then a safe computation rule (to decide recursive function expansions) is *still* a fixpoint rule. Proofs of these conjectures is a topic for future research. Nevertheless, it is on the basis of these conjectures that we develop the operational semantics for PowerFuL.

4.4 Operational Semantics of PowerFuL

Instead of a single recursive function, we have four mutually-recursive functions, $\mathcal{E}$, which maps a syntactic expression and an environment to a semantic object, $\mathcal{F}$ which maps a syntactic expression to a semantic object (without need of the environment), $\mathcal{D}$, which maps a syntactic expression and an environment to a new environment, and $\mathcal{FIX}$, whose recursive definition is taken to define the least fixpoint operator. The program is written via a set of recursive equations, making use of pattern-matching. To fit it into Vuillemin's computational scheme as extended above with the new computation rule, we should rewrite the equations as a single large case statement, using special syntax primitives to recognize the outermost syntactic construct, and to replace references to the pattern-matching variables in the right-hand side by expressions referencing relevant portions of the abstract syntax tree. To compute an expression of the form

$\mathcal{E}[\![syntax]\!]\ \rho$,

we would:

1) substitute the piece of syntax and the environment into the large case statement;

2a) simplify the primitives manipulating the syntax tree (producing an expression resembling the right-hand side of one of the equations); and

2b) simplify semantic primitives and do β-reduction (where applicable).

Cluttering up the denotational semantics with such syntax-tree primitives would be unnecessarily tedious. Therefore, in the remainder of this work, we shall work from the semantic equations directly, combining steps 1 and 2a into a single step.

Because functions are written as lambda expressions, some semantic equations

introduce lambda variables. These variables may have to be renamed at times to avoid variable capture; however, this is standard in lambda calculus based languages.

As will be shown in the next chapter, semantic primitive simplification rules will terminate on any finite argument (or infinite argument that is yet computed only only to a finite degree). To implement the denotational equations as a recursive program, we want a safe computation rule for which β-reductions will always terminate.

4.4.1 Termination of β-Reduction

In the sense that our entire formalism assumes the ability to compose functions, function application is not technically a primitive. However, since functions are represented as lambda expressions, and executed via β-reduction, operationally, β-reduction must be treated as a primitive. That is, we simplify lambda expression via the rewrite rule

$$(\lambda\ var.body)arg\ =\ body[arg\ /\ var],$$

Using (untyped) lambda expressions to represent functions is dangerous, as there exist lambda-expressions whose simplification will fail to terminate. This can only happen when a lambda expression is applied to another lambda expression, in which one β-reduction creates opportunities for additional β-reductions. Consider the evaluation of:

$\mathbf{func}!(\mathcal{E}[\![\lambda \mathrm{x.x\ x}]\!]\ \rho)(\mathcal{E}[\![\lambda \mathrm{x.x\ x}]\!]\ \rho)$.

If we simplify both arguments of this β-reduction simultaneously, we eventually get:

$(\lambda\ y.y\ y)(\lambda\ y.y\ y)$,

whose β-reduction will never terminate. Such inherently nonterminating lambda-expressions should be treated as $\perp$. However, in our computational paradigm, this should be through a non-terminating computation sequence, and not by nonterminating simplifications within a single computation step. This expression may be only a small piece of the main expression, and we do not want endless simplification to prevent computation of the other parts.

Suppose we systematically delay computation within the body of a lambda expression until after the expression has been applied. This means that, following

every β reduction, elements of the function body must be expanded before any new β-reductions can be done there. This ensures that, within each computation step, β-reduction will terminate. At no step will an infinity of β-reductions be called for. Every approximation generated by the computation sequence, the above example reduces to $\perp$, as desired.

If the computation rule ensures that β-reduction will always terminate, then we can treat function application like the terminating semantic primitives. A user-defined function application may be treated as a primitive that is strict in the first argument (the function begin applied), and which simplifies (β-reduces) as soon as the outermost constructor (the introduction of the lambda-variable) is computed. That is, given an application of the form

$$(\textbf{func!}\ (\mathcal{E}[\![expr_1]\!]\ \rho_1))\mathcal{E}[\![expr_2]\!]\ \rho_2,$$

we would compute the left portion to produce an expression of the form

$$(\lambda x.\ (\mathcal{E}[\![body]\!]\ \rho_3))\mathcal{E}[\![expr_2]\!]\ \rho_2.$$

Note that these computation steps may have extended the environment ρ_1; hence we use ρ_3 in its place. This would immediately reduce to

$$(\mathcal{E}[\![body]\!]\ [(\mathcal{E}[\![expr_2]\!]\ \rho_2)/x]\rho_3)).$$

Note that the body of the lambda-expression is not be computed until after application (after which it is no longer the body of a lambda-expression!).

The prohibition against computing the body of an unapplied lambda-expression means that we will not be able to compute a program which denotes an unapplied function, nor a structure which contains an unapplied function as a part. The program as a whole must denote an element of E, where

$$\mathtt{E} = (\mathtt{B} + \mathtt{A} + \mathtt{E}\times\mathtt{E} + \mathcal{P}(\mathtt{E}))_\perp.$$

though individual parts of the program may freely denote objects from D, where

$$\mathtt{D} = (\mathtt{B} + \mathtt{A} + \mathtt{D}\times\mathtt{D} + \mathtt{D}\mapsto\mathtt{D} + \mathcal{P}(\mathtt{D}))_\perp.$$

That is, we are only considering higher-order programs in which functions are defined for the purpose of application, but not as final values per se.

4.4.2 Desiderata for the Computation Rule

Aside from the restriction against expanding recursive functions in the body

of a lambda-expression, we should not need to always expand *all* outermost occurrences of the recursive function in every step. Consider an expression headed by the conditional primitive (**if**). We would prefer to restrict computation to the first argument, the condition, and postpone evaluation of the other two arguments, until we know which one will be needed. For efficiency, one would like to limit computation to any primitive's strict arguments, at least until we have enough information about the argument to simplify the primitive, and to delay evaluation of non-strict arguments, whose computation may not be needed. Even if the primitive is strict in more than one argument, we may wish to concentrate on just one argument at a time. By distinguishing essential from non-essential parallelism, we can define a more efficient computation rule.

One computation rule often used for lazy functional languages is the *leftmost* rule. Of the outermost occurrences of the recursive function(s), a substitution step expands only the leftmost. Vuillemin proves that the leftmost computation rule is safe for very simple languages.

Theorem 4.4 [V74]: If all the primitives are strict, except for the **if** (which is strict in the first argument), and the semantic domain is flat, and assuming that primitive simplifications are made as early as possible, then the leftmost computation rule is safe.

Proof: At the outermost, an expression must be either an atom, an occurrence of the recursive function begin computed, or a primitive (we do not need to consider constructors at the outermost, as each element of a flat domain may be considered as an individual atom). If the expression is an atom, then there is no occurrence for the computation rule to choose, and any computation step is trivially safe. If an occurrence of F is at the outermost, then a leftmost substitution is the same as a parallel-outermost substitution, already proven to be safe. The only alternative is a primitive at the outermost. Consider the primitives leftmost argument. If we replace all occurrences in the argument chosen by the leftmost rule with Ω, the result is either an atom or $\perp$. If the result is an atom, then the expression was obviously amenable to simplification (since the primitive is strict in that argument). However, we have assumed that all possible simplifications have already been carried out. Therefore, the result is $\perp$. Replacing *all* occurrences in this argument by Ω yield the same result ($\perp$). Since we get the same result either way, the substitution

is safe, by definition. **End of Proof**

Many interesting languages do have nonflat domains, and thus do not meet Vuillemin's criteria. For a higher-order language, problems occur when we try to evaluate a function outside the context of its application. In the following example in which we are computing *an unapplied function* as a topmost goal. Assume exp_1 denotes an infinite list, and the interpreter is asked to evaluate the function

$\lambda f.(\textbf{if } f(expr_1) \textbf{ then } expr_2 \textbf{ else } expr_3).$

Using the parallel-outermost rule, we would evaluate all three arguments of the **if** expression simultaneously, producing a sequence of partial functions whose limit is the denoted function. The leftmost rule would produce better and better approximations of $expr_1$, but would never get around to computing $expr_2$ or $expr_3$. Though evaluation of $f(expr_1)$ fails to terminate, we cannot say that the expression as a whole denotes bottom; its value depends on the hypothetical value bound to f. Since the lambda expression is not being applied to any argument here, the first argument of **if** will not reduce to an element of the semantic domain. It remains as a "parameterized" description of a domain element.

In spite of this problem, leftmost evaluation is used in implementing higher-order functional languages. The implementor simply acknowledges that, though it is useful to pass functions as arguments and results, we do so only for the sake of applying them in the computation of other objects. We have no need to expand an unapplied function for its own sake, no need to expand in the body of a function until after that function has been applied (in which case the lambda parameter has been replaced). Indeed, as we have seen above, this is also necessary to ensure that β-reduction terminates.

The leftmost rule has yet another deficiency when dealing with non-flat domains. Consider a language with a hierarchical domain, built using the ordered pair constructor '$<,>$'. Suppose we are trying to compute an ordered pair consisting of two infinite sublists, $F(exp_1)$ and $F(exp_2)$:

$< F(exp_1),\ F(exp_2) >.$

A safe computation rule should produce a sequence of approximations whose limit is the denoted pair of infinite lists. Using the left-most rule, however, no part of the right side would ever be computed. The limit of the approximation sequence

would be a pair of objects, the first an infinite list, and the second object completely undefined. Lazy functional languages *do* permit infinite lists in the domain, nevertheless, they are usually implemented with a *leftmost* computation rule. This works so long as one is only concerned with computing finite objects (though finite portions of infinite objects may be used during the computation).

The limitation to programs which denote finite objects is inadequate for PowerFuL, a language which contains the expressive power of logic programming. Many useful logic programs will denote an infinite set of solutions; even if the set of solutions is finite, the search space may contain nonterminating branches. Even with a complete breadth-first search strategy, the computation procedure need not terminate. Yet, even if computation of the set never terminates, certain elements of the set might be computed with only a finite amount of computation, and the user might wish to see those elements as they appear. Even though standard Prolog does not provide a complete search strategy for Horn logic (though, in principle, complete breadth-first Prolog interpreters could be built), Prolog does make a serious effort to compute infinite sets. Consider the following Prolog program, which denotes an infinite set of correct answer substitutions:

```
app([], Y, Y).
app([H|T], Y, [H|Z]) :- app(T, Y, Z).
?- app([1,2], X, Y), app(X, [1,2],Y).
```

Rather than waiting for the entire set to be computed (which may never happen), the system suspends and turns control over to the user, each time another member of this set is computed. With each new solution, the user has a better approximation to the complete set.

Traditionally, lazy languages have been *demand-driven*, in that they compute only those finite parts of an infinite object specifically requested by the user. Languages computing infinite sets, however, cannot be demand driven. The problem is that there is no referentially-transparent command or operation that a user may invoke to reduce an infinite set to a single finite piece. The user cannot ask for the "first" element of the set (as he could ask for the first element of a list), because sets have no implicit ordering. All the system can do is to provide increasingly better approximations, in which any finite element should eventually appear, until

the user decides he has seen enough, and terminates the computation. This could be done interactively, with the system suspending each time a new element is ready for output, and resuming at the option of the user. This leads to the topic of user interfaces for PowerFuL. Perhaps in an interactive implementation, the programmer will be able to direct where in the set expression the computational effort should be concentrated. As the desire for referential transparency prevents such commands from being part of the language per se, they could be provided in the meta-linguistic environment, analogous to online-debugger commands. Details of such an environment are beyond the scope of this work.

Therefore, we would like our implementation to be complete, if possible, even for programs which are inherently non-terminating. That is, we want a fair driver which, upon learning that the program denotes a union of two subsets, computes both subsets simultaneously. Similarly, the top-level driver simultaneously compute both sides of an ordered pair.

In summary, the leftmost computation rule is more efficient than parallel-outermost in that the delay in expanding some outermost occurrences, permits some subcomputations to be avoided completely. Use of the leftmost computation rule for a lazy higher-order language requires two compromises: The first is that we never try to "compute" a function alone, though we may compute a function applied to some argument. The second compromise is that we only wish to compute objects for which the computation sequence will terminate, namely finite objects. We accept the first compromise, but not the second. Therefore, we must develop a computation rule which is a compromise between the leftmost and the parallel-outermost.

4.4.3 PowerFuL's Reduced Parallel-Outermost Rule

The computation rule must consider four separate and exhaustive cases: (i) when the expression is a recursive function call (not a primitive or constructor); (ii) when the expression is headed by a data constructor (other than λ); (iii) when the expression is headed by a primitive function not within the context of a lambda expression (so we need not consider the presence of unbound lambda variables); and (iv) when the expression is an unapplied λ expression.

We only require that the computation rule be safe when computing objects

from a domain such as 'E', where

$$\mathtt{E} = (\mathtt{B} + \mathtt{A} + \mathtt{E}\times\mathtt{E} + \mathcal{P}(\mathtt{E}))_{\perp}.$$

This is the subset of PowerFuL's semantic domain D, where

$$\mathtt{D} = (\mathtt{B} + \mathtt{A} + \mathtt{D}\times\mathtt{D} + \mathtt{D}\mapsto\mathtt{D} + \mathcal{P}(\mathtt{D}))_{\perp}.$$

Domain E excludes those objects from D which either are or contain functions (though we may use these excluded element of D in computing elements of E). We also require that primitives satisfy the following property.

Definition 4.7: We say that a primitive is *eager* if it can simplify (either to the final result or to another primitive expression) as soon as the outermost constructor of any strict argument is known, or, in the case of a type-checking primitive, as soon as the type of the argument is known (e.g. by noticing the output type of the arguments outermost function).

Definition 4.8: The *reduced parallel-outermost computation rule* chooses function calls according to the following:

Case (i): If the expression is a function call, then expand only the main (single outermost) function call. For example, in $F(1, F(2, F))$, only the first F would be expanded.

Case (ii): If the expression is headed by a data constructor other than λ (e.g. the ordered-pair or set union), then expand the union of sets of function calls chosen by applying a safe computation rule individually to each argument. For example, in $< F(1, F(1, 2)),\ F(3, 4) >$, the first and last occurrences would be expanded.

Case (iii): Suppose the expression is headed by a primitive function (and is not within the context of a lambda expression). Let *arg* be any of the arguments in which the primitive is strict. The computation rule chooses just those occurrences in the primitive expression that would be chosen by applying the computation rule to *arg* alone. (Note that if the primitive is strict in several arguments, this computation rule gives us a choice of substitution steps.) For example, if addition is a primitive strict in both arguments, then in $+(F(1, 2),\ F(2, 3))$, either of the two occurrences could be chosen.

Case (iv): Suppose the occurrences are within a lambda expression, such as $\lambda x.\ \mathbf{if}(F(x, 1),\ F(1, 2),\ 7)$. Given the the program denotes an object from sub-domain E, and given that all primitives are eager, and assuming that primitives are

simplified as much as possible between recursive function expansions steps, then an unapplied lambda expression can only occur either within a nonstrict argument of a primitive, or, as a subexpression of a recursive function call. In neither case would the computation rule look to the lambda expression for recursive function calls to expand. Therefore, this case can be eliminated out of hand. (Furthermore, since this case will not occur, we will never compute the bodies of functions except after application, and therefore we may be assured that β-reduction will always terminate).

Theorem 4.5: If all the semantic primitives are eager, and assuming that primitive simplifications are made as early as possible, then a computation rule which chooses from among the above substitution steps (depending upon the situation) is safe.

Proof: A safe computation rule is, by definition, one which uses only safe substitution steps. Note that all four cases above describe safe substitution steps. Case (i) is a parallel outermost substitution step, a substitution already proven to be safe [V74]. Case (ii) can be proven by induction on the height of the term. If the substitution steps calculated for each subterm are safe, then safety holds individually for each argument, and therefore must also hold for the expression as a whole. Case (iii): If we replace the chosen recursive function occurrences with least-defined values of the appropriate functionality, then the argument in which the occurrences were found evaluates to $\perp$ (the primitives being eager, there otherwise would have been opportunity to simplify; but we assumed the term was already simplified as far a possible). Since the primitive is strict in this argument, replacement of these occurrences by $\perp$ makes the primitive application evaluate to the least element of the primitive's output domain. Replacement of *all* occurrences in all arguments by $\perp$ must give the save result, proving that this is a safe substitution. Case (iv): This case will not occur, as mentioned above. **End of Proof**

Note that if the primitive is strict in several arguments, this computation rule gives us a choice of substitution steps. The two advantages of this approach over simple parallel outermost is that β-reductions will always terminate (for programs denoting elements of domain E), and also that we can sometimes avoid computation in a primitive's nonstrict arguments. This gives us some of the computational advantages of the leftmost (outermost) rule, without sacrificing safety.

4.4.4 Example

We have proven that the reduced-parallel-outermost rule is safe (and therefore *probably* complete for programs of interest), assuming that semantic primitives simplify as soon as they have enough information to do so, and assuming that all semantic primitives terminate. In Chapter 5, we examine the semantic primitives and justify these assumptions. As an aside, note that all simplifications would terminate under any computation rule given a first-order restriction of PowerFuL, so for a first-order PowerFuL, our safe computation rule is indeed known to be a fixpoint rule.

Below is a sample program execution to translate into the semantic domain the object program:

```
letrec
   inf be cons('joe, inf)
in
   car(inf).
```

We start with an empty environment, so the initial input is:

$\mathcal{E}[\![\texttt{letrec inf be cons('joe,inf) in car(inf)}]\!]\ \Lambda.$

Expanding the outermost call yields:

$\mathcal{E}[\![\texttt{car(inf)}]\!](\mathcal{D}[\![\texttt{inf be cons('joe,inf)}]\!]\ \Lambda).$

There are still no simplifications to be performed, so we again expand the outermost function call, yielding:

$\textbf{left}(\textbf{pair!}(\mathcal{E}[\![\texttt{inf}]\!](\mathcal{D}[\![\texttt{inf be cons('joe,inf)}]\!]\ \Lambda))).$

Expanding the outermost function call yields:

$\textbf{left}(\textbf{pair!}((\mathcal{D}[\![\texttt{inf be cons('joe,inf)}]\!]\ \Lambda)\texttt{inf})),$

and then:

$\textbf{left}(\textbf{pair!}([[(\mathcal{FIX}\ \lambda X.\ (\mathcal{E}[\![\texttt{cons('joe,inf)}]\!]\ [X/\texttt{inf}]))/\texttt{inf}]\texttt{inf})).$

Note that when introducing new lambda variables, one must be careful to standardize variables apart (rename bound variables so as not to confuse them with pre-existing lambda variables). Simplifying (applying the environment) yields

$\textbf{left}(\textbf{pair!}(\mathcal{FIX}(\lambda X.(\mathcal{E}[\![\texttt{cons('joe,inf)}]\!]\ [X/\texttt{inf}])))).$

Expanding the outermost call yields:

$$\textbf{left}(\textbf{pair}!((\lambda X.\ (\mathcal{E}[\![\texttt{cons}('\texttt{joe}, \texttt{inf})]\!]\ [X/\texttt{inf}]))$$
$$(\mathcal{FIX}(\lambda X.\ (\mathcal{E}[\![\texttt{cons}('\texttt{joe}, \texttt{inf})]\!]\ [X/\texttt{inf}])))))).$$

A β-reduction yields:

$$\textbf{left}(\textbf{pair}!(\mathcal{E}[\![\texttt{cons}('\texttt{joe}, \texttt{inf})]\!]\ \rho)),$$

where ρ is:

$$[(\mathcal{FIX}(\lambda Y.(\mathcal{E}[\![\texttt{cons}('\texttt{joe}, \texttt{inf})]\!]\ [Y/\texttt{inf}])))/\texttt{inf}].$$

Expanding the outermost function call yields:

$$\textbf{left}(\textbf{pair}!(<(\mathcal{E}[\!['\texttt{joe}]\!]\ \rho),\ (\mathcal{E}[\![\texttt{inf}]\!]\ \rho)>)).$$

This simplifies to:

$$\mathcal{E}[\!['\texttt{joe}]\!]\ \rho.$$

Expanding the remaining function call yields:

'joe.

4.6 Summary

When proposing a language, it is good to show that it *can* be correctly implemented, at least theoretically. In this chapter we adapted Vuillemin's methodology to interpret a language's denotational description as an interpreter, thus ensuring that the operational semantics is consistent with the denotational semantics. We developed the reduced-parallel-outermost computation rule, which we believe is complete for those programs in which an unapplied function is not part of the final result.

Because we chose earlier to map set abstractions onto angelic powerdomains, an approximation to the denoted set consists of those elements of the denoted set which we have proven to be contained therein. If a set is described as a union of two subsets, then whatever is contained in one subset must surely be contained in the union. For completeness, we must therefore compute of both parts of a union in parallel. Had we used demonic powerdomains, an approximation to the denoted set would consist of those elements which we have not yet ruled out as being members. To prove that an element is not a member of a union, we must prove for both subsets that the element is not a member. Had we been interested in this

sort of computations, the computation of a union could be sequentialized. With demonic powerdomains, an undefined subset makes the whole set undefined, so we could compute the subsets in sequence. Had we used the Egli-Milner powerdomain, our approximations have to give more information, i.e., an approximation would have to indicate whether the approximation is capable of being enlarged with additional elements, or whether the computation has in fact terminated. This is added precision is not required for any of the features in PowerFuL, though it would be necessary if PowerFuL contained a predicate to test whether a set were empty.

In writing PowerFuL's denotational semantics, we have carefully chosen the semantic primitives in such a way that a correct implementation can be derived directly. For an efficient implementation, we do not recommend that this procedure be followed too literally. Many optimizations are needed to make the implementation efficient. So long as each optimization maintains correctness, then the resulting efficient operational semantics will also be correct with respect to the normative denotational description. Much research has already been done on techniques to implement lazy functional languages (see [P87]), and we will not discuss these techniques here. This thesis will concern itself only with one very special optimization to introduce logical variables. This optimization, to be discussed in Chapter 6, will avoid blind generating and testing when the set denoted by `terms` is a relative set abstractions generator.

5 POWERFUL SEMANTIC PRIMITIVES

This chapter formally defines the semantic primitives of PowerFuL. To justify the operational semantics developed in the previous chapter, we must:

1) identify which arguments of each primitive are strict;

2) verify that the primitive simplifies upon knowing the outermost constructor of any strict argument, (or, in the case of a type-checking primitive, as soon as the type of the argument is known);

3) verify that after any application of the computation rule, primitive simplifications will terminate; and

4) verify that all primitive functions are continuous.

These requirements affected the way the semantic equations were written. For instance, the fixpoint operator would never do as a primitive, as it would lead to non-terminating simplifications. Instead, we implemented the fixpoint operator using the denotational equations themselves. That way, only a finite amount of work is called for within each computation step. Creating a denotational description suitable for direct interpretation requires this kind of special care.

We define the primitives via equations. Each equation is also a simplification rule, rewriting from left to right. These functions are well-defined. Their values at the limit points are defined by the continuous extension.

5.1 Boolean Input Primitives

A boolean input primitive is one which requires one or more of its arguments be from the subdomain $\mathtt{B}_\perp$. This is a flat domain consisting two elements, **TRUE** and **FALSE**, and the least element $\perp$. PowerFuL has two such primitives, the conditional and the negation. Each is strict in its boolean argument, and each simplifies when the outermost constructor of that argument is available.

The equations defining **if**: $B_\perp \times D \times D \mapsto D$ are:

if(**TRUE**, *arg2*, *arg3*) = *arg2*
if(**FALSE**, *arg2*, *arg3*) = *arg3*
if($\perp$, *arg2*, *arg3*) = $\perp$.

For clarity when writing nested conditionals, we shall feel free to express this primitive using the alternative **if**...**then**...**else**...**fi** notation.

The equations defining **not**: $B_\perp \mapsto B_\perp$ are:

not(**TRUE**) = **FALSE**
not(**FALSE**) = **TRUE**
not($\perp$) = $\perp$.

Simplifications for these primitives obviously terminate, as only one rewriting is needed.

The function **not** is a monotonic function over a discrete domain, and therefore is continuous (by Theorem 3.1.). To show that the function **if** is continuous, we must show that it is continuous in each argument. It is continuous in the first argument by Theorem 3.1. For the second argument, we must split into cases, according to the value of the first argument. If the first argument is either $\perp$ or $FALSE$, then the second argument is completely ignored, and so **if** is continuous in that argument by default. If the first argument is **TRUE**, then **if** simplifies to the identity function (which is continuous) applied to the second argument.

5.2 Atomic Input Primitives

From the syntax of each program, we determine a finite set of atoms, each beginning with a quote ('). The set of atoms is always assumed to include `'nil`. For each such atom A_i in the syntax there exists a corresponding semantic zero-arity constructor $\mathbf{A}_i$. These objects, together with the bottom element $\perp$, make up the subdomain of atoms, $A_\perp$.

For each atom $\mathbf{A}_i$ is a primitive function **is**$\mathbf{A}_i$**?**: $A_\perp \mapsto B_\perp$, strict in its only argument. The simplification rules are:

is$\mathbf{A}_i$**?**($\perp$) = $\perp$
is$\mathbf{A}_i$**?**($\mathbf{A}_i$) = **TRUE**
is$\mathbf{A}_i$**?**($\mathbf{A}_j$) = **FALSE** for $i \neq j$.

Note that the last rewrite rule actually represents a number of rewrite rules, one for each pair of distinct atoms. For instance, the primitive **is'nil?** tests whether an

atom is equal to **'nil**. Each primitive of the form $\mathbf{isA}_i\mathbf{?}$ is strict in its argument, and simplifies whenever the outermost constructor of that argument (the atom itself) is available. Simplifications for these primitives obviously terminate, as only one rewriting is needed.

Using the primitives just described as a base, we define a primitive to compare two atoms for equality. The simplification rules for

atomeq?: $\mathtt{A}_\perp \times \mathtt{A}_\perp \mapsto \mathtt{B}_\perp$

are:

atomeq?($\perp$, *arg2*) = $\perp$
atomeq?(*arg1*, $\perp$) = $\perp$
atomeq?($\mathbf{A}_i$, *arg2*) = $\mathbf{isA}_i$**?**(*arg2*)
atomeq?(*arg1*, $\mathbf{A}_i$) = $\mathbf{isA}_i$**?**(*arg1*).

This primitive is strict in both arguments, and simplifies whenever the outermost constructor (an atom) of either argument is available. With one simplification, it either terminates or simplifies to another primitive ($\mathbf{isA}_i$) which terminates. Note that the third and fourth rules are actually rule schemas, each defining a rewrite rule for each atom $\mathbf{A}_i$.

These primitives are also continuous by theorem 3.1.

5.3 List Primitives

The primitive **left**: $\mathtt{D} \times \mathtt{D} \mapsto \mathtt{D}$ is strict in its only argument; likewise for the primitive **right**. Each needs only a single simplification rule:

left(<*1st*, *2nd*>) = *1st*
right(<*1st*, *2nd*>) = *2nd*.

Even before we have fully computed the argument, as soon as we have broken it into an ordered pair (computed the outermost constructor), the primitive simplifies. Termination is obvious, as only one rewriting is needed.

The proof continuity of these functions may be found in [S86].

5.4 The Powerdomain Primitive

The primitive '+' lets us iterate a function of type $\mathtt{D} \mapsto \mathcal{P}(\mathtt{D})$ over the elements of an input set, combining the results via union into a single new set. It is strict in the second argument. We can define '+' recursively via the rules:

$F^+(\phi) = \phi$
$F^+(\{Expr\}) = F(Expr)$
$F^+(Set_1 \cup Set_2) = (F^+(Set_1) \cup F^+(Set_2))$

Note that it simplifies immediately as soon as the outermost constructor (either ϕ, { }, or $\cup$) is available. It is not quite strict in the first argument, since

$(\perp_{D \mapsto \mathcal{P}(D)})^+(set)$

could produce $\{\perp\}$, which, though less defined than most powerdomain elements, is not less defined than $\perp_{\mathcal{P}(D)} = \phi$ (ϕ can be interpreted as, "We do not know any elements of this set, nor even if any exist," whereas $\{\perp\}$ can be interpreted as, "We know this set has at least one element, but we do not know anything about it.")

At any stage of computation, any union tree will have been constructed to only a finite height. Though '+' is defined recursively, each recursion goes deeper into the finite union tree, so simplifications must terminate. In other words, though it might not terminate when applied to a fully-computed infinite set, in practice no infinite set is ever fully computed. The best one can produce are arbitrarily good finite approximations.

The proof continuity of this function may be found in [S86].

5.5 Coercions

Chapter 3 introduced the coercions, and showed their use in handling error conditions. That is, they protect typed primitives from having to deal with arguments of an inappropriate type. The rewrite rules defining them are listed below. All are computed with a single simplification, guaranteeing termination. All simplify when the argument's outermost constructor is known, as can be verified by looking at the equations.

The coercions, first described in Chapter 3, are **bool!**, **atom!**, **pair!**, **func!** and **set!**.

The function **bool!**: $D \mapsto B_\perp$ maps *arg* to itself if *arg* is a member of $B_\perp$, and to $\perp$ otherwise, and is implemented using these equations:

bool!$(\perp) = \perp$
bool!$(\mathbf{TRUE}) = \mathbf{TRUE}$
bool!$(\mathbf{FALSE}) = \mathbf{FALSE}$
bool!$(A_i) = \perp$
bool!$(< exp_1, exp_2 >) = \perp$

$\mathbf{bool!}(\lambda\ \ldots) = \bot$
$\mathbf{bool!}(\phi) = \bot$
$\mathbf{bool!}(\{\ldots\}) = \bot$
$\mathbf{bool!}(\ldots \cup \ldots) = \bot$

The function **atom!**: $\mathtt{D} \mapsto \mathtt{A}_\bot$ maps *arg* to itself if *arg* is a member of $\mathtt{A}_\bot$, and to $\bot$ otherwise.

$\mathbf{atom!}(\bot) = \bot$
$\mathbf{atom!}(\mathbf{TRUE}) = \bot$
$\mathbf{atom!}(\mathbf{FALSE}) = \bot$
$\mathbf{atom!}(\mathbf{A}_i) = \mathbf{A}_i$
$\mathbf{atom!}(< exp_1,\ exp_2 >) = \bot$
$\mathbf{atom!}(\lambda\ \ldots) = \bot$
$\mathbf{atom!}(\phi) = \bot$
$\mathbf{atom!}(\{\ldots\}) = \bot$
$\mathbf{atom!}(\ldots \cup \ldots) = \bot$

The function **pair!**: $\mathtt{D} \mapsto \mathtt{D}\times\mathtt{D}$ maps *arg* to itself if *arg* is a member of $\mathtt{D}\times\mathtt{D}$, and to $\bot_{\mathtt{D}\times\mathtt{D}}$ (that is, $< \bot,\ \bot >$) otherwise.

$\mathbf{pair!}(\bot) = < \bot,\ \bot >$
$\mathbf{pair!}(\mathbf{TRUE}) = < \bot,\ \bot >$
$\mathbf{pair!}(\mathbf{FALSE}) = < \bot,\ \bot >$
$\mathbf{pair!}(\mathbf{A}_i) = < \bot,\ \bot >$
$\mathbf{pair!}(< exp_1,\ exp_2 >) = < exp_1,\ exp_2 >$
$\mathbf{pair!}(\lambda\ \ldots) = < \bot,\ \bot >$
$\mathbf{pair!}(\phi) = < \bot,\ \bot >$
$\mathbf{pair!}(\{\ldots\}) = < \bot,\ \bot >$
$\mathbf{pair!}(\ldots \cup \ldots) = < \bot,\ \bot >$

The function **func!**: $\mathtt{D} \mapsto [\mathtt{D}\mapsto\mathtt{D}]$ maps *arg* to itself if *arg* is a member of $\mathtt{D}\mapsto\mathtt{D}$ and to $\bot_{\mathtt{D}\mapsto\mathtt{D}}$ (that is, $\lambda \mathbf{x}.\ \ \bot_{\mathtt{D}}$, also written as Ω) otherwise.

$\mathbf{func!}(\bot) = \Omega$
$\mathbf{func!}(\mathbf{TRUE}) = \Omega$
$\mathbf{func!}(\mathbf{FALSE}) = \Omega$
$\mathbf{func!}(\mathbf{A}_i) = \Omega$
$\mathbf{func!}(< exp_1,\ exp_2 >) = \Omega$
$\mathbf{func!}(\lambda\ var.\ body) = \lambda\ var.\ body$
$\mathbf{func!}(\phi) = \Omega$
$\mathbf{func!}(\{\ldots\}) = \Omega$
$\mathbf{func!}(\ldots \cup \ldots) = \Omega$

The function **set!**: $\mathtt{D} \mapsto \mathcal{P}(\mathtt{D})$ maps *arg* to itself if *arg* is a member of $\mathcal{P}(\mathtt{D})$ and to $\bot_{\mathcal{P}(\mathtt{D})}$ (that is, ϕ) otherwise.

$\mathbf{set!}(\bot) = \phi$
$\mathbf{set!}(\mathbf{TRUE}) = \phi$
$\mathbf{set!}(\mathbf{FALSE}) = \phi$

$\mathbf{set!}(\mathbf{A}_i) = \phi$
$\mathbf{set!}(< exp_1,\ exp_2 >) = \phi$
$\mathbf{set!}(\lambda\ \ldots) = \phi$
$\mathbf{set!}(\phi) = \phi$
$\mathbf{set!}(\{element\}) = \{element\}$
$\mathbf{set!}(set_1 \cup set_2) = set_1 \cup set_2$

We can reduce sequences of coercions by noting that for two coercions C_a and C_b:

$C_a(C_b(arg)) = C_a(arg)$ when C_a and C_b are the same coercions.
$C_a(C_b(arg)) = C_a(\perp)$ when C_a and C_b are different coercions.

For instance, we can simplify **set!**(**set!**(arg)) to **set!**(arg), and **set!**(**func!**(arg)) to ϕ. The continuity of these functions was discussed at the end of Chapter 3.

5.6 Run-time Type-checking

PowerFuL is an untyped language. To allow the programmer to specify run-time type-checking, provide these primitive semantic functions over $\mathtt{D} \mapsto \mathtt{B}_\perp$: **bool?**, **atom?**, **pair?**, **func?** and **set?**. We can determine an object's type by viewing the outermost constructor, or in some cases, by the output type of the primitive heading the argument (i.e. if the primitive *never* returns a result that is completely undefined. For instance, the expression **set?**(**set!**(arg)) always denotes **TRUE**).

The equations defining **bool?** are:

$\mathbf{bool?}(\perp) = \perp$
$\mathbf{bool?}(\mathbf{TRUE}) = \mathbf{TRUE}$
$\mathbf{bool?}(\mathbf{FALSE}) = \mathbf{TRUE}$
$\mathbf{bool?}(\mathbf{A}_i) = \mathbf{FALSE}$
$\mathbf{bool?}(< exp_1,\ exp_2 >) = \mathbf{FALSE}$
$\mathbf{bool?}(\lambda\ \ldots) = \mathbf{FALSE}$
$\mathbf{bool?}(\phi) = \mathbf{FALSE}$
$\mathbf{bool?}(\{\ldots\}) = \mathbf{FALSE}$
$\mathbf{bool?}(\ldots \cup \ldots) = \mathbf{FALSE}.$
$\mathbf{bool?}(\mathbf{pair!}(arg)) = \mathbf{FALSE}.$
$\mathbf{bool?}(\mathbf{func!}(arg)) = \mathbf{FALSE}.$
$\mathbf{bool?}(\mathbf{set!}(arg)) = \mathbf{FALSE}.$
$\mathbf{bool?}(\ldots{}^{+}\ldots) = \mathbf{FALSE}.$

In other words, **bool?** returns **TRUE** if the argument is a boolean, **FALSE** if the argument is an atom, an ordered pair, a function or a set, and $\perp$ if the object's type is unknown. The primitive is computed with just one simplification, so termination is guaranteed. When the outermost constructor of the object is known, simplification can proceed.

The other type-checking primitives are similar, and have the same termination and simplification properties. The equations for **atom?** are:

$$\begin{aligned}
&\textbf{atom?}(\bot) = \bot \\
&\textbf{atom?}(\textbf{TRUE}) = \textbf{FALSE} \\
&\textbf{atom?}(\textbf{FALSE}) = \textbf{FALSE} \\
&\textbf{atom?}(\mathbf{A}_i) = \textbf{TRUE} \\
&\textbf{atom?}(< exp_1,\ exp_2 >) = \textbf{FALSE} \\
&\textbf{atom?}(\lambda\ \ldots) = \textbf{FALSE} \\
&\textbf{atom?}(\phi) = \textbf{FALSE} \\
&\textbf{atom?}(\{\ldots\}) = \textbf{FALSE} \\
&\textbf{atom?}(\ldots \cup \ldots) = \textbf{FALSE}. \\
&\textbf{atom?}(\textbf{pair!}(arg)) = \textbf{FALSE}. \\
&\textbf{atom?}(\textbf{func!}(arg)) = \textbf{FALSE}. \\
&\textbf{atom?}(\textbf{set!}(arg)) = \textbf{FALSE}. \\
&\textbf{atom?}(\ldots{}^{+}\ldots) = \textbf{FALSE}.
\end{aligned}$$

The equations for **pair?** are:

$$\begin{aligned}
&\textbf{pair?}(\bot) = \bot \\
&\textbf{pair?}(\textbf{TRUE}) = \textbf{FALSE} \\
&\textbf{pair?}(\textbf{FALSE}) = \textbf{FALSE} \\
&\textbf{pair?}(\mathbf{A}_i) = \textbf{FALSE} \\
&\textbf{pair?}(< exp_1,\ exp_2 >) = \textbf{TRUE} \\
&\textbf{pair?}(\lambda\ \ldots) = \textbf{FALSE} \\
&\textbf{pair?}(\phi) = \textbf{FALSE} \\
&\textbf{pair?}(\{\ldots\}) = \textbf{FALSE} \\
&\textbf{pair?}(\ldots \cup \ldots) = \textbf{FALSE}. \\
&\textbf{pair?}(\textbf{pair!}(arg)) = \textbf{TRUE}. \\
&\textbf{pair?}(\textbf{func!}(arg)) = \textbf{FALSE}. \\
&\textbf{pair?}(\textbf{set!}(arg)) = \textbf{FALSE}. \\
&\textbf{pair?}(\ldots{}^{+}\ldots) = \textbf{FALSE}.
\end{aligned}$$

The equations for **func?** are:

$$\begin{aligned}
&\textbf{func?}(\bot) = \bot \\
&\textbf{func?}(\textbf{TRUE}) = \textbf{FALSE} \\
&\textbf{func?}(\textbf{FALSE}) = \textbf{FALSE} \\
&\textbf{func?}(\mathbf{A}_i) = \textbf{FALSE} \\
&\textbf{func?}(< exp_1,\ exp_2 >) = \textbf{FALSE} \\
&\textbf{func?}(\lambda\ \ldots) = \textbf{TRUE} \\
&\textbf{func?}(\phi) = \textbf{FALSE} \\
&\textbf{func?}(\{\ldots\}) = \textbf{FALSE} \\
&\textbf{func?}(\ldots \cup \ldots) = \textbf{FALSE}. \\
&\textbf{func?}(\textbf{pair!}(arg)) = \textbf{FALSE}. \\
&\textbf{func?}(\textbf{func!}(arg)) = \textbf{TRUE}. \\
&\textbf{func?}(\textbf{set!}(arg)) = \textbf{FALSE}. \\
&\textbf{func?}(\ldots{}^{+}\ldots) = \textbf{FALSE}.
\end{aligned}$$

The equations for **set?** are:

$$\begin{aligned}
&\textbf{set?}(\bot) = \bot \\
&\textbf{set?}(\textbf{TRUE}) = \textbf{FALSE}
\end{aligned}$$

set?(FALSE) = FALSE
set?(A_i) = FALSE
set?(< exp_1, exp_2 >) = FALSE
set?(λ ...) = FALSE
set?(ϕ) = TRUE
set?({...}) = TRUE
set?(... $\cup$ ...) = TRUE.
set?(pair!(*arg*)) = FALSE.
set?(func!(*arg*)) = FALSE.
set?(set!(*arg*)) = TRUE.
set?(...$^+$...) = TRUE.

To prove continuity, note that, because these functions are monotonic, the continuity requirement holds for all finite chains. For any infinite chain in D, the least upper bound of the chain and every chain element (with the possible exception of $\perp_D$ are all contained within one of these subdomains: $B_\perp$, $A_\perp$, $D \times D$, $D \mapsto D$ and $\mathcal{P}(D)$. When applied to members of the same subdomain, each type-checking always returns the same constant result, permitting continuity to be easily verified.

5.7 Equality

We define *first-order* equality by saying that two first-order objects are equal if and only if they are identical. First-order objects include atoms, booleans, and nested ordered pairs whose leaves are atoms and booleans. We wish to define equality over all first-order objects, not just atoms. There are several ways we could have done this. We could have given the programmer access to the **atomeq?** primitive, letting the user include a definition for equality within his program. Alternatively, we could have defined equality construct via the denotation equations. However, an important optimization technique developed in the next chapter requires that equality be made a primitive. As a primitive strict in both arguments, there must be a simplification available whenever the outermost constructor of either argument is available. This motivates the following equations, broken into groups for discussion. The first two equations simply indicate that the primitive is strict in both arguments.

equal?($\perp$, *arg2*) = $\perp$
equal?(*arg1*, $\perp$) = $\perp$.

If we know anything at all about either argument, we know whether it is a member of B (a boolean), A (an atom), D×D (an ordered pair), D↦D (a function) or $\mathcal{P}$(D) (a set). As soon as we know this about an argument (either by having computed the

outermost constructor, or by computing it to the form of a primitive application headed by **pair!**, **func!**, **set!** or '+'), we can apply a simplification rule.

If B is known to be a boolean, then:

$$\textbf{equal?}(B,\ exp) = \textbf{if bool?}(exp)\ \textbf{then if}(B, \textbf{bool!}(exp), \textbf{not}(\textbf{bool!}(exp)))\ \textbf{else FALSE fi}$$

and by symmetry:

$$\textbf{equal?}(exp,\ B) = \textbf{if bool?}(exp)\ \textbf{then if}(\textbf{bool!}(exp), B, \textbf{not}(B))\ \textbf{else FALSE fi.}$$

If A is an atom, then

$$\textbf{equal?}(A,\ exp) = \textbf{if atom?}(exp)\ \textbf{then atomeq?}(A, \textbf{atom!}(exp))\ \textbf{else FALSE fi}$$

and by symmetry

$$\textbf{equal?}(exp,\ A) = \textbf{if atom?}(exp)\ \textbf{then atomeq?}(\textbf{atom!}(exp), A)\ \textbf{else FALSE fi}$$

If F is a function, then

$$\textbf{equal?}(F,\ exp) = \textbf{if func?}(exp)\ \textbf{then}\ \perp\ \textbf{else FALSE fi}$$
$$\textbf{equal?}(exp,\ F) = \textbf{if func?}(exp)\ \textbf{then}\ \perp\ \textbf{else FALSE fi}$$

If S is a set, then

$$\textbf{equal?}(S,\ exp) = \textbf{if set?}(exp)\ \textbf{then}\ \perp\ \textbf{else FALSE fi}$$
$$\textbf{equal?}(exp,\ S) = \textbf{if set?}(exp)\ \textbf{then}\ \perp\ \textbf{else FALSE fi}$$

If P is an ordered pair, then:

$$\begin{aligned}&\textbf{equal?}(P,\ exp) = \\ &\quad \textbf{if not}(\textbf{pair?}(exp))\ \textbf{then FALSE} \\ &\quad \textbf{elseif not}(\textbf{equal?}(\textbf{left}(P),\ \textbf{left}(\textbf{pair!}(exp))))\ \textbf{then FALSE} \\ &\quad \textbf{else equal?}(\textbf{right}(P),\ \textbf{right}(\textbf{pair!}(exp)))\ \textbf{fi}\end{aligned}$$

and by symmetry:

$$\begin{aligned}&\textbf{equal?}(exp,\ P) = \\ &\quad \textbf{if not}(\textbf{pair?}(exp))\ \textbf{then FALSE} \\ &\quad \textbf{elseif not}(\textbf{equal?}(\textbf{left}(\textbf{pair!}(exp)),\ \textbf{left}(P)))\ \textbf{then FALSE} \\ &\quad \textbf{else equal?}(\textbf{right}(\textbf{pair!}(exp)),\ \textbf{right}(P))\ \textbf{fi}\end{aligned}$$

Though the simplification rules for **equal?** are recursive, simplifications will always terminate. The recursion only occurs when comparing two ordered pairs. At any stage of computation, any ordered-pair tree will have been constructed to only a

finite height. Each recursion goes deeper into the finite union tree, so simplifications must terminate. In other words, though it might not terminate when comparing two identical fully-computed lists, in practice no infinite list is ever *fully* computed. The best one can produce are arbitrarily good finite approximations.

Essentially, **equal?** is the least fixpoint of a recursive functional built using the function composition of the function variable and continuous primitives, and is therefore itself a continuous function [S86].

5.8 Summary

In this chapter, we have defined the primitives through equations which also serve a rewrite rules. We have shown that each primitive does indeed simplify upon availability of the outermost constructor of any argument in which it is strict. We have also shown that the simplification stage of every computation step must terminate.

With these definitions of the primitives, the definition of our language is complete. Because they satisfy the above properties, as required by the developments in chapter 4, we have an operational semantics as well. The next chapter will improve the interpretation procedure, and there these primitives play a central role.

6 OPTIMIZATIONS

This chapter improves the basic operational procedure described in the last chapter. The problem being attacked is the inefficiency of relative set abstraction when the set of terms acts as a generator. Rather than enumerating terms blindly, we would like to treat the enumerated parameter as a logical variable, analogous to the implementation of absolute set abstraction. This chapter explains why this is proper to do, and provides a scheme for how to implement this optimization.

6.1 Avoiding Generate-and-Test

In Section 2, we specified a Horn logic program using `letrec` (the feature for creating recursive definitions), set abstraction, the conditional and the equality primitive. Executing this program using PowerFuL's denotational equations as the interpreter would be analogous to using Herbrand's method [Q82] to solve problems in logic. Herbrand's "generate-and-test" approach is simple but inefficient.

The resolution method avoids blind generation of instantiations, preferring to do as much work as possible on non-ground expressions. In logic programming, a logical variable denotes an element from the set of terms (the Herbrand Universe). In resolution, a logical variable becomes instantiated only to the extent necessary to satisfy the inference rule's equality test. Partial instantiation narrows the set of candidate bindings, without necessarily settling on a single choice. When performed to ensure equality of non-ground terms, partial instantiation is called *unification*, and the substitution implementing the partial instantiation is called a *unifier*.

In PowerFuL, to compute a relative set expression, we normally begin by computing the generator set. As an example, consider the following expression.

```
letrec
   append be λ l1 l2. if null?(l1) then l2
                      else cons(car(l1), append(cdr(l1),l2)) fi
in {X :  X ∈ terms,  append(X, ['a,'b]) = ['c,'d,'a,'b]}
```

Whenever we isolate an expression denoting an element of the generator, in this case a term, a copy of the relative set abstraction is created, with the generator element expression replacing the enumeration parameter. In this case, the set `terms` evaluates to a union tree, whose leaves are each a singleton set containing one term. The set abstraction reduces to a union tree, each of whose leaves contains the value denoted by

```
letrec
   append be λ l1 l2. if null?(l1) then l2
                      else cons(car(l1), append(cdr(l1),l2)) fi
in
   { term : append(term, ['a,'b]) = ['c,'d,'a,'b]}
```

where *term* represents one element of the set of terms. It is clear that each such instantiation can easily be computed. Where *term* satisfies the condition, the result is a singleton set containing this term. For terms that do not satisfy the condition, the result will be ϕ. The complete result is union of all these subsets. We would like to modify this procedure so that when `terms` is the generator set, we treat the enumeration parameter as a logical variable, rather than blindly enumerating its many simple objects. Similarly, an enumeration parameter generated by *atoms*, or *bools* can be viewed as a partially-instantiated, or constrained, logical variable.

In Horn logic, each correct answer substitution provides ground bindings for the goal's logical variables. Members of this set can be grouped into families. Within a family, all answer substitutions share common aspects, with the remaining details varying freely. The Herbrand derivation of one member of the family is almost identical to the Herbrand derivation for any other member. For each family of correct answer substitutions, Herbrand's method would derive each member individually, with an infinity of essentially similar derivations. Resolution, however, produces *most general computed* answer substitutions, one per family. A general computed answer substitution only partially instantiates a goal's logical variables, and does

so in such a way that for *any* ground completion of the general computed answer substitution would result in a correct answer substitution. The derivation of the general answer substitution resembles a parameterized Herbrand derivation.

Resolution performs modus ponens inferences on the non-ground clauses directly, rather than first instantiating them. Logical variables become partially instantiated (via a most general unifier) only to the extent necessary to satisfy the inference rule's equality requirement. Pure Horn logic ignores logical variable bindings which make the equality false. PowerFuL primitives are more complex, in that we are interested in all possible logical variable bindings to produce all possible results. However, some versions of Prolog [N85] add the *inequality* predicate as well. Rather than actively generating bindings for logical variables, the inequality primitive constrains future bindings.

In a sense, program execution is left unfinished. Though it is easy to extend a most general answer substitution, to produce (ground) correct answer substitutions, this is not done. Reporting results in the general form is more economical than individually reporting each of the infinite ways in which each most general answer can be extended.

6.2 Logical Variable Abstraction

To treat an enumeration parameter as a logical variable, we must recognize that its generator is the set of first-order terms (or part of this set). An expression of the form

$$(\lambda x.body)^{+}\ \mathcal{F}[\![\texttt{terms}]\!]$$

or equivalently

$$(\lambda x.body)^{+}\ \textbf{set!}(\mathcal{F}[\![\texttt{terms}]\!])$$

is rewritten at run-time to

$$term(x).body$$

to indicate that x is to be treated as a logical variable, rather than blindly enumerating its generator. We can look out for this situation by noting when the semantic equation

$$\mathcal{E}([\![\{expr\ :\ id \in genrtr, qualifierlist\}]\!]\ \rho)$$
$$= (\lambda\ X.\ \mathcal{E}[\![\{expr\ :\ qualifierlist\}]\!]\ \rho[X/id])^{+}(\textbf{set!}(\mathcal{E}[\![genrtr]\!]\ \rho))$$

is used, and checking to see whether *genrtr* refers to `terms`, `atoms` or `bools`. The expressions $atom(x).body$ and $bool(x).body$ are constructed analogously. In an expression of the form

$$atom(x).body$$

x is a logical variable representing a term, under the constraint that this term must be an atom (not a boolean or ordered pair). Rather than recomputing the *body* for each trivial instantiation, we will evaluate *body* in its uninstantiated form, leaving it parameterized by the enumeration variable, computing a parameterized set expression. This parameterized set expression stands for the union of all possible instantiations, allowing us to express results more compactly.

For uniformity, we will keep parameterized sets in an analog of logic programming's clause form. That is, we will strive to represent a parameterized set as a union tree, each leaf of which is a possibly parameterized singleton (or empty) set. Therefore, a '+' expression with a parameterized set as the second argument will be rewritten as a parameterized '+' expression, and a parameterized union of subsets will be replaced by a union of parameterized subsets. That is, an expression of the form:

$$(\lambda x.\ body_1)^+(term(y).\ body_2),$$

will be rewritten as:

$$term(y).\ ((\lambda x.\ body_1)^+(body_2)).$$

This transformation is valid, because the first expression is an alternate notation for:

$$(\lambda x.\ body_1)^+((\lambda y.\ body_2)^+(\mathcal{F}[\![\texttt{terms}]\!])),$$

and the second is an alternate notation for

$$((\lambda x.\ body_1)^+(\lambda y.\ body_2))^+(\mathcal{F}[\![\texttt{terms}]\!]),$$

and these two are equal (due to the associativity of set union). When breaking up a parameterized union, an expression of the form

$$term(x).(exp_1 \cup exp_2)$$

becomes

$$term(x).exp_1 \ \cup \ term(x).exp_2.$$

A parameterized empty set, such as $term(x).\phi$, can be simplified to ϕ, if desired.

One computes a parameterized set expression by expanding the occurrences of the semantic function ($\mathcal{E}$, $\mathcal{F}$ or $\mathcal{D}$) in the parameterized body as specified by the computation rule, and doing all possible primitive simplifications between expansion steps. In other words, one strives to compute the parameterized body just as one would if each logical variable had been replaced by a ground term. Though expansion of the semantic function is unaffected by the fact that some terms are represented by logical variables, simplification of primitives becomes more difficult. Simplification of primitives applied to logical variables is discussed in the next section.

6.3 Simplifying Primitives with Logical Variables

Suppose a parameterized body contains a subexpression headed by a primitive, one of whose strict arguments is a logical variable. If the logical variable were replaced by a term, we could simplify the primitive with the appropriate primitive rewrite rule chosen according to the value of the term. But how do we choose which rewrite rule to use, when the logical variable represents any of a whole range of possible terms? For any such case, one of three techniques will permit simplification of the primitive. Cases motivating each technique are listed below.

1) When the same rewrite rule would be chosen for any possible instantiation of the logical variable, Technique 1 allows us to simplify directly, without any instantiation of the logical variable.

2) When the choice of rewrite rule depends upon whether a term represents a boolean, an atom or an ordered pair, and the logical variable is unconstrained (i.e. it could represent any of these), the second technique allows us to split into three subsets, each of which handles one of the possibilities.

3) When two logical variables are being compared for equality, the third technique splits the set into two subsets, one in which the equality is assumed to hold, and the other in which an inequality constraint is generated.

We describe each case in the notation:

$term(u).(\ldots prim(u)\ldots)$,

where *prim* is some specific primitive. The ellipses indicate that the primitive subex-

pression occurs anywhere in the body of the parameterized set expression. Between any two expansion steps, *all* possible primitive simplifications are performed, so the primitive's position in the expression is irrelevant.

6.3.1 Technique 1: Simple Reduction

Often, due to implied or stated constraints on the logical variable, the same rewrite rule would be chosen regardless of which term the logical variable represents. In this case, simplification of the primitive is straightforward. One applies the single applicable rewrite rule to the parameterized subexpression. For example, consider a parameterized expression containing a subexpression headed by the primitive **func?**:

$term(u).(\ldots \textbf{func?}(u) \ldots).$

We can simplify **func?**(u) to **FALSE**, without knowing the value of u, since any value would certainly not be a function. Below is a comprehensive list of similar situations. The rationale for these cases is that a logical variable never represents a function or a set, and a logical variable constrained to one subtype will never represent a term of another type.

$term(u).(\ldots \textbf{func?}(u) \ldots) \rightarrow term(u).(\ldots \textbf{FALSE} \ldots)$
$term(u).(\ldots \textbf{func!}(u) \ldots) \rightarrow term(u).(\ldots \Omega \ldots)$
$atom(u).(\ldots \textbf{func?}(u) \ldots) \rightarrow atom(u).(\ldots \textbf{FALSE} \ldots)$
$atom(u).(\ldots \textbf{func!}(u) \ldots) \rightarrow atom(u).(\ldots \Omega \ldots)$
$bool(u).(\ldots \textbf{func?}(u) \ldots) \rightarrow bool(u).(\ldots \textbf{FALSE} \ldots)$
$bool(u).(\ldots \textbf{func!}(u) \ldots) \rightarrow bool(u).(\ldots \Omega \ldots)$
$term(u).(\ldots \textbf{set?}(u) \ldots) \rightarrow term(u).(\ldots \textbf{FALSE} \ldots)$
$term(u).(\ldots \textbf{set!}(u) \ldots) \rightarrow term(u).(\ldots \phi \ldots)$
$atom(u).(\ldots \textbf{set?}(u) \ldots) \rightarrow atom(u).(\ldots \textbf{FALSE} \ldots)$
$atom(u).(\ldots \textbf{set!}(u) \ldots) \rightarrow atom(u).(\ldots \phi \ldots)$
$bool(u).(\ldots \textbf{set?}(u) \ldots) \rightarrow bool(u).(\ldots \textbf{FALSE} \ldots)$
$bool(u).(\ldots \textbf{set!}(u) \ldots) \rightarrow bool(u).(\ldots \phi \ldots)$
$atom(u).(\ldots \textbf{bool?}(u) \ldots) \rightarrow atom(u).(\ldots \textbf{FALSE} \ldots)$
$atom(u).(\ldots \textbf{bool!}(u) \ldots) \rightarrow atom(u).(\ldots \perp \ldots)$
$bool(u).(\ldots \textbf{bool?}(u) \ldots) \rightarrow bool(u).(\ldots \textbf{TRUE} \ldots)$
$bool(u).(\ldots \textbf{bool!}(u) \ldots) \rightarrow bool(u).(\ldots u \ldots)$
$atom(u).(\ldots \textbf{atom?}(u) \ldots) \rightarrow atom(u).(\ldots \textbf{TRUE} \ldots)$
$atom(u).(\ldots \textbf{atom!}(u) \ldots) \rightarrow atom(u).(\ldots u \ldots)$
$bool(u).(\ldots \textbf{atom?}(u) \ldots) \rightarrow bool(u).(\ldots \textbf{FALSE} \ldots)$

$bool(u).(\ldots \textbf{atom!}(u)\ldots) \rightarrow bool(u).(\ldots \perp \ldots)$
$atom(u).(\ldots \textbf{pair?}(u)\ldots) \rightarrow atom(u).(\ldots \textbf{FALSE}\ldots)$
$atom(u).(\ldots \textbf{pair!}(u)\ldots) \rightarrow atom(u).(\ldots < \perp,\ \perp > \ \ldots)$
$bool(u).(\ldots \textbf{pair?}(u)\ldots) \rightarrow bool(u).(\ldots \textbf{FALSE}\ldots)$
$bool(u).(\ldots \textbf{pair!}(u)\ldots) \rightarrow bool(u).(\ldots < \perp,\ \perp > \ \ldots)$
$bool(u).(\ldots \textbf{equal?}(u,\ ex)\ldots) \rightarrow \textbf{if}(\ \textbf{bool?}(ex),\ \textbf{if}(u, ex, \textbf{not}(\textbf{bool!}(ex))),\ \textbf{FALSE})$
$bool(u).(\ldots \textbf{equal?}(ex,\ u)\ldots) \rightarrow \textbf{if}(\ \textbf{bool?}(ex),\ \textbf{if}(\textbf{bool!}(ex), u, \textbf{not}(u)),\ \textbf{FALSE})$
$atom(u).(\ldots \textbf{equal?}(u,\ ex)\ldots) \rightarrow \textbf{if}(\textbf{atom?}(ex), \textbf{atomeq?}(u, \textbf{atom!}(ex)), \textbf{FALSE})$
$atom(u).(\ldots \textbf{equal?}(ex,\ u)\ldots) \rightarrow \textbf{if}(\textbf{atom?}(ex), \textbf{atomeq?}(\textbf{atom!}(ex), u), \textbf{FALSE})$
$term(u).(\ldots \textbf{equal?}(u,\ u)\ldots) \rightarrow term(u).(\ldots \textbf{TRUE}\ldots)$
$atom(u).(\ldots \textbf{atomeq?}(u,\ u)\ldots) \rightarrow atom(u).(\ldots \textbf{TRUE}\ldots).$

6.3.2 Technique 2: Splitting by Type

Given any of the primitives: **bool?**, **bool!**, **atom?**, **atom!**, **pair?** and **pair!**, the rewrite rule chosen depends upon the type of the argument (boolean, atom, ordered pair, function or set). Unless otherwise constrained, a logical variable can represent three of these types (atom, boolean and ordered pair); in order to simplify, we must consider each possibility separately. We divide the parameterized set expression into three subsets, so that for each subset, Technique 1 (simple reduction) will apply.

Let *prim* represent one of these four primitives. An expression of the form

$term(u).(\ldots prim(u)\ldots)$

is replaced by:

$bool(u).(\ldots prim(u)\ldots)$
$\cup\ atom(u).(\ldots prim(u)\ldots)$

$\cup\ term(v).term(w).(\ldots prim(u)\ldots)[< v, w > /u].$

In the first branch of the union, we have partially instantiated the logical variable by constraining it to represent a boolean; in the second branch, we have constrained it to represent an atom; in the third branch, we have constrained it to represent a term which is an ordered pair of subterms. The primitive function simplifies immediately in each subset, thus computes each branch of the union separately.

To prove that this technique is correct, note that the original expression is an alternate notation for:

$(\ldots prim(u)\ldots)^{+}(\mathcal{F}[\![\texttt{terms}]\!])$.

Expanding $\mathcal{F}$ yields:

$(\mathcal{F}[\![\texttt{bools}]\!])$
$\cup\ (\mathcal{F}[\![\texttt{atoms}]\!])$

$\cup\ (term(u).term(v).\ < u, v >)$.

Simplifying $^{+}$ by distributing $(\ldots prim(u)\ldots)$ over the union yields:

$(\ldots prim(u)\ldots)^{+}(\mathcal{F}[\![\texttt{bools}]\!])$
$\cup\ (\ldots prim(u)\ldots)^{+}(\mathcal{F}[\![\texttt{atoms}]\!])$

$\cup\ (\ldots prim(u)\ldots)^{+}(term(v).term(w).\ < v, w >)$.

Putting this into the standard format for a parameterized set yields the replacement expression.

This technique is analogous to narrowing in constructor-based term-rewriting systems [R85], where, to permit further reduction of a non-ground term (a term containing logical variables), one instantiates a variable in all possible ways which would enable further reduction. This technique is also analogous to the use of most general unifiers in Horn logic resolution. Unification prepares two clauses for modus ponens by instantiating them no more than is necessary to satisfy the equality requirement. One difference is that traditional Horn logic does not use negative information. Horn logic only considers instantiations to make the equality true. In PowerFuL, we are concerned with *all* possible outcomes. Some variations of Horn logic do consider negative information through the use of a inequality predicate and negative unifiers [N85] [K84]. We discuss primitives based on equality next.

6.3.3 Technique 3: Splitting on Equality

This section describes what to do when an argument of an equality primitive (**equal?** or **atomeq?**) is a logical variable. Combined with the rewrite rules for equality and the techniques given earlier, this technique implements syntactic unification. Actually, a generalization of unification results, as negative bindings (to make the equality false) are also considered.

Chapter 5 provided a comprehensive set of **equal?**'s rewrite rules. It simplifies to a conditional expression whenever the type of either argument becomes known.

If the logical variable is constrained to one particular type, a simple reduction can be performed, as was shown in a previous section.

If the lambda variable can represent more than one type, then, theoretically, one *could* split according to type (divide into three cases, and then do simple reductions for each case). In each case, the equality primitive would simplify to a conditional, whose strictness demands that the type of the other argument be next ascertained. However, a more efficient way is to *delay* the reduction of equality until the type of the other argument (the one which is *not* a logical variable) is known, and simplify according to *its* type. This way, preliminary computation of the other argument needs be done only once, instead of three times.

When Both Arguments are Logical Variables

If *both* arguments of the equality predicate are logical variables unconstrained in type, then we *must* simplify. Consider an expression of the form:

$$term(u).(\ldots term(v).(\ldots \textbf{equal?}(v,\ u)\ldots)\ldots)$$

This parameterized set expression contains somewhere in its body a primitive subexpression comparing two unconstrained logical variables. Theoretically, one could break this into an infinity of special cases, in each case u and v each being replaced by an element of the set of terms. For some combinations the predicate **equal?** would simplify to **TRUE**, and **FALSE** for other combinations. This could also have been done with the primitives described earlier, but it is better to deal with a few large subsets, than an infinity of individual cases. Splitting them into atoms, booleans and ordered pairs does not help. We must recognize that the instantiations fall into two cases: those for which the two terms are equal, and those for which they are not. The subset handling the cases in which the two terms are equal can be summarized by replacing all occurrences of v with occurrences of u:

$$term(u).(\ldots term(v).(\ldots \textbf{equal?}(v,\ u)\ldots)\ [v/u]\ldots).$$

Since there are no more occurrences of v in the subexpression

$$term(v).(\ldots \textbf{equal?}(v,\ u)\ldots)\ [v/u],$$

we can simplify this to

$$(\ldots.(\ldots \textbf{TRUE}\ldots)\ [v/u]).$$

(In fact, this simplification can be performed whenever a body does not depend on the enumerating variable. The special case of $term(x).\phi$ being replaced by ϕ was given earlier.)

Inequality Constraints

We also need to summarize the cases when u and v are *not* equal. This could be summarized by

$term(u).(\ldots term(v).$**if not(equal?**$(v,\ u))$ **then** $(\ldots$**FALSE**$\ldots)$ **else** $\phi\ \ldots).$

This summarizes the elements of the set for which which the two terms u and v are *not equal.* Is there a way to compute this further, without trying individually all possible combinations of unequal terms?

Lee Naish [N85] proposes for Prolog a inequality predicate, defined on terms. His inequality predicate would fail when two terms are identical, succeed when two terms cannot be unified, and delay when two terms are unifiable, but not identical. In the last case, the other subgoals would be executed first, until the values of logic variables have been instantiated enough to prove either the terms' equality or their inequality. If all other subgoals succeed, without instantiating the variables enough, Naish's Prolog gives an error message. This is not ideal behavior, since unequal instantiations can certainly be computed. A better alternative would be to make the inequality part of the solution, as a kind of negative unifier. Khabaza describes a way in which this can be done [K84]. In essence, the inequality becomes part of the general solution. Specific ground solutions can be generated from the general solutions by instantiating logical variables in all possible ways *subject to the inequality constraint.* Constraint logic programming [JL87] sets another precedent for this approach. Computation of general non-ground solutions greatly improves efficiency, and because replacing term variables by arbitrary ground terms is such a trivial operation it matters little that the computation has terminates prematurely. Requiring such term enumerations to satisfy a few inequalities adds little to the complexity of the output, and makes it more compact.

To express such a constraint, we could write the above subset as:

$term(u).(\ldots term(v)u \neq v.(\ldots$**FALSE**$\ldots)).$

We have simplified the equality predicate by splitting into two expressions: one expression representing the cases for which the equality holds, and the other expression representing the cases for which it is false, without the need to consider every case individually.

Solving subsequent inequalities result in a constraint which is a conjunction of inequalities. If the satisfaction of other predicates cause u and v to become refined into the ordered pairs, $< u_1, u_2 >$ and $< v_1, v_2 >$, respectively, then the inequality $u \neq v$ will become $< u_1, u_2 > \neq < v_1, v_2 >$, which simplifies to $or(u_1 \neq v_1,\ u_2 \neq v_2)$. In general, the total constraint will be an and/or tree of simple inequalities. As these simple constraints are satisfied, they can be replaced by **TRUE**. Those inequalities which become unsatisfiable can be replaced by **FALSE**, leading to further simplifications of the and/or tree. If the whole tree simplifies to **FALSE**, then we are enumerating an empty set, and the whole expression within can be replaced by ϕ.

Similar techniques are used for the predicates **atomeq?** and **isA$_i$?** (where **A$_i$** represents an arbitrary atom). To be thorough, the cases are itemized below.

List of Cases for Equality Splitting

$term(u).(\ldots term(v)\ldots constraint.(\ldots \mathbf{equal?}(v,\ u)\ldots)) \rightarrow$
$\quad term(u).(\ldots constraint(\ldots \mathbf{TRUE}\ldots)\ [v/u])$
$\quad \cup\ term(u).(\ldots term(v)\mathbf{and}(constraint, (u \neq v)).(\ldots \mathbf{FALSE}\ldots)).$

$atom(u).(\ldots atom(v)\ldots constraint.(\ldots \mathbf{atomeq?}(v,\ u)\ldots)) \rightarrow$
$\quad atom(u).(\ldots constraint(\ldots \mathbf{TRUE}\ldots)\ [v/u])$
$\quad \cup\ atom(u).(\ldots atom(v)\mathbf{and}(constraint, (u \neq v)).(\ldots \mathbf{FALSE}\ldots)).$

$atom(u).(\ldots constraint.(\ldots \mathbf{isA_i?}(u)\ldots)) \rightarrow$
$\quad (\ldots constraint(\ldots \mathbf{TRUE}\ldots)\ [u/\mathbf{A}_i]).$
$\quad \cup\ atom(u).(\ldots \mathbf{and}(constraint, (u \neq \mathbf{A}_i)).(\ldots \mathbf{FALSE}\ldots)).$

These optimizations are of course symmetrical in the order of arguments to **equal?** and **atomeq?**.

Note that we consider the binding of logical variables separate from the definition of the equality primitive itself. Robinson also split unification into these

components [BRS82]. We have generalized the approach to also consider negative unification.

6.3.3 Discussion

We have already considered these primitives applied to logical variables: **bool?**, **atom?**, **pair?**, **func?**, **set?**, **bool!**, **atom!**, **pair!**, **func!**, **set!**, **if**, **not**, **isA$_i$?** (for each atom **A$_i$**) and **equal?**. The primitives not discussed are: **left**, **right**, **not**, **if**, β-reduction and '+'.

According to the denotational equations, **left** and **right** are always applied in conjunction with the coercion **pair!**. Since we have already considered logical variables as arguments to **pair!**, we need no special mechanism for **left** and **right**.

The denotational equation producing function applications coerces the first argument via the **func!** primitive. Since we have already considered logical variables as arguments to **func!**, we need no special mechanism for β-reduction, either.

Analogously, the coercion **set!** intercedes between '+' and its strict second argument. Actually, since the second argument of '+' must be a set, one should not expect to see a logical variable (which represents a term) in that position. We have already discussed the case in which the second argument is a parameterized set expression.

When a logical variable is an argument to **not**, or the first argument to **if**, these arguments are protected by the coercion **bool!**, so the logical variable must already have been constrained to be a boolean. In that case, we simply split into two subsets, one for which the logical variable is **TRUE** and one for which it is **FALSE**. Though this involves a split, this is not really a special optimization technique. It is the default procedure when `bools` is a relative set abstraction generator! The only difference is that we delay the enumeration of `bools` until a boolean primitive needs to know which one.

The only case not covered under any of these techniques is the case when a logical variable represents a *nonstrict* argument of a primitive, e.g. if a logical variable is the second or third argument of **if**. A semantic primitive of PowerFuL will always simplify if it is given the outermost constructor of any strict argument. If the primitive is *not* strict in that argument, however, the primitive will *not* simplify, until something about a strict argument is known. So, consider

$term(u).(\ldots \mathbf{if}((\mathcal{E}[\![exp_1]\!]\ \rho), u, exp_3)\ldots).$

Assuming the outermost constructor of exp_1 is not available, the primitive **if** would not yet simplify no matter which term might replace u. Therefore, the occurrence of a logical variable in this position does not call for any special consideration.

6.4 Example

Suppose we wish to unify these two non-ground first-order terms: `[A|B]` and `[C |'d]` . i.e., to find replacements for logical variables `A`, `B` and `C` so that `[A|B] = [C |'d]`. It is clear that there are an infinite number of possible unifiers, each of the form `[A/D, B/'d, C/D]` for some term `D`. In fact, this parameterized substitution (parameterized by logical variable `C`) `[A/D, B/'d, C/D]` is the *most general unifier.* It is a *general* unifier in that replacing `C` by any term will result in a unifier, and *most* general in that any unifier is an instantiation of this form.

To calculate the set of unifiers, one might execute the Powerful program:

```
{ [A|[B|C]]: A,B,C ∈ terms, [A|B]=[C|'d] }
```

That is, each unifier is represented by a list, whose elements are the respective bindings for `A`, `B` and `C`. Removing some syntactic sugars, the denoted object is defined as

$\mathcal{E}[\![\{\mathtt{cons}(\mathtt{A},\mathtt{cons}(\mathtt{B},\mathtt{C})) : \mathtt{A},\mathtt{B},\mathtt{C} \in \mathtt{terms}, \mathtt{cons}(\mathtt{A},\mathtt{B}) = \mathtt{cons}(\mathtt{C},'\,\mathtt{d})\}]\!]\ \Lambda.$

Without the optimizations, this program would produce a set represented by an infinite union tree with a leaf for each possible assignment of terms to `A`, `B` and `C`. Where the assignment is a unifier, the leaf would be a singleton set containing these bindings in a list. Where the assignment is not a unifier, the leaf would be the empty set. The following derivation, however, uses the optimizations discussed in this chapter, and succeeds in avoiding much redundant work. As we shall see, it will construct a finite union tree, with a few leaves representing the empty set, and a single leaf containing a parameterized singleton set. This parameterized singleton set will represent the most general unifier `[A/C, 'd/B]`. Expanding the outermost call, the above expression becomes

$(\lambda X.\ \ \mathcal{E}[\![\{\mathtt{cons}(\mathtt{A},\mathtt{cons}(\mathtt{B},\mathtt{C})) : \mathtt{B},\mathtt{C} \in \mathtt{terms}, \mathtt{cons}(\mathtt{A},\mathtt{B}) = \mathtt{cons}(\mathtt{C},'\,\mathtt{d})\}]\!]\ [X/A]$
$)^{+}\mathbf{set!}(\mathcal{E}[\![\mathtt{terms}]\!]\ \Lambda).$

Since '+' is strict in the second argument, we rewrite its outermost occurrence of $\mathcal{E}$, yielding

$(\lambda X.\ \ \mathcal{E}[\![\{\mathtt{cons}(\mathtt{A},\mathtt{cons}(\mathtt{B},\mathtt{C})) : \mathtt{B},\mathtt{C} \in \mathtt{terms}, \mathtt{cons}(\mathtt{A},\mathtt{B}) = \mathtt{cons}(\mathtt{C},'\,\mathtt{d})\}]\!]\ \Lambda$
$)^{+}\mathbf{set!}(\mathcal{F}[\![\mathtt{terms}]\!])$.

Putting the whole expression into the new notations yields

$term(X).\mathcal{E}[\![\{\mathtt{cons}(\mathtt{A},\mathtt{cons}(\mathtt{B},\mathtt{C})) : \mathtt{B},\mathtt{C} \in \mathtt{terms}, \mathtt{cons}(\mathtt{A},\mathtt{B}) = \mathtt{cons}(\mathtt{C},'\,\mathtt{d})\}]\!]\ [X/A]$.

Evaluation of the body proceeds as before, eventually producing

$term(X).term(Y).term(Z).$ **set!**(
  if(**equal?**($\mathcal{E}[\![\mathtt{cons}(\mathtt{A},\mathtt{B})]\!]\ \rho_1, \mathcal{E}[\![\mathtt{cons}(\mathtt{C},'\,\mathtt{d})]\!]\ \rho_1$)
  $\mathcal{E}[\![\{\mathtt{cons}(\mathtt{A},\mathtt{cons}(\mathtt{B},\mathtt{C})) :\}]\!]\ \rho_1$,
  ϕ)),

where ρ_1 is $[X/\mathtt{A}, Y/\mathtt{B}, Z/\mathtt{C}]$. The computation rule allows us to expand either the first or second occurrence of $\mathcal{E}$. We choose the first and simplify. Continuing in this manner eventually yields

$term(X).term(Y).term(Z).$ **set!**(
  if(**if**(**not**(**pair?**($\mathcal{E}[\![\mathtt{cons}(\mathtt{C},'\,\mathtt{d})]\!]\ \rho_1$)),
    FALSE,
    if(**not**(**equal?**(
      $\mathcal{E}[\![\mathtt{A}]\!]\ \rho_1$,
      left(**pair!**($\mathcal{E}[\![\mathtt{cons}(\mathtt{C},'\,\mathtt{d})]\!]\ \rho_1$))))
      FALSE,
      equal?($\mathcal{E}[\![\mathtt{B}]\!]\ \rho_1$, **right**(**pair!**($\mathcal{E}[\![\mathtt{cons}(\mathtt{C},'\,\mathtt{d})]\!]\ \rho_1$))))),
  $\mathcal{E}[\![\{\mathtt{cons}(\mathtt{A},\mathtt{cons}(\mathtt{B},\mathtt{C})) :\}]\!]\ \rho_1$,
  ϕ)).

The leftmost function call expands to an ordered pair. Then, the primitive **pair?** simplifies to **TRUE**, and the primitive **not** simplifies to **FALSE**. The inner **if** simplifies, producing

$term(X).term(Y).term(Z).$ **set!**(
  if(**if**(**not**(**equal?**(
      $\mathcal{E}[\![\mathtt{A}]\!]\ \rho_1$,
      left(**pair!**($\mathcal{E}[\![\mathtt{cons}(\mathtt{C},'\,\mathtt{d})]\!]\ \rho_1$)))),
    FALSE,
    equal?($\mathcal{E}[\![\mathtt{B}]\!]\ \rho_1$, **right**(**pair!**($\mathcal{E}[\![\mathtt{cons}(\mathtt{C},'\,\mathtt{d})]\!]\ \rho_1$)))),
  $\mathcal{E}[\![\{\mathtt{cons}(\mathtt{A},\mathtt{cons}(\mathtt{B},\mathtt{C})) :\}]\!]\ \rho_1$,
  ϕ)).

Expanding the leftmost function call yields

$$
\begin{array}{l}
term(X).term(Y).term(Z).\ \mathbf{set!}(\\
\quad \mathbf{if}(\quad \mathbf{if}(\quad \mathbf{not}(\mathbf{equal?}(X, \mathbf{left}(\mathbf{pair!}(\mathcal{E}[\![\mathtt{cons(C,'\,d)}]\!]\ \rho_1)))), \\
\qquad\qquad\qquad \mathbf{FALSE}, \\
\qquad\qquad\qquad \mathbf{equal?}(\mathcal{E}[\![\mathtt{B}]\!]\ \rho_1, \mathbf{right}(\mathbf{pair!}(\mathcal{E}[\![\mathtt{cons(C,'\,d)}]\!]\ \rho_1)))), \\
\qquad\qquad \mathcal{E}[\![\{\mathtt{cons(A, cons(B, C))}\ \mathtt{:}\}]\!]\ \rho_1, \\
\qquad\qquad \phi)).
\end{array}
$$

The body is strict in the outermost **if**, which is strict in its leftmost argument. The leftmost argument is itself an **if**, strict in its leftmost argument. That argument is a **not** which is strict in an expression headed by **equal?**. This first occurrence of **equal?** is strict in both arguments, one of which is a parameter enumerated by terms, the other a function call. No matter how the equality would simplify, were a term provided in place of the parameter, further simplification would be require evaluation of the other argument. Therefore, we delay simplification of the equality to compute the other argument, yielding

$$
\begin{array}{l}
term(X).term(Y).term(Z).\ \mathbf{set!}(\\
\quad \mathbf{if}(\quad \mathbf{if}(\quad \mathbf{not}(\mathbf{equal?}(X, \mathbf{left}(\mathbf{pair!}(< \mathcal{E}[\![\mathtt{C}]\!]\ \rho_1, \mathcal{E}[\![\mathtt{'d}]\!]\ \rho_1 >)))), \\
\qquad\qquad\qquad \mathbf{FALSE}, \\
\qquad\qquad\qquad \mathbf{equal?}(\mathcal{E}[\![\mathtt{B}]\!]\ \rho_1, \mathbf{right}(\mathbf{pair!}(\mathcal{E}[\![\mathtt{cons(C,'\,d)}]\!]\ \rho_1)))), \\
\qquad\qquad \mathcal{E}[\![\{\mathtt{cons(A, cons(B, C))}\ \mathtt{:}\}]\!]\ \rho_1, \\
\qquad\qquad \phi)).
\end{array}
$$

Now we can simplify to

$$
\begin{array}{l}
term(X).term(Y).term(Z).\ \mathbf{set!}(\\
\quad \mathbf{if}(\quad \mathbf{if}(\quad \mathbf{not}(\mathbf{equal?}(X, \mathcal{E}[\![\mathtt{C}]\!]\ \rho_1)), \\
\qquad\qquad\qquad \mathbf{FALSE}, \\
\qquad\qquad\qquad \mathbf{equal?}(\mathcal{E}[\![\mathtt{B}]\!]\ \rho_1, \mathbf{right}(\mathbf{pair!}(\mathcal{E}[\![\mathtt{cons(C,'\,d)}]\!]\ \rho_1)))), \\
\qquad\qquad \mathcal{E}[\![\{\mathtt{cons(A, cons(B, C))}\ \mathtt{:}\}]\!]\ \rho_1, \\
\qquad\qquad \phi)).
\end{array}
$$

The equality still requires evaluation of the second argument, which finally evaluates to Z, yielding

$$
\begin{array}{l}
term(X).term(Y).term(Z).\ \mathbf{set!}(\\
\quad \mathbf{if}(\quad \mathbf{if}(\quad \mathbf{not}(\mathbf{equal?}(X, Z)), \\
\qquad\qquad\qquad \mathbf{FALSE}, \\
\qquad\qquad\qquad \mathbf{equal?}(\mathcal{E}[\![\mathtt{B}]\!]\ \rho_1, \mathbf{right}(\mathbf{pair!}(\mathcal{E}[\![\mathtt{cons(C,'\,d)}]\!]\ \rho_1)))), \\
\qquad\qquad \mathcal{E}[\![\{\mathtt{cons(A, cons(B, C))}\ \mathtt{:}\}]\!]\ \rho_1, \\
\qquad\qquad \phi)).
\end{array}
$$

This simplifies to either **TRUE** or **FALSE**, depending upon whether or not X equals Z. The first subset is

$term(X).term(Y).term(Z).X \neq Z.$ **set!**(
  if(**if**(**not**(**FALSE**),
  FALSE,
  equal?($\mathcal{E}[\![$B$]\!]\ \rho_1$, **right**(**pair!**($\mathcal{E}[\![$**cons**(C,'d)$]\!]\ \rho_1$)))),
  $\mathcal{E}[\![\{$**cons**(A, **cons**(B, C)) :$\}]\!]\ \rho_1$,
  ϕ)).

which simplifies to $(term(X).term(Y).term(Z).X \neq Y.\phi)$, or simply ϕ. The second subset in the union is

$term(X).term(Y).$ **set!**(
  if(**if**(**not**(**equal?**(X, X)),
  FALSE,
  equal?($\mathcal{E}[\![$B$]\!]\ \rho_2$, **right**(**pair!**($\mathcal{E}[\![$**cons**(C,'d)$]\!]\ \rho_2$)))),
  $\mathcal{E}[\![\{$**cons**(A, **cons**(B, C)) :$\}]\!]\ \rho_2$,
  ϕ)),

where ρ_2 is $[X/$A$, Y/$B$, X/$C$]$. This simplifies to

$term(X).term(Y).$ **set!**(
  if(**equal?**($\mathcal{E}[\![$B$]\!]\ \rho_2$, **right**(**pair!**($\mathcal{E}[\![$**cons**(C,'d)$]\!]\ \rho_2$)))),
  $\mathcal{E}[\![\{$**cons**(A, **cons**(B, C)) :$\}]\!]\ \rho_2$,
  ϕ)),

The body is headed by an **if**, strict in its first argument, which is headed by an **equal?**. Computing as before rewrites the first argument of **equal?** to the parameter Y, and the second argument to **'d**, yielding

$term(X).term(Y).$ **set!**(
  if(**equal?**(Y,'d),
  $\mathcal{E}[\![\{$**cons**(A, **cons**(B, C)) :$\}]\!]\ \rho_2$,
  ϕ)),

Simplifying the equality yields

$term(X).term(Y).$ **set!**(
  if(**if**(**atom?**(Y),
  atomeq?(Y,'d),
  FALSE)
  $\mathcal{E}[\![\{$**cons**(A, **cons**(B, C)) :$\}]\!]\ \rho_2$,
  ϕ)),

Simplifying the **atom?** primitive requires splitting into the union of three cases:

$term(X).bool(Y)$. **set!**(
  if(**if**(**FALSE**,
  atomeq?$(Y,$'**d**$)$,
  FALSE)
  $\mathcal{E}[\![\{\texttt{cons(A,cons(B,C))}:\}]\!]\ \rho_2$,
  ϕ)),

$term(X).atom(Y)$. **set!**(
  if(**if**(**TRUE**,
  atomeq?$(Y,$'**d**$)$,
  FALSE)
  $\mathcal{E}[\![\{\texttt{cons(A,cons(B,C))}:\}]\!]\ \rho_2$,
  ϕ)),

and

$term(X).term(Y_1).term(Y_2)$ **set!**(
  if(**if**(**atom?**$(< Y_1, Y_2 >)$,
  atomeq?$(< Y_1, Y_2 >,$'**d**$)$,
  FALSE)
  $\mathcal{E}[\![\{\texttt{cons(A,cons(B,C))}:\}]\!]\ \rho_3$,
  ϕ)),

where ρ_3 is $[X/\texttt{A}, < Y_1, Y_2 > /\texttt{B}, X/\texttt{C}]$. The first and third subsets simplify to ϕ, but the second simplifies to

$term(X).atom(Y)$. **set!**(
  if(**is'd**(Y),
  $\mathcal{E}[\![\{\texttt{cons(A,cons(B,C))}:\}]\!]\ \rho_2$,
  ϕ)).

Simplifying **is'd?** requires splitting Y into two cases:

$term(X).atom(Y).Y \neq$ '**d**. **set!**(ϕ)).

which simplifies to ϕ, and

$term(X)$. **set!**$(\mathcal{E}[\![\{\texttt{cons(A,cons(B,C))}:\}]\!]\ [X/A,$'**d**$/B, X/C])$.

This latter clearly evaluates to

$term(X)$. $(\{< X, <$ '**d**$, X >\})$.

Theoretically, we should still compute this expression as the union of all possible instantiations, each instantiation replacing the logical variable X by a first-order term (or approximate the result by doing some of these instantiations). For practical

purposes, such instantiation is trivial, so computation would terminate here. The final result of this calculation is therefore a finite union tree, with ϕ for the first few leaves (for the branches which split off earlier) and the last leaf being the above.

6.5 Correctness Observations

Soundness: If t_i is a partially computed parameterized set expression, and $t_i[\Omega/\mathcal{E}]$ is an approximation produced by setting all unevaluated function calls ($\mathcal{D}$, $\mathcal{E}$ or $\mathcal{F}$) to $\perp$, then for every instantiation σ replacing logical variables with terms satisfying the constraints, $t_i[\Omega/\mathcal{E}]$ σ approximates a subset of $\bigsqcup\{t_i[\Omega/\mathcal{E}]\}$.

Proof: The theorem is true because of the meaning of a parameterized expression (in terms of '+'), and the fact that all steps in a parameterized derivation replace expressions by equals. **End of Proof**

Completeness: Any element of a set which can be computed by a non-optimized derivation can be computed as an instantiation of a parameterized derivation.

Proof: This theorem is true because when dividing a parameterized expression into cases (for the purpose of simplifying a primitive), every possible instantiation of logical variables which satisfies the constraints is a possible instantiation of one of the subcases. No possible instantiation is ever lost. Furthermore, any computation which can be performed after replacement of the enumeration variable by a term can be performed on the parameterized body. **End of Proof**

6.6 Summary

A type 1 simplification (simple reduction) does not require splitting; a type 2 simplification (splitting by type) requires a three-way split; type 3 simplifications (splitting on equality) requires a two-way split. When performing primitive simplifications, it is efficient to do first those simplifications which do not split the computation into subcases, then those which split into two subcases and save for last those requiring a three-way split. These optimizations avoid blind enumeration of the sets `terms`, `atoms` and `bools` when used as relative set abstraction generators. Instead, the enumeration parameter becomes a logical variable. An enumeration variable from the set `atoms` is treated as a logical variable carrying the constraint that it can be bound only to an atom. Enumeration variables from the set `bools` are handled analogously. Inequality constraints relating two logical variables are also

used. The logical variables are instantiated only to the extent needed to simplify the body. Wherever a logical variable is the argument of a primitive function, and the primitive function needs more information about its argument to execute, the generating set is divided into a few subsets, thereby dividing the whole expression into subsets. In each subset, the range of the logical variable is narrowed enough that the primitive has enough information to execute. Computing with logical variables and constraints gives the set abstraction facility "resolution-like" efficiency. Treating an enumeration parameter as a logical variable is practical because the generating set `terms` is so simple in structure.

The optimizations described in this chapter modify the simple (though inefficient) operational semantics derived from the denotational equations in Chapter 4. Even with these optimizations, the result is far from a production-level implementation! The purpose was rather to explain the role of the logical variable in functional programming with set abstraction. The syntax and denotational semantics of PowerFuL made no reference to logical variables. The logical variable is merely an *operational* concept to improve the execution efficiency when `terms` is used as a generator. More complicated sets are also permitted as generators (e.g. such as sets of functions, sets of sets, etc.), and in these cases, the default mechanism (generate, instantiate, and continue) is used.

7 CONCLUSIONS

During the past decade, proponents of declarative programming languages have discussed the possibility of combining of functional and logic programming styles within one declarative language. Many proposals have been based on equational logic, reasoning that equations could be used to define both functions and relations. A weakness of this approach is the difficulty of maintaining functions (and other higher-order constructions) as first-class objects. Testing higher-order objects for equality is very difficult, and is not always computable, so a higher-order extension will either relinquish referential transparency (through the use of an efficient but unreliable higher-order equality test), or will require a very difficult and perhaps impractical, operational primitives such as higher-order unification, general theorem-proving and unrestricted narrowing. Traditional functional programming, in contrast, does not require tests of equality when passing arguments to procedures, and so avoids this problem. Since our approach is based on ordinary functional programming, incorporation of higher-order programming proved to be no problem. Our approach supports both functions and sets as first-class higher-order constructs. The underlying theses of our approach are:

1) Functional programming with relative set abstraction subsumes the expressiveness of logic programming.

2) The resulting language does *not* require higher-order unification to maintain first-class higher-order objects.

3) The use of logical variables is most properly viewed as an implementation tool rather than as part of the language definition.

7.1 Results and Contributions

The principal results of this dissertation are:

(i) *Relative set abstraction* can combine lazy higher-order functional programming not only with first-order Horn logic, but also with a useful subset of higher-order Horn logic. Sets, as well as functions, can be treated as first-class objects.

(ii) *Angelic powerdomains* provide the semantic foundation for relative set abstraction.

(iii) The computation rule appropriate for this language is a modified parallel-outermost, rather than the more familiar left-most rule.

(iv) Optimizations incorporating ideas from narrowing and resolution greatly improve the efficiency of the interpreter, while maintaining correctness.

We are not the first to advocate set abstraction as a means of incorporating Horn logic capability into higher-order functional programming. However, we do believe this design is the first to be rigorously described via denotational semantics, mapping the syntax onto computable semantic primitives. The brevity and simplicity of the denotational description attests to the elegance and integrity of the design. To consider set-valued functions and sets as objects, we had to incorporate angelic powerdomains into the complete semantic domain (each object in a powerdomain represents a set of elements from some simpler domain). Our use of powerdomains is novel in that we make the set an explicit data type, rather than the implicit result of a non-deterministic control structure.

Horn logic programming can be described by either the fixed-point semantics (closely related to Horn logic's model-theoretic semantics) or by its operational semantics (SLD resolution). Soundness and completeness proofs attest the equivalence of these two descriptions. We feel that this result is an important feature of Horn logic programming, and that our language should have a similar property. Extending Vuillemin's theory of correct implementation of recursion, and applying the resulting technique to the denotational equations themselves, we derived an operational semantics equivalent to the denotational description. We think this novel technique may be an important addition to the methodology of functional programming, independent of the set abstraction problem.

To derive equivalent operational semantics by this technique, the denotational semantics must handle most recursion explicitly, ensuring that primitives will always terminate. Primitives must be rigorously defined via rewrite rules, rewriting

whenever the outermost constructor of any strict argument is available. Though these restrictions make writing the denotational semantics more difficult, they ensure the rigor of the definition. In extending Vuillemin's theory, we propose a new computation rule which is a compromise between the parallel-outermost and the leftmost computation rules. Like the leftmost rule, it permits evaluation to concentrate in a primitive's strict arguments, but provides for parallel evaluation where necessary due to the presence of non-flat domains.

Of special interest for logic programming is the set of terms, objects for which identity is synonymous with equality. We showed that, when the set of terms is used as a generating set in a relative set abstraction, the enumeration parameter can be computed as a logical variable, instead of using the default blind generate-and-test procedure which would otherwise result. In the general case, however, the enumeration parameters *are* instantiated by the various generator set elements as these elements are computed. Thus, generators need *not* be arbitrarily restricted to contain only first-order types.

Incorporating logical variables into the operational procedure complicates the simplification of semantic primitives. That is, simplification sometimes requires knowing more about a primitive's argument than that it is a term. It may depend upon whether the logical variable represents an atom, a boolean or an ordered pair. In such a case, we split the set into three subsets, one for each assumption. In each subset, the primitive can simplify. A similar splitting procedure is used for handling equality/inequality primitives.

7.2 Further Work

Several areas of additional research seem apparent:

1) Our decision to derive the operational semantics from the denotational equations was motivated by a desire to ensure that the operational semantics remained true to the denotational definition, and not by a conviction that this kind of interpretation would be most efficient. Practical software to implement this language should make use of a more efficient strategy. The optimizations described in chapter 6 do not address the inefficiencies of interpreting from semantic equations in general; rather, they solve a separate problem specific to set abstraction, i.e. the desire to use logical variables in computing set abstractions generated by the set of

first-order terms. A challenging problem would be to describe this optimization in terms of a more conventional operational semantics. In this dissertation we strove to unify the declarative aspects of functional and logic programming; combining the efficient implementation strategies developed for these kinds of languages [W83b] [P87] is yet another topic.

2) Possible inefficiency aside, we feel that generating operational semantics from denotational equations is an interesting idea, giving much insight as to the relation between denotation and computation. We would like to study further the semantics of *denotational semantics as a programming language.* From our experience in this research, we feel that next time it might be better to define the semantics of the language in terms of *typed* lambda calculus. This should permit greater rigor in proving the relationship between operational and denotational semantics.

3) We would like to supplement PowerFuL with a polymorphic type system like those provided for many other modern functional languages. This would lead to many *types* of constructors. The most interesting aspect would be its effect on the optimizations providing for logical variables. Would it increase the complexity of computing with logical variables? Our feeling is that it might make the interpreter more complex, but ought not to hurt efficiency.

4) Computation of infinite sets requires closer interaction between user and interpreter. This places additional demands on the language environment, so new programming environments may also need to be developed.

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