Determination of the Quadratic Coefficients of a Projected Conic Section[1]

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Abstract

This report describes two methods used to derive the quadratic coefficients of the perspective projection of a conic section lying in an arbitrary plane in the three space onto the image plane. The content of that report should be especially useful for the scan conversion phase of a new graphic primitive on the *Pxpl5* graphics system [Fuchs 89], namely the conic spline defined fonts embedded in three space . I will first state the problem, describe how this work relates to the scan conversion of conic fonts and define some notations. I will then determine the equation of the conic section that is obtained by perspective projection using the two methods. The second method, although slower at first sight, provides much more intuition as to how the coefficients of the initial conic section get transformed. It also gives a clear insight on how the different parameters affect that transformation. Finally, it is much clearer to program since it uses only matrix multiplications.

Introduction

In a recent paper, we presented an algorithm used for the fast rendering of fonts on the Pxpl5 graphics system [Fuchs 89, Tardif 89]. This technique provides a decomposition of any character as a difference of convex regions whose boundaries are composed of straight line segments and/or arcs of conic. Each arc of conic is in turn represented by a set of quadratic coefficients. Yet, this set of coefficients determines the equation of the arc of conic with respect to a coordinate system attached to the plane where the character lies. Although the theory tells us that conic sections are invariant under projective maps, it would be interesting to have an efficient method to compute the quadratic coefficients corresponding to the conic section obtained by perspective projection onto the image plane. This is precisely the purpose of this report to provide efficient methods for the determination of the quadratic coefficients of a projected conic section. Using the Quadratic Expression Evaluator in the processor enhanced memories of *Pxpl5*, the area bounded by the projected conic section is then scan-converted by simply broadcasting its corresponding set of quadratic coefficients obtained by the methods described in this report.

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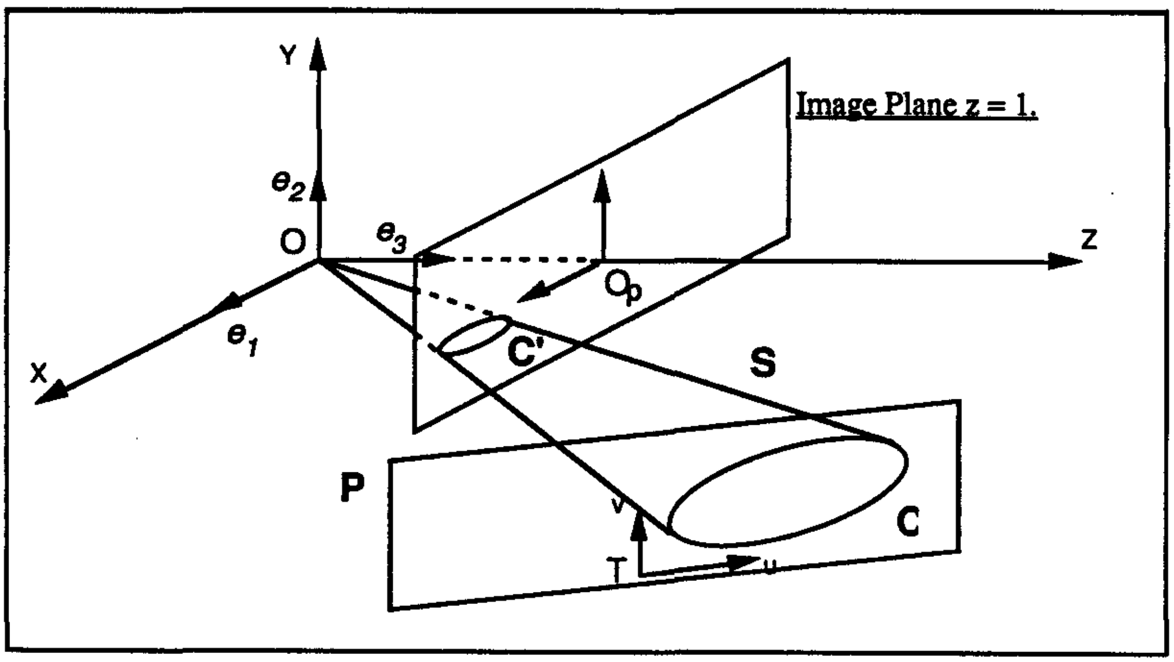
Notation

We consider the Euclidean space E^3 and the eye space coordinate system (O, e_1, e_2, e_3) where O is the camera origin, e_3 is the viewing direction. By convention, (x, y, z) will only be used to designate the coordinates of a point in the eye space coordinate system. We assume that the equation of the image plane is $z = 1$.

Consider an arbitrary plane **P** in the Euclidean space E^3 and a coordinate system (T, u, v) defined on that plane. Now, consider a conic section **C** lying on **P**. We assume that the equation of **C** in the "local" coordinate system (T, u, v) is determined by the quadratic coefficients (A, B, C, D, E, F).

We want to obtain the quadratic coefficients (A', B', C', D', E', F') of the projection of that conic on the image plane with respect to the coordinate system (O, e_1, e_2). These coefficients could then be sent to the frame buffer for parallel evaluation at all pixels. The situation is depicted in the following figure.

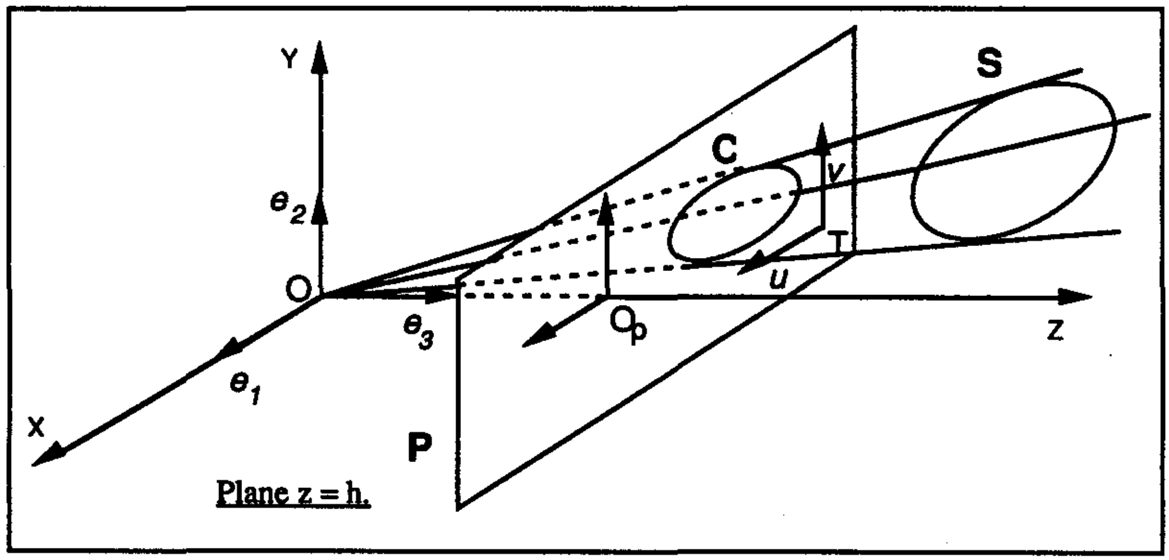


Geometrically, the solution to our problem consists in finding the equation of the curve **C'** that is the intersection of the plane z = 1 and the surface **S** obtained by casting rays from the camera origin O to all points of the conic section **C**. Hence the problem amounts to finding the equation of the surface **S**.

Determination of the equation of the surface S:

<u>Method 1:</u>
Before we treat the general case, we solve the problem in the case where **P** is perpendicular to the Z axis. The equation of the surface **S** which is easily derived in that case will be used in the more general case.

<u>case 1:</u> When **P** is perpendicular to the Z axis:
The equation of the surface S is very easy to determine when the plane P that supports the conic is perpendicular to the z axis (equation z = h). To see this, let's examine the following figure:



Let (ξ, η) be the coordinates of a point expressed in the coordinate system (T, u, v). The equation of the conic C in (T, u, v) is:

$$A\xi + B\eta + C + D\xi^2 + E\xi\eta + F\eta^2 = 0;$$

If (x_T, y_T) denotes the coordinates of T in (O_p, e_1, e_2), the equation of C is then:

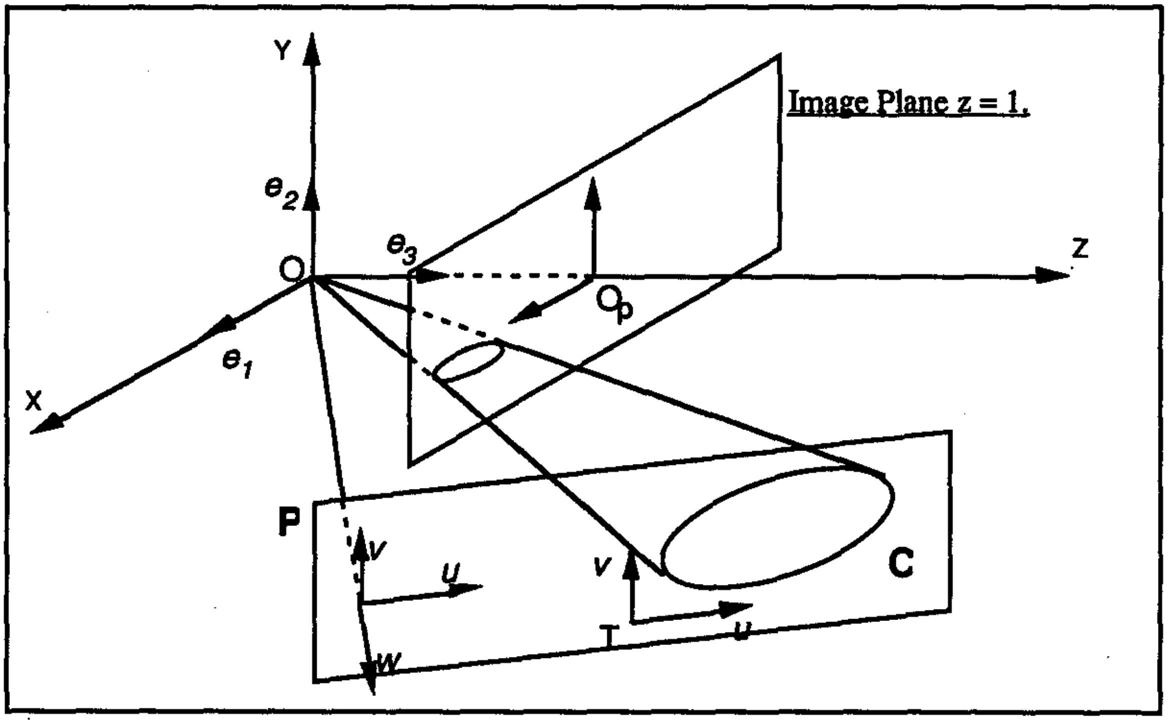
$$A(x-x_T) + B(y-y_T) + C + D(x-x_T)^2 + E(x-x_T)(y-y_T) + F(y-y_T)^2 = 0;$$

We immediately deduce the equation of S in the eye space coordinate system:

$$A(hx/z - x_T) + B(hy/z - y_T) + C + D(hx/z - x_T)^2 + E(hx/z - x_T)(hy/z - y_T) + F(hy/z - y_T)^2 = 0;$$

Setting z = 1 in that equation would give us the equation of the conic section obtained by projection of C onto the image plane z = 1.

General case: **P** is an arbitrary plane in E^3:
The situation is illustrated by the following figure:



Let w be a unit vector such that (u, v, w) forms an orthonormal basis. w is just the normal vector of plane **P**. Let (A, B, C, D, E, F) be the quadratic coefficients that determine the equation of the conic section **C** in the coordinate system (T, u, v).

Here is how we proceed in order to determine the equation of S in the eye space coordinate system:
- We first determine the equation of the surface S in the coordinate system (O, u, v, w), which is exactly what we did previously in case 1.
- We then perform a change of variables to obtain the equation of S in the eye space coordinate system (O, e_1, e_2, e_3).

Let (ξ, η, τ) be the coordinates of a point in the coordinate system (O, u, v, w). Assuming (ξ_T, η_T, τ_T) are the coordinates of T in (O, u, v, w), the equation of S in (O, u, v, w) is given by:

$$(1)\quad A(\tau_T\xi/\tau - \xi_T) + B(\tau_T\eta/\tau - \eta_T) + C +$$
$$D(\tau_T\xi/\tau - \xi_T)^2 + E(\tau_T\xi/\tau - \xi_T)(\tau_T\eta/\tau - \eta_T) + F(\tau_T\eta/\tau - \eta_T)^2 = 0;$$

Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$ be the coordinates of (u, v, w) in (O, e_1, e_2, e_3).

We can perform the following change of variables:

$$\xi = u_1x + u_2y + u_3z;$$
$$\eta = v_1x + v_2y + v_3z;$$
$$\tau = w_1x + w_2y + w_3z;$$

and plug these in equation (1) in order to obtain the equation of the surface S in the eye space coordinate system.

Setting $z = 1$ in this equation, gives us the equation of the conic obtained by projection of the conic section C onto the image plane.

The equation of the projected conic is thus:

$$A'x + B'y + C' + D'x^2 + E'xy + F'y^2 = 0;$$

where:

$$A' = 2w_1w_3c(T) + \tau_T w_1(\alpha u_3 + \beta v_3) + \tau_T w_3(\alpha u_1 + \beta v_1) + 2\tau_T^2(Du_1u_3 + Eu_1v_3 + Eu_3v_1 + Fv_1v_3)$$
$$B' = 2w_2w_3c(T) + \tau_T w_2(\alpha u_3 + \beta v_3) + \tau_T w_3(\alpha u_2 + \beta v_2) + 2\tau_T^2(Du_2u_3 + Eu_2v_3 + Eu_3v_2 + Fv_2v_3)$$
$$E' = 2w_1w_2c(T) + \tau_T w_1(\alpha u_2 + \beta v_2) + \tau_T w_2(\alpha u_1 + \beta v_1) + 2\tau_T^2(Du_1u_2 + Eu_1v_2 + Eu_2v_1 + Fv_1v_2)$$
$$C' = w_3^2c(T) + \tau_T w_3(\alpha u_3 + \beta v_3) + \tau_T^2(Du_3^2 + Eu_3v_3 + Fv_3^2)$$
$$D' = w_1^2c(T) + \tau_T w_1(\alpha u_1 + \beta v_1) + \tau_T^2(Du_1^2 + Eu_1v_1 + Fv_1^2)$$
$$F' = w_2^2c(T) + \tau_T w_2(\alpha u_2 + \beta v_2) + \tau_T^2(Du_2^2 + Eu_2v_2 + Fv_2^2)$$

with:

$$c(T) = -A\xi_T - B\eta_T + C + D\xi_T^2 + E\xi_T\eta_T + F\eta_T^2$$
$$\alpha = A - 2D\xi_T - E\eta_T,$$

and

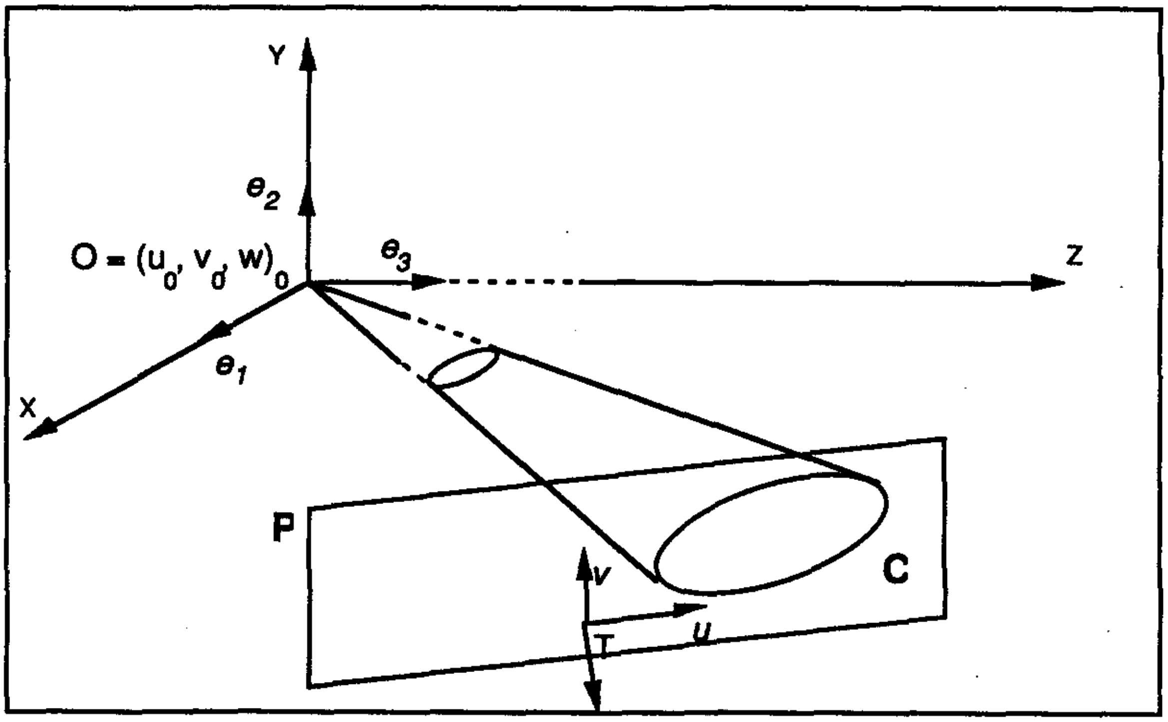
$$\beta = B - 2F\eta_T - E\xi_T.$$

The quadratic coefficients may then be sent to the Quadratic Expression Evaluator for evaluation at all pixels.As it is described in this section, it takes roughly 70 multiplications and 40 additions to obtain the coefficients that we would then send to the frame buffer.

Note: For comparison, it takes 30 multiplications, 21 additions and two divisions to obtain the linear coefficients (A, B, C) in screen space from a segment of points given in world space.

Method 2:

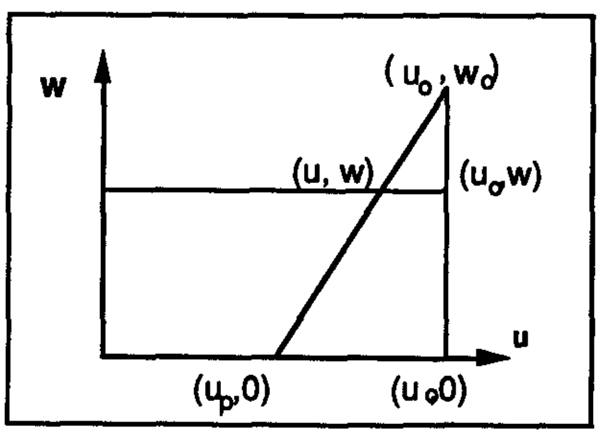
Yet another method is inspired by using similar triangles. Consider the following figure where the camera origin is expressed in the coordinate system (u, v, w) as O = (u_0, v_0, w_0)



Consider the (u, w) plane and let (u_p, v_p) be a point on the conic C. We deduce the following relation illustrated by the figure below:

$$(w_0 - w) / (u_o - u) = w_0 / (u_0 - u_p) \text{ that we rewrite:}$$

(2) $$u_p = (u\, w_0 - u_0\, w) / (w_0 - w) = u' / w'.$$



Now applying this again to the (v,w) plane, we get the relation:

(3) $$v_p = (v\, w_0 - v_0\, w) / (w_0 - w) = v' / w'.$$

Since (u_p, v_p) belongs to the conic **C**, (u_p, v_p) verify the equation of the conic **C** in the coordinate system (T, u, v, w), that is:

(4) $$Au_p + Bv_p + C + Du_p^2 + Eu_pv_p + Fv_p^2 = 0;$$

Assuming that the camera does not lie in the plane **P**, we obtain:

$$Au'w' + Bv'w' + Cw'^2 + Du'^2 + Eu'v' + Fv'^2 = 0;$$

which we rewrite in matrix notation as:

$$U' \, Q \, U'^t = 0$$

where Q is the following symmetric matrix:

$$Q = \begin{vmatrix} D & E/2 & A/2 & 0 \\ E/2 & F & B/2 & 0 \\ A/2 & B/2 & C & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Now (2) and (3) can also be expressed in matrix notation:

$$(u', v', w', 1) = (u, v, w, 1) \begin{vmatrix} w_0 & 0 & 0 & 0 \\ 0 & w_0 & 0 & 0 \\ -u_0 & -v_0 & -1 & 0 \\ 0 & 0 & w_0 & 1 \end{vmatrix} \quad \text{i.e. } U' = U\,N$$

If N denotes the matrix on the right hand side of the equation above, equation (4) leads to:

$$U \;\; (N.Q.N^t) \, U'^t = 0$$

This is the equation of the surface **S** obtained by casting rays from the camera origin to all the points that belong to the conic section lying in **P**. This equation is given in the coordinate system (T, u, v, w). In order to obtain the equation of **S** in (O, e_1, e_2, e_3) we simply perform the following change of variables:

$$(u, v, w, 1) = (x, y, z, 1) \;\; M;$$

If $T = (T_x, T_y, T_z)$, $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$ are the coordinates of (T, u, v, w) in (O, e_1, e_2, e_3), we then have M = T R where

$$T = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -T_x & -T_y & -T_z & 1 \end{vmatrix} \qquad R = \begin{vmatrix} u_1 & v_1 & w_1 & 0 \\ u_2 & v_2 & w_2 & 0 \\ u_3 & v_3 & w_3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Hence the equation of the surface S is given by the following equation:

$$X \quad (T.R.N.Q.N^t.R^t.T^t) \quad X^t = 0$$

Since Q is symmetric, the product $H = T.R.N.Q.N^t.R^t.T^t$ also forms a symmetric matrix. Assuming that the image plane is the plane $= 1$, the equation of the projected conic on the image plane is thus given by:

$$A'x + B'y + C' + D'x^2 + E'xy + F'y^2 = 0;$$

where:

$$A' = 2\,a_{13} + 2\,a_{14},$$
$$B' = 2\,a_{23} + 2\,a_{24},$$
$$C' = a_{33} + 2\,a_{34} + a_{44},$$
$$D' = a_{11},$$
$$E' = 2\,a_{12},$$
$$F' = a_{22},$$

given that $H = (a_{ij}, 0 < i, j < 5)$ and $a_{ij} = a_{ji}$.
In this method, the determination of the quadratic coefficients of the projection of the initial conic section onto the image plane amounts to compute the coefficients of the symmetric matrix H. However, because of the way H is derived, it is much easier to see the effects of each of the parameters on the transformation. Such partitioning of the matrix H also provides better clarity for later programming.

Conclusion

Given the following

1. A point T in E^3, and two vectors (u, v) in the eye space coordinate system,
2. The quadratic coefficients (A, B, C, D, E, F) of a conic section C in the coordinate system (T, u, v),
3. The equation of the image plane in eye space (usually $z = 1$),

We described two methods to determine the quadratic coefficients (A', B', C', D', E', F') of the conic section obtained by the perspective projection of C onto the image plane with respect to the center of projection (or camera position) O. Although these methods of computing the quadratic coefficients of a projected conic can certainly be optimized, they give an idea of the speed at which we could scan convert conic sections on Pxpl5.

Acknowledgments
I wish to thank Jack Goldfeather for suggesting the second method that is presented in that report.

References:

[Fuchs 89] Pixel-Planes 5: A Heterogeneous Multiprocessor Graphics System Using Processor-Enhanced Memories. *Computer Graphics, Volume 23, Number 3, July 1989.*

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