On the Random Covering Problem

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John H. Halton

The University of North Carolina at Chapel Hill
Department of Computer Science
CB#3175, Sitterson Hall
Chapel Hill, NC 27599-3175

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by

John H. Halton

Computer Science Department
The University of North Carolina
Chapel Hill, NC 27599

ABSTRACT

N identical k-dimensional objects (in the original, two-dimensional problem, coins) are independently, uniformly, randomly distributed in a k-dimensional hyper-rectangle (originally, a table). The problem is to determine the statistics of the ratio of the k-dimensional volume covered by N such "coins" to that of the "table". In this paper, we obtain asymptotic results, for large N, and some exact results for $k = 1$.

1. INTRODUCTION

The original problem is to find the total area of a rectangular table that is *covered*, when N identical coins (i.e., circular discs) are randomly placed on it. To be more specific, let us say that the table-top is in a horizontal plane, and has length a and width b; and that the discs are of uniform thickness and density, and have radius r. Then the centres of the discs are independently and uniformly distributed over the table-top (if the centre of a coin falls outside the table-top, the coin falls off and is recycled).

Variations on this problem include the one-dimensional problem, in which segments of length $2r$ are randomly placed on an interval of length a; the two-dimensional problem, in which the discs are replaced by squares, rectangles, ellipses, or other shapes (irregular shapes all being identical and congruently oriented with respect to the rectangular table-top); the same, when, the identical shapes are randomly oriented; the extension to table-tops of arbitrary shape; and the corresponding k-dimensional problems.

It is also of interest to solve the problem *asymptotically*, when the identical objects are small and numerous, specifically, when $N \to \infty$, while the total k-dimensional volume of the N objects (i.e., N times the volume of one of these objects) remains constant. This is the situation of interest, for example, when calculating the effective *cross-section* of objects, scattered at random through a volume of material. (This is important in atomic and nuclear physics.)

The literature [see, e.g., (**3**)] mainly deals with optimally, rather than randomly, distributed objects; arranged, for example, in lattices.

2. THE ONE-DIMENSIONAL PROBLEM

We are given an interval of length a, upon which N segments of length $2r$ are placed, their mid-points distributed independently and uniformly at random over the interval. Let the interval be $[0, a]$ and let the center of the segment fall at c. We shall suppose that the interval is quite a bit greater than the segments; more specifically, we suppose that

$$a > 4r. \tag{1}$$

Let us define the *indicator function*, with variable x, of the *exterior* of a segment of length $2r$, with centre at c by

$$\chi_r(c;\, x) = \left\{ \begin{array}{ll} 0 & \text{if } \; |x-c| \le r \\ 1 & \text{if } \; |x-c| > r \end{array} \right\}. \tag{2}$$

Note that, by the symmetry of (2), this may also be viewed as the indicator function, with variable c, of the exterior of a segment of length $2r$, with centre at x. Also, for any $p > 0$,

$$\chi_{pr}(pc;\, px) = \chi_r(c;\, x). \tag{3}$$

In particular, if we scale our parameters to a, by writing

$$z = \frac{x}{a}, \quad \theta = \frac{c}{a}, \quad \lambda = \frac{2Nr}{a}, \quad \text{and} \quad \mu = \frac{r}{a} = \frac{\lambda}{2N} < \frac{1}{4}, \tag{4}$$

we conclude that

$$\chi_r(c;\, x) = \chi_\mu(\theta;\, z). \tag{5}$$

We observe that, if the N segments have their centres at $c_1, c_2, \ldots, c_N$, and $\mathbf{c}$ is the vector $[c_1, c_2, \ldots, c_N]$ of the N centres, then the function

$$X_r(\mathbf{c};\, x) = X_r(c_1, c_2, \ldots, c_N;\, x) = \prod_{i=1}^{N} \chi_r(c_i;\, x) \tag{6}$$

equals 0 if x lies in *any* of the N segments, and equals 1 if x lies outside *all* of the segments. Thus x is covered by at least one of the segments if and only if $X_r(\mathbf{c}; x) = 0$. Therefore, the ratio $C_\lambda(N, \Theta)$ of the measure ("length"), of the set of points of the interval $[0, a]$ covered by the N segments, to the length, a, of the interval (i.e., the *probability* of covering any given point) is, by (4), (5), and (6),

$$C_\lambda(N, \Theta) = \frac{1}{a}\int_0^a \mathrm{d}x\ [1 - X_r(\mathbf{c}; x)] = 1 - \frac{1}{a}\int_0^a \mathrm{d}x \prod_{i=1}^{N} \chi_r(c_i; x)$$

$$= 1 - \int_0^1 \mathrm{d}x \prod_{i=1}^{N} \chi_\mu(\theta_i; z), \tag{7}$$

where

$$\Theta = [\theta_1, \theta_2, \ldots, \theta_N] = \frac{1}{a}\mathbf{c} = \left[\frac{c_1}{a}, \frac{c_2}{a}, \ldots, \frac{c_N}{a}\right]. \tag{8}$$

Note that $C_\lambda(N, \Theta)$ remains unchanged, if we consider, instead, the *scaled* problem, of distributing N segments of length 2μ, with centres at θ_i $(i = 1, 2, \ldots, N)$, over the unit interval, $[0, 1]$.

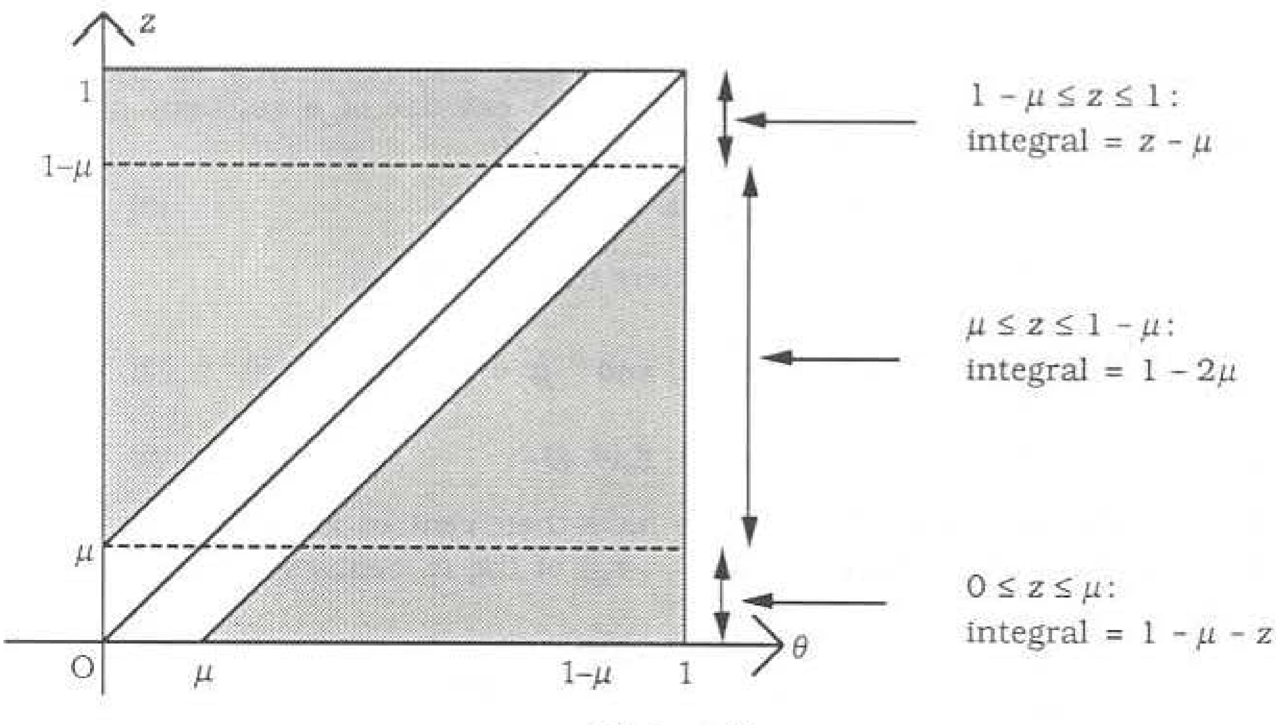

Figure 1

Values of integral $\int_0^1 \mathrm{d}\theta\, \chi_\mu(\theta; z)$ for different values of x.

In Figure 1, the shaded region indicates where $\chi_\mu(\theta;\, z) = 1$. The integral from (7), $J_\mu(z) = \int_0^1 d\theta\, \chi_\mu(\theta;\, z)$, is the width of the shaded region at the height z. When $0 \le z \le \mu$, this region is the interval $[\mu + z,\, 1]$, whose width is $1 - \mu - z$; when $\mu \le z \le 1 - \mu$, the region consists of the *two* intervals $[0,\, z - \mu]$ and $[z + \mu,\, 1]$, of widths $(z - \mu)$ and $(1 - z - \mu)$, for a total of $1 - 2\mu$; and when $1 - \mu \le z \le 1$, the region is the interval $[0,\, z - \mu]$, whose width is $z - \mu$. Thus,

$$J_\mu(z) \;=\; \int_0^1 d\theta\, \chi_\mu(\theta;\, z) \;=\; \left\{ \begin{array}{ll} 1 - \mu - z & \text{if } \; 0 \le z \le \mu \\ 1 - 2\mu & \text{if } \; \mu \le z \le 1 - \mu \\ z - \mu & \text{if } \; 1 - \mu \le z \le 1 \end{array} \right\}. \tag{9}$$

All the θ_i are uniformly, independently distributed in the interval $[0,\, 1]$. Therefore, by (9), the expected value of $C_\lambda(N,\, \Theta)$ is

$$\mathcal{E}\Big[C_\lambda(N,\, \Theta)\Big] \;=\; 1 - \int_0^1 dz \int_0^1 d\theta_1 \int_0^1 d\theta_2 \ldots \int_0^1 d\theta_N \prod_{i=1}^{N} \chi_\mu(\theta_i;\, z)$$

$$= 1 - \int_0^1 dz \left\{ \int_0^1 d\theta\, \chi_\mu(\theta;\, z) \right\}^N$$

$$= 1 - \int_0^1 dz\, (1 - 2\mu)^N - \int_0^\mu dz\, \left[(1 - \mu - z)^N - (1 - 2\mu)^N\right]$$

$$- \int_{1-\mu}^1 dz\, \left[(z - \mu)^N - (1 - 2\mu)^N\right]$$

$$= 1 - (1 - 2\mu)^N - 2\left\{ \frac{(1 - \mu)^{N+1}}{N + 1} - \frac{(1 - 2\mu)^{N+1}}{N + 1} - \mu\,(1 - 2\mu)^N \right\}$$

$$= 1 - \left(1 - \frac{\lambda}{N}\right)^N - \left\{ \frac{2}{N + 1}\left[\left(1 - \frac{\lambda}{2N}\right)^{N+1} - \left(1 - \frac{\lambda}{N}\right)^{N+1}\right] - \frac{\lambda}{N}\left(1 - \frac{\lambda}{N}\right)^N \right\}$$

$$= 1 - \frac{N - 1}{N + 1}\left(1 - \frac{\lambda}{N}\right)^{N+1} - \frac{2}{N + 1}\left(1 - \frac{\lambda}{2N}\right)^{N+1}. \tag{10}$$

As $N \to \infty$, with λ kept constant, by (A7) (the Corollary to the Lemma in Appendix A), we get the asymptotic behaviour

$$\begin{aligned}\mathcal{E}\Big[C_\lambda(N, \Theta)\Big] &= 1 - \left(1 - \frac{2}{N}\right) e^{-\lambda}\left[1 - \frac{1}{N}(\lambda + \tfrac{1}{2}\lambda^2)\right] - \frac{2}{N} e^{-\lambda/2} + O\!\left(\frac{1}{N^2}\right) \\ &= 1 - e^{-\lambda} + \frac{1}{2N}\left[(4 + 2\lambda + \lambda^2)\, e^{-\lambda} - 4\, e^{-\lambda/2}\right] + O\!\left(\frac{1}{N^2}\right) \\ &= 1 - e^{-\lambda} + O\!\left(\frac{1}{N}\right). \qquad (11)\end{aligned}$$

For small λ, as we might expect,

$$\mathcal{E}\Big[C_\lambda(N, \Theta)\Big] = \lambda + O(\lambda^2) + O(\tfrac{1}{N^2}); \qquad (12)$$

i.e., sparse segments (of total length only λ) are unlikely to overlap.

To obtain the variance of $C_\lambda(N, \Theta)$, we observe that

$$\begin{aligned}\mathcal{V}ar\Big[C_\lambda(N, \Theta)\Big] &= \mathcal{E}\Big[\Big\{C_\lambda(N, \Theta) - \mathcal{E}[C_\lambda(N, \Theta)]\Big\}^2\Big] \\ &= \mathcal{E}\Big[\Big\{1 - C_\lambda(N, \Theta)\Big\}^2\Big] - \Big\{1 - \mathcal{E}\Big[C_\lambda(N, \Theta)\Big]\Big\}^2 \\ &= \mathcal{E}\Big[\Big\{1 - C_\lambda(N, \Theta)\Big\}^2\Big] \\ &\qquad - \left\{\frac{N-1}{N+1}\left(1 - \frac{\lambda}{N}\right)^{N+1} + \frac{2}{N+1}\left(1 - \frac{\lambda}{2N}\right)^{N+1}\right\}^2. \qquad (13)\end{aligned}$$

Now, by (7), as in the derivation of (10),

$$\begin{aligned}\mathcal{E}\Big[\Big\{1 - C_\lambda(N, \Theta)\Big\}^2\Big] &= \int_0^1 dz \int_0^1 dz' \prod_{i=1}^{N} \left\{\int_0^1 d\theta_i\; \chi_\mu(\theta_i; z)\, \chi_\mu(\theta_i; z')\right\} \\ &= \int_0^1 dz \int_0^1 dz' \left\{\int_0^1 d\theta\, \chi_\mu(\theta; z)\, \chi_\mu(\theta; z')\right\}^N \\ &= \int_0^1 dz \int_0^1 dz'\, I_\mu(z, z')^N. \qquad (14)\end{aligned}$$

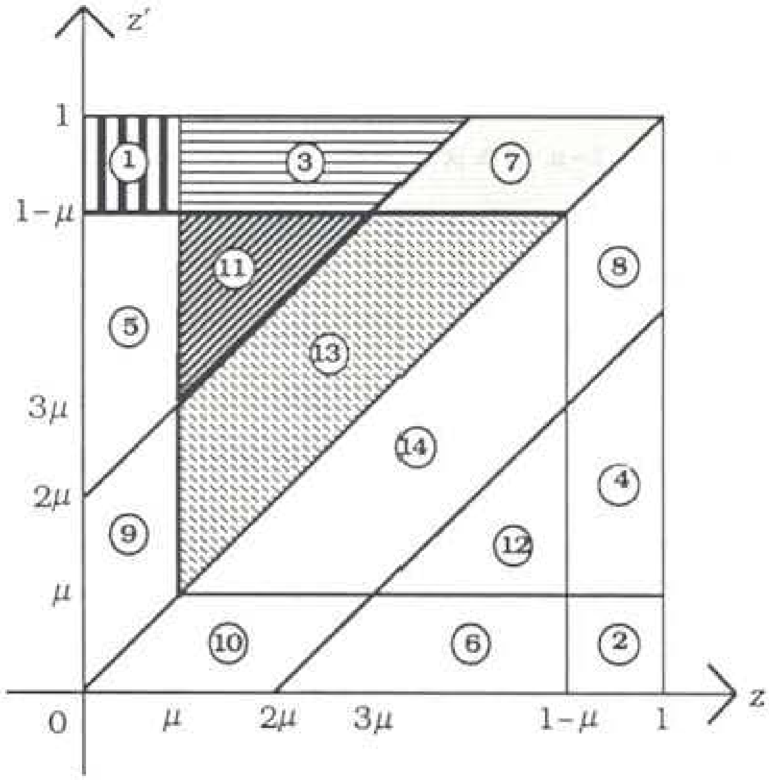

Figure 2

Analysis of the θ-integral, $I = I_\mu(z, z') = \int_0^1 d\theta\, \chi_\mu(\theta; z)\, \chi_\mu(\theta; z')$ in the fourteen critical regions of the square.

where, again, we define μ, z, and z′ as in (4) and write

$$I = I_\mu(z, z') = \int_0^1 d\theta\, \chi_\mu(\theta; z)\, \chi_\mu(\theta; z'). \tag{15}$$

We note, by (2), that the product $\chi_\mu(\theta; z)\, \chi_\mu(\theta; z')$ equals 0 or 1 everywhere in its domain, the (θ, z, z')-cube of side 1. Observe that there are fourteen distinct regions of the (z, z')-square, for which I takes different forms. The situation is described, and the regions are numbered, in Figure 2. It is clear that, throughout this square,

$$0 \leq I_\mu(z, z') \leq 1. \tag{16}$$

The fourteen regions are characterized as follows:

1:	$z \le \mu,\ z' \ge 1-\mu,$	**2:**	$z' \le \mu,\ z \ge 1-\mu,$
3:	$z'-z \ge 2\mu,\ z' \ge 1-\mu,\ z \ge \mu,$	**4:**	$z-z' \ge 2\mu,\ z \ge 1-\mu,\ z' \ge \mu,$
5:	$z'-z \ge 2\mu,\ z \le \mu,\ z' \le 1-\mu,$	**6:**	$z-z' \ge 2\mu,\ z' \le \mu,\ z \le 1-\mu,$
7:	$0 \le z'-z \le 2\mu,\ z' \ge 1-\mu,$	**8:**	$0 \le z-z' \le 2\mu,\ z \ge 1-\mu,$
9:	$0 \le z'-z \le 2\mu,\ z \le \mu,$	**10:**	$0 \le z-z' \le 2\mu,\ z' \le \mu,$
11:	$z'-z \ge 2\mu,\ z' \le 1-\mu,\ z \ge \mu,$	**12:**	$z-z' \ge 2\mu,\ z \le 1-\mu,\ z' \ge \mu,$
13:	$0 \le z'-z \le 2\mu,\ z' \le 1-\mu,\ z \ge \mu,$	**14:**	$0 \le z-z' \le 2\mu,\ z \le 1-\mu,\ z' \ge \mu$

The corresponding values of $I = I_\mu(z, z')$, with the ranges in which they lie, as z and z′ vary, are tabulated below. For example: when (z, z′) is in the region '**1**', $z \le \mu$ and $z' \ge 1 - \mu$, so that $\chi_\mu(\theta; z)\, \chi_\mu(\theta; z') = 1$ in the θ-interval $[z + \mu, z' - \mu]$, whose length is $I = z' - z - 2\mu$; when (z, z′) is in '**3**', $z' - z \ge 2\mu$, $z' \ge 1 - \mu$, and $z \ge \mu$, so that $\chi_\mu(\theta; z)\, \chi_\mu(\theta; z') = 1$ in the θ-intervals $[0, z - \mu]$ and $[z + \mu, z' - \mu]$, whose summed lengths are $I = (z - \mu) + (z' - z - 2\mu) = z' - 3\mu$; when (z, z′) is in '**7**', $0 \le z' - z \le 2\mu$ and $z' \ge 1 - \mu$, so that $\chi_\mu(\theta; z)\, \chi_\mu(\theta; z') = 1$ in the θ-interval $[0, z - \mu]$, whose length is $I = z - \mu$; and, finally, when (z, z′) is in the region '**13**', $0 \le z' - z \le 2\mu$, $z' \le 1 - \mu$, and $z \ge \mu$, so that $\chi_\mu(\theta; z)\, \chi_\mu(\theta; z') = 1$ in the θ-intervals $[0, z - \mu]$ and $[z' + \mu, 1]$, whose summed lengths are $I = (z - \mu) + (1 - z' - \mu) = 1 - z' + z - 2\mu$.

1:	$I = z'-z-2\mu \in [1-4\mu, 1-2\mu];$	**2:**	$I = z-z'-2\mu \in [1-4\mu, 1-2\mu];$
3:	$I = z'-3\mu \in [1-4\mu, 1-3\mu];$	**4:**	$I = z-3\mu \in [1-4\mu, 1-3\mu];$
5:	$I = 1-z-3\mu \in [1-4\mu, 1-3\mu];$	**6:**	$I = 1-z'-3\mu \in [1-4\mu, 1-3\mu];$
7:	$I = z-\mu \in [1-4\mu, 1-\mu];$	**8:**	$I = z'-\mu \in [1-4\mu, 1-\mu];$
9:	$I = 1-z'-\mu \in [1-4\mu, 1-\mu];$	**10:**	$I = 1-z-\mu \in [1-4\mu, 1-\mu];$
11:	$I = 1-4\mu;$	**12:**	$I = 1-4\mu;$
13:	$I = 1-z'+z-2\mu \in [1-4\mu, 1-2\mu];$	**14:**	$I = 1-z+z'-2\mu \in [1-4\mu, 1-2\mu].$

We observe from this that, in regions '**11**' and '**12**', whose total area is $(1-4\mu)^2$,

$$I_\mu(z,\, z') \;=\; 1 - 4\mu, \tag{17}$$

while, throughout the entire $(z,\, z')$-square,

$$1 - 4\mu \;\le\; I_\mu(z,\, z') \;\le\; 1 - \mu. \tag{18}$$

Considerations of symmetry show that certain regions yield the same values of the partial $(z,\, z')$-integrals making up (14). These partial integrals are:

1 & 2:
$$\int_{1-\mu}^{1} dz' \int_0^{\mu} dz\, (z' - z - 2\mu)^N = \frac{1}{N+1}\int_{1-\mu}^{1} dz'\, [(z' - 2\mu)^{N+1} - (z' - 3\mu)^{N+1}]$$
$$= \frac{1}{(N+1)(N+2)}[(1-2\mu)^{N+2} - 2(1-3\mu)^{N+2} + (1-4\mu)^{N+2}].$$

3 & 4 & 5 & 6:
$$\int_{1-\mu}^{1} dz' \int_{\mu}^{z'-2\mu} dz\, (z' - 3\mu)^N$$
$$= \int_{1-\mu}^{1} dz'\, (z' - 3\mu)^{N+1} = \frac{1}{N+2}[(1-3\mu)^{N+2} - (1-4\mu)^{N+2}].$$

7 & 8 & 9 & 10:
$$\int_{1-\mu}^{1} dz' \int_{z'-2\mu}^{z'} dz\, (z-\mu)^N = \frac{1}{N+1}\int_{1-\mu}^{1} dz'\, [(z'-\mu)^{N+1} - (z' - 3\mu)^{N+1}]$$
$$= \frac{1}{(N+1)(N+2)}[(1-\mu)^{N+2} - (1-2\mu)^{N+2} - (1-3\mu)^{N+2} + (1-4\mu)^{N+2}].$$

11 & 12:
$$\int_{3\mu}^{1-\mu} dz' \int_{\mu}^{z'-2\mu} dz\, (1-4\mu)^N = \int_{3\mu}^{1-\mu} dz'\, (1-4\mu)^N (z'-3\mu) = \tfrac{1}{2}(1-4\mu)^{N+2}.$$

13 & 14:
$$\int_{\mu}^{3\mu} dz' \int_{\mu}^{z'} dz\, (1 - z' + z - 2\mu)^N + \int_{3\mu}^{1-\mu} dz' \int_{z'-2\mu}^{z'} dz\, (1 - z' + z - 2\mu)^N$$
$$= \frac{1}{N+1}\left\{\int_{\mu}^{3\mu} dz'\, \Big[(1-2\mu)^{N+1} - (1 - z' - \mu)^{N+1}\Big] + \int_{3\mu}^{1-\mu} dz'\, [(1-2\mu)^{N+1} - (1-4\mu)^{N+1}]\right\}$$
$$= \frac{1}{N+1}\left(1 - \frac{1}{N+2}\right)[(1-2\mu)^{N+2} - (1-4\mu)^{N+2}] = \frac{1}{N+2}[(1-2\mu)^{N+2} - (1-4\mu)^{N+2}].$$

Thus, by (13) and (14), we get

$$\mathcal{V}ar\left[C_\lambda(N, \Theta)\right] = \frac{2}{(N+1)(N+2)}\left[(1-2\mu)^{N+2} - 2(1-3\mu)^{N+2} + (1-4\mu)^{N+2}\right]$$

$$+ \frac{4}{N+2}\left[(1-3\mu)^{N+2} - (1-4\mu)^{N+2}\right]$$

$$+ \frac{4}{(N+1)(N+2)}\left[(1-\mu)^{N+2} - (1-2\mu)^{N+2} - (1-3\mu)^{N+2} + (1-4\mu)^{N+2}\right]$$

$$+ (1-4\mu)^{N+2} + \frac{2}{N+2}\left[(1-2\mu)^{N+2} - (1-4\mu)^{N+2}\right]$$

$$- \left\{\frac{N-1}{N+1}\left(1-\frac{\lambda}{N}\right)^{N+1} - \frac{2}{N+1}\left(1-\frac{\lambda}{2N}\right)^{N+1}\right\}^2$$

$$= \frac{4}{(N+1)(N+2)}\left(1-\frac{\lambda}{2N}\right)^{N+2} + \frac{2N}{(N+1)(N+2)}\left(1-\frac{\lambda}{N}\right)^{N+2}$$

$$+ \frac{4(N-1)}{(N+1)(N+2)}\left(1-\frac{3\lambda}{2N}\right)^{N+2} + \frac{(N-1)(N-2)}{(N+1)(N+2)}\left(1-\frac{2\lambda}{N}\right)^{N+2}$$

$$- \left\{\frac{N-1}{N+1}\left(1-\frac{\lambda}{N}\right)^{N+1} + \frac{2}{N+1}\left(1-\frac{\lambda}{2N}\right)^{N+1}\right\}^2. \tag{19}$$

Again we apply the asymptotic formula (A7), for λ fixed and $N \to \infty$, as we did in (11). Then

$$\mathcal{V}ar\left[C_\lambda(N, \Theta)\right] = \frac{2}{N}\, e^{-\lambda} + \frac{4}{N}\, e^{-3\lambda/2} + \left(1-\frac{6}{N}\right)e^{-2\lambda}\left[1-\frac{1}{N}(4\lambda+2\lambda^2)\right]$$

$$- \left\{e^{-\lambda} - \frac{1}{2N}\left[(4+2\lambda+\lambda^2)\, e^{-\lambda} + 4\, e^{-\lambda/2}\right]\right\}^2 + O\!\left(\frac{1}{N^2}\right)$$

$$= \frac{1}{N}\left\{2\, e^{-\lambda} - (2+2\lambda+\lambda^2)\, e^{-2\lambda}\right\} + O\!\left(\frac{1}{N^2}\right). \tag{20}$$

Thus, we see that, while the mean $\mathcal{E}\left[C_\lambda(N, \Theta)\right]$ of the relative measure $C_\lambda(N, \Theta)$ of the interval covered tends to the constant $1 - e^{-\lambda}$ as $N \to \infty$ [see (11)], the standard deviation $\left\{\mathcal{V}ar\left[C_\lambda(N, \Theta)\right]\right\}^{1/2}$ tends to zero like $O(N^{-1/2})$ as $N \to \infty$.

LEMMA 1. *If a random variable X_N, with parameter N, has a mean value $\mu_N \to K$ and a standard deviation $\sigma_N \to 0$, as $N \to \infty$, then $X_N \to K$* ***in quadratic mean*** *and* ***in probability****, as $N \to \infty$.*

Proof. We have that

$$\begin{aligned}\mathcal{E}[(X_N - K)^2] &= \mathcal{E}\left[\left\{(X_N - \mu_N) + (\mu_N - K)\right\}^2\right] \\ &= \mathcal{E}\left[(X_N - \mu_N)^2\right] + 2(\mu_N - K)\,\mathcal{E}\left[X_N - \mu_N\right] + (\mu_N - K)^2 \\ &= \sigma_N^2 + (\mu_N - K)^2 \to 0, \quad \text{as} \quad N \to \infty. \end{aligned} \tag{21}$$

Thus, by definition,

$$X_N \to K \quad (\textit{in quadratic mean}) \quad \text{as} \quad N \to \infty; \tag{22}$$

and therefore [see, e.g., (**1**), p. 176, or (**2**), p. 160],

$$X_N \to K \quad (\textit{in probability}) \quad \text{as} \quad N \to \infty. \tag{23}$$

QED.

We recall that (23) means that

$$(\forall \varepsilon > 0)\ \ \mathcal{P}rob\left[\left|X_N - K\right| \geq \varepsilon\right] \to 0 \quad \text{as} \quad N \to \infty. \tag{24}$$

An immediate consequence of this lemma is:

THEOREM 1. *The random variable $C_\lambda(N, \Theta)$ tends to $1 - e^{-\lambda}$* ***in quadratic mean****, and hence* ***in probability****, as $N \to \infty$.*

Proof. We have shown [see (11) and (20)] that, if

$$X_N = C_\lambda(N, \Theta), \tag{25}$$

then
$$\mu_N = \mathcal{E}[X_N] = \mathcal{E}\left[C_\lambda(N, \Theta)\right] \to 1 - e^{-\lambda} \tag{26}$$

and
$$\sigma_N^2 = \mathcal{V}ar[X_N] = \mathcal{V}ar\left[C_\lambda(N, \Theta)\right] \to 0. \tag{27}$$

as $N \to \infty$. Thus, Lemma 1 applies to $C_\lambda(N, \Theta)$ and so the theorem holds. QED.

For small λ, (20) becomes [compare (12)]

$$\mathcal{V}ar\Big[C_\lambda(N, \Theta)\Big] = \frac{\lambda^3}{3N} + O(\frac{\lambda^4}{N}) + O\Big(\frac{1}{N^2}\Big). \tag{28}$$

If our principal concern is for the asymptotic forms, such as (11), (12), (20), and (28), then we can considerably streamline our derivations. First, we observe that, by (9) and the third line of (10),

$$\mathcal{E}\Big[C_\lambda(N, \Theta)\Big] = 1 - (1 - 2\mu)^N - \int_0^\mu \mathrm{d}z\,\big[(1 - \mu - z)^N - (1 - 2\mu)^N\big]$$
$$- \int_{1-\mu}^1 \mathrm{d}z\,\big[(z - \mu)^N - (1 - 2\mu)^N\big],$$

and, if [with $2\mu = \lambda/N$, by (4)] we write $z = (u - \lambda/2)/N$ in the first integral, and $z = 1 - (u - \lambda/2)/N$ in the second, we get, by (A7), that

$$\begin{aligned}\mathcal{E}\Big[C_\lambda(N, \Theta)\Big] &= 1 - \Big(1 - \frac{\lambda}{N}\Big)^N - \frac{2}{N}\int_{\lambda/2}^{\lambda} \mathrm{d}u\Big\{\Big(1 - \frac{u}{N}\Big)^N - \Big(1 - \frac{\lambda}{N}\Big)^N\Big\} \\ &= 1 - \Big(1 - \frac{\lambda}{N}\Big)^{N+1} - \frac{2}{N}\int_{\lambda/2}^{\lambda} \mathrm{d}u\, \mathrm{e}^{-u}\Big\{1 + O\Big(\frac{1}{N}\Big)\Big\} \\ &= 1 - \mathrm{e}^{-\lambda}\Big[1 - \frac{1}{N}(\lambda + \tfrac{1}{2}\lambda^2)\Big] - \frac{2}{N}(\mathrm{e}^{-\lambda/2} - \mathrm{e}^{-\lambda}) + O\Big(\frac{1}{N^2}\Big) \\ &= 1 - \mathrm{e}^{-\lambda} + \frac{1}{2N}\big[(4 + 2\lambda + \lambda^2)\,\mathrm{e}^{-\lambda} - 4\,\mathrm{e}^{-\lambda/2}\big] + O\Big(\frac{1}{N^2}\Big);\end{aligned} \tag{29}$$

in agreement, of course, with (11).

Secondly, we similarly use (14) - (18) to simplify the variance computation even more markedly.

$$\mathcal{E}\left[\left\{1 - C_\lambda(N, \Theta)\right\}^2\right] = \int_0^1 dz \int_0^1 dz'\, I_\mu(z, z')^N$$

$$= \int_0^1 dz \int_0^1 dz'\, (1 - 4\mu)^N + \Gamma_\lambda(N)$$

$$= \left(1 - \frac{2\lambda}{N}\right)^N + \Gamma_\lambda(N). \qquad (30)$$

where $$0 \le \Gamma_\lambda(N) = \underset{\substack{\text{Complement of regions} \\ \text{'}\mathbf{11}\text{' and '}\mathbf{12}\text{'}}}{\int dz \int dz'} \left\{I_\mu(z, z')^N - (1 - 4\mu)^N\right\}$$

$$\le [1 - (1 - 4\mu)^2]\,[(1 - \mu)^N - (1 - 4\mu)^N]$$

$$\le 8\mu = \frac{4\lambda}{N}; \qquad (31)$$

that is, $$\Gamma_\lambda(N) = O\left(\frac{1}{N}\right). \qquad (32)$$

Therefore, by (A7),

$$\mathcal{E}\left[\left\{1 - C_\lambda(N, \Theta)\right\}^2\right] = e^{-2\lambda} + O\left(\frac{1}{N}\right);$$

while $$\left\{1 - \mathcal{E}\left[C_\lambda(N, \Theta)\right]\right\}^2 = \left\{e^{-\lambda} + O\left(\frac{1}{N}\right)\right\}^2 = e^{-2\lambda} + O\left(\frac{1}{N}\right),$$

by (11); which, with (13), shows that

$$\mathcal{Var}\left[C_\lambda(N, \Theta)\right] = O\left(\frac{1}{N}\right). \qquad (33)$$

This, in turn, suffices to prove Theorem 1.

3. THE k-DIMENSIONAL PROBLEM, WITH HYPER-RECTANGLES

The most direct generalisation of the foregoing problem replaces the interval $[0, a]$ by a hyper-rectangular interval,

$$\mathop{\times}_{j=1}^{k} [0, a_j], \text{ of } k\text{-dimensional volume } A = \prod_{j=1}^{k} a_j , \tag{34}$$

and replaces the segment $[c - r, c + r]$ by the hyper-rectangular interval,

$$\mathop{\times}_{j=1}^{k} [c_j - r_j, c_j + r_j], \text{ of } k\text{-dimensional volume } v = 2^k \prod_{j=1}^{k} r_j . \tag{35}$$

When there are N such "bricks", we denote their centres by $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_N$, with $\boldsymbol{c}_i = [c_{i1}, c_{i2}, \ldots, c_{ik}]$, for $i = 1, 2, \ldots, N$, and write

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \cdots \\ \boldsymbol{c}_N \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ c_{N1} & c_{N2} & \cdots & c_{Nk} \end{bmatrix}. \tag{36}$$

The bricks are supposed to be congruent and similarly oriented; so the dimension, parallel to the j-th coordinate axis, of *every* brick is r_j, and we may write $\boldsymbol{r} = [r_1, r_2, \ldots, r_k]$. The indicator function of the exterior of the i-th brick is then clearly

$$\bar{\chi}_{\boldsymbol{r}}(\boldsymbol{c}_i ; \boldsymbol{x}) = 1 - \prod_{j=1}^{k} \left[1 - \chi_{r_j}(c_{ij} ; x_j)\right]; \tag{37}$$

and the function

$$X_{\boldsymbol{r}}(\mathbf{C};\boldsymbol{x}) = \prod_{i=1}^{N}\left\{\bar{\chi}_{\boldsymbol{r}}(\boldsymbol{c}_i;\boldsymbol{x})\right\} = \prod_{i=1}^{N}\left\{1-\prod_{j=1}^{k}\left[1-\chi_{r_i}(c_{ij};x_j)\right]\right\} \quad (38)$$

equals 0, if $\boldsymbol{x}$ lies in *any* of the N bricks, and equals 1 otherwise. If we scale our variables as in (4) and (8), in the forms

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \dots & \theta_{1k} \\ \theta_{21} & \theta_{22} & \dots & \theta_{2k} \\ \dots & \dots & \dots & \dots \\ \theta_{N1} & \theta_{N2} & \dots & \theta_{Nk} \end{bmatrix}, \quad \text{with} \quad (\forall i, j) \;\; \theta_{ij} = c_{ij}/a_j, \quad (39)$$

$$\Lambda = [\lambda_1, \lambda_2, \dots, \lambda_k] = 2N^{1/k}[r_1/a_1, r_2/a_2, \dots, r_k/a_k], \quad (40)$$

and
$$(\forall j) \;\; z_j = \frac{x_j}{a_j} \quad \text{and} \quad \mu_j = \frac{r_j}{a_j} = \frac{\lambda_j}{2N^{1/k}}, \quad (41)$$

we see that
$$\omega = \frac{Nv}{A} = \prod_{j=1}^{k}\lambda_j. \quad (42)$$

The generalisation of (7) is then

$$\begin{aligned} C_\Lambda(N,\Theta) &= \frac{1}{A}\int_0^{a_1}\mathrm{d}x_1\int_0^{a_2}\mathrm{d}x_2\dots\int_0^{a_k}\mathrm{d}x_k\;[1-X_{\boldsymbol{r}}(\mathbf{C};\boldsymbol{x})] \\ &= 1-\frac{1}{A}\int_0^{a_1}\mathrm{d}x_1\int_0^{a_2}\mathrm{d}x_2\dots\int_0^{a_k}\mathrm{d}x_k\;X_{\boldsymbol{r}}(\mathbf{C};\boldsymbol{x}), \end{aligned} \quad (43)$$

and therefore the analogue of the first line of (10) is

$$\begin{aligned} \mathcal{E}\left[C_\Lambda(N,\Theta)\right] = 1 &- \int_0^1\mathrm{d}z_1\int_0^1\mathrm{d}z_2\dots\int_0^1\mathrm{d}z_k \\ &\times\int_0^1\mathrm{d}\theta_{11}\int_0^1\mathrm{d}\theta_{12}\dots\int_0^1\mathrm{d}\theta_{1k}\int_0^1\mathrm{d}\theta_{21}\int_0^1\mathrm{d}\theta_{22}\dots\int_0^1\mathrm{d}\theta_{2k} \\ &\times\dots\times\int_0^1\mathrm{d}\theta_{N1}\int_0^1\mathrm{d}\theta_{N2}\dots\int_0^1\mathrm{d}\theta_{Nk}\prod_{i=1}^{N}\left\{1-\prod_{j=1}^{k}\left[1-\chi_{\mu_j}(\theta_{ij};z_j)\right]\right\}. \end{aligned} \quad (44)$$

which simplifies, much as before, by (9), to

$$\mathcal{E}\left[C_\Lambda(N, \Theta)\right] = 1 - \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \cdots \int_0^1 \mathrm{d}z_k \prod_{i=1}^{N} \left\{ 1 - \prod_{j=1}^{k} \left[1 - \int_0^1 \mathrm{d}\theta_{ij}\, \chi_{\mu_j}(\theta_{ij}; z_j)\right] \right\}$$

$$= 1 - \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \cdots \int_0^1 \mathrm{d}z_k \left\{1 - \prod_{j=1}^{k} \left[1 - J_{\mu_j}(z_j)\right]\right\}^N$$

$$= 1 - \sum_{q=0}^{N} (-1)^q \binom{N}{q} \prod_{j=1}^{k} \left\{ \int_0^1 \mathrm{d}z_j\, (2\mu_j)^q \right.$$

$$\left. - \int_0^{\mu_j} \mathrm{d}z_j \left[(2\mu_j)^q - (\mu_j + z_j)^q\right] - \int_{1-\mu_j}^{1} \mathrm{d}z_j \left[(2\mu_j)^q - (1 + \mu_j - z_j)^q\right] \right\}$$

$$= 1 - \sum_{q=0}^{N} (-1)^q \binom{N}{q} \prod_{j=1}^{k} (2\mu_j)^q \left\{1 - 2\mu_j \left[1 - \frac{2}{q+1} + \frac{1}{(q+1)2^q}\right]\right\}$$

$$= 1 - \sum_{q=0}^{N} (-1)^q \binom{N}{q} \prod_{j=1}^{k} \left(\frac{\lambda_j}{N^{1/k}}\right)^q \left\{1 - \frac{\lambda_j}{N^{1/k}} \left[1 - \frac{2}{q+1} + \frac{1}{(q+1)2^q}\right]\right\}$$

$$= 1 - \sum_{q=0}^{N} (-1)^q \binom{N}{q} \left(\frac{\omega}{N}\right)^q \prod_{j=1}^{k} \left\{1 - \frac{\lambda_j}{N^{1/k}} \left[1 - \frac{2}{q+1} + \frac{1}{(q+1)2^q}\right]\right\}. \tag{45}$$

Since, for $q \geq 1$,

$$0 \leq 1 - \frac{2}{q+1} + \frac{1}{(q+1)2^q} < 1 \tag{46}$$

[the expression vanishes for $q = 0$], and since, for $\omega > 0$,

$$0 < \sum_{q=0}^{N} \binom{N}{q} \left(\frac{\omega}{N}\right)^q = \left(1 + \frac{\omega}{N}\right)^N$$

$$= 1 + \omega + \frac{1}{2} \frac{N-1}{N} \omega^2 + \frac{1}{3!} \frac{(N-1)(N-2)}{N^2} \omega^3 + \ldots$$

$$< 1 + \omega + \frac{1}{2} \omega^2 + \frac{1}{3!} \omega^3 + \ldots < e^{\omega}. \tag{47}$$

and putting
$$\tau = \sum_{j=1}^{k} \lambda_j, \tag{48}$$

we have from (45) that

$$\begin{aligned}
&\mathcal{E}\left[C_\lambda(N,\Theta)\right] \\
&= 1 - \sum_{q=0}^{N} (-1)^q \binom{N}{q} \left(\frac{\omega}{N}\right)^q \left\{1 - \sum_{j=1}^{k} \frac{\lambda_j}{N^{1/k}}\left[1 - \frac{2}{q+1} + \frac{1}{(q+1)2^q}\right]\right\} + O\!\left(\frac{1}{N^{2/k}}\right) \\
&= 1 - \sum_{q=0}^{N} (-1)^q \binom{N}{q} \left(\frac{\omega}{N}\right)^q \left\{1 - \frac{\tau}{N^{1/k}}\left[1 - \frac{2}{q+1} + \frac{1}{(q+1)2^q}\right]\right\} + O\!\left(\frac{1}{N^{2/k}}\right).
\end{aligned} \tag{49}$$

Now note that

$$\sum_{q=0}^{N} (-1)^q \binom{N}{q}\left(\frac{\omega}{N}\right)^q = \left(1 - \frac{\omega}{N}\right)^N; \tag{50}$$

and that, for any $\rho > 0$,

$$\begin{aligned}
&\sum_{q=0}^{N} (-1)^q \binom{N}{q}\left(\frac{\omega}{N}\right)^q \frac{1}{(q+1)\rho^q} \\
&= \sum_{q=0}^{N} \binom{N}{q}\left(\frac{-\omega}{\rho N}\right)^q \left[\frac{t^{q+1}}{q+1}\right]_{t=0}^{t=1} \\
&= \int_0^1 dt \sum_{q=0}^{N} \binom{N}{q}\left(\frac{-\omega t}{\rho N}\right)^q = \int_0^1 dt \left(1 - \frac{\omega t}{\rho N}\right)^N \\
&= \left[\left(-\frac{\rho N}{\omega}\right)\frac{1}{N+1}\left(1 - \frac{\omega t}{\rho N}\right)^{N+1}\right]_{t=0}^{t=1} \\
&= \left(\frac{\rho}{\omega}\right)\left(\frac{N}{N+1}\right)\left\{1 - \left(1 - \frac{\omega}{\rho N}\right)^{N+1}\right\};
\end{aligned} \tag{51}$$

so that, for $k \geq 2$ (whence $1/N \geq 1/N^{2/k}$), with all λ_j fixed, as $N \to \infty$, we obtain, using (A7), an asymptotic formula, similar to (11):

$$\mathcal{E}\left[C_\Lambda(N,\Theta)\right] = 1 - \left(1 - \frac{\omega}{N}\right)^N + \frac{\tau}{N^{1/k}}\left\{\left(1 - \frac{\omega}{N}\right)^N + \left(\frac{2}{\omega}\right)\left(\frac{N}{N+1}\right)\left[\left(1 - \frac{\omega}{N}\right)^{N+1} - \left(1 - \frac{\omega}{2N}\right)^{N+1}\right]\right\} + \mathcal{O}\left(\frac{1}{N^{2/k}}\right)$$

$$= 1 - e^{-\omega} + \frac{\tau}{N^{1/k}}\left[e^{-\omega}\left(1 + \frac{2}{\omega}\right) - \frac{2}{\omega}e^{-\omega/2}\right] + \mathcal{O}\left(\frac{1}{N^{2/k}}\right)$$

$$= 1 - e^{-\omega} + \mathcal{O}\left(\frac{1}{N^{1/k}}\right) \tag{52}$$

The variance calculation uses [compare (13)]

$$\mathcal{V}ar\left[C_\Lambda(N,\Theta)\right] = \mathcal{E}\left[\left\{C_\Lambda(N,\Theta) - \mathcal{E}[C_\Lambda(N,\Theta)]\right\}^2\right]$$

$$= \mathcal{E}\left[\left\{1 - C_\Lambda(N,\Theta)\right\}^2\right] - \left\{1 - \mathcal{E}\left[C_\Lambda(N,\Theta)\right]\right\}^2. \tag{53}$$

By (43), we get [compare (14)] that

$$\mathcal{E}\left[\left\{1 - C_\Lambda(N,\Theta)\right\}^2\right] = \int_0^1 dz_1 \int_0^1 dz_2 \ldots \int_0^1 dz_k \int_0^1 dz'_1 \int_0^1 dz'_2 \ldots \int_0^1 dz'_k$$

$$\times \int_0^1 d\theta_{11} \int_0^1 d\theta_{12} \ldots \int_0^1 d\theta_{1k} \int_0^1 d\theta_{21} \int_0^1 d\theta_{22} \ldots \int_0^1 d\theta_{2k}$$

$$\times \ldots \times \int_0^1 d\theta_{N1} \int_0^1 d\theta_{N2} \ldots \int_0^1 d\theta_{Nk}$$

$$\times \prod_{i=1}^{N}\left\{1 - \prod_{j=1}^{k}\left[1 - \chi_{\mu_j}(\theta_{ij}; z_j)\right]\right\}\left\{1 - \prod_{j=1}^{k}\left[1 - \chi_{\mu_j}(\theta_{ij}; z'_j)\right]\right\}$$

$$= \int_0^1 dz_1 \int_0^1 dz_2 \ldots \int_0^1 dz_k \int_0^1 dz'_1 \int_0^1 dz'_2 \ldots \int_0^1 dz'_k$$

$$\times \prod_{i=1}^{N}\left\{1 - \prod_{j=1}^{k}\left[1 - \int_0^1 d\theta_{ij}\, \chi_{\mu_j}(\theta_{ij}; z_j)\right] - \prod_{j=1}^{k}\left[1 - \int_0^1 d\theta_{ij}\, \chi_{\mu_j}(\theta_{ij}; z'_j)\right]\right.$$

$$\left. + \prod_{j=1}^{k}\left[1 - \int_0^1 d\theta_{ij}\, \chi_{\mu_j}(\theta_{ij}; z_j) - \int_0^1 d\theta_{ij}\, \chi_{\mu_j}(\theta_{ij}; z'_j) + \int_0^1 d\theta_{ij}\, \chi_{\mu_j}(\theta_{ij}; z_j)\, \chi_{\mu_j}(\theta_{ij}; z'_j)\right]\right\}. \tag{54}$$

We note that the initial integrand in (54) is always in [0, 1]; whence the result of integrating over all the unit θ_{ij}-intervals (the integrand of the z_j) is also always in [0, 1].

In the regions '**11**' and '**12**', of total area $(1 - 4\mu)^2$, defined for the (z, z')-square in Figure 2,

$$J_\mu(z) = J_\mu(z') = 1 - 2\mu \quad \text{and} \quad I_\mu(z, z') = 1 - 4\mu, \tag{55}$$

by (9), (15), and (17). Thus, (54), with (9), (15), and (41), yields that

$$\mathcal{E}\left[\left\{1 - C_\Lambda(N, \Theta)\right\}^2\right] = \int_0^1 dz_1 \int_0^1 dz_2 \ldots \int_0^1 dz_k \int_0^1 dz'_1 \int_0^1 dz'_2 \ldots \int_0^1 dz'_k$$

$$\times\left\{1 - \prod_{j=1}^{k}\left[1 - J_{\mu_j}(z_j)\right] - \prod_{j=1}^{k}\left[1 - J_{\mu_j}(z'_j)\right] + \prod_{j=1}^{k}\left[1 - J_{\mu_j}(z_j) - J_{\mu_j}(z'_j) + I_{\mu_j}(z_j, z'_j)\right]\right\}^N$$

$$= \left(1 - 2\prod_{j=1}^{k}(2\mu_j)\right)^N + \Gamma_\Lambda(N) = \left(1 - \frac{2\omega}{N}\right)^N + \Gamma_\Lambda(N), \tag{56}$$

say; and we see that

$$0 \le \Gamma_\Lambda(N) \le \prod_{j=1}^{k}\left[1 - (1 - 4\mu_j)^2\right]^k\left\{1 - \left(1 - \frac{2\omega}{N}\right)^N\right\} < \prod_{j=1}^{k}(8\mu_j)^k = 4^k\left(\frac{\omega}{N}\right). \tag{57}$$

Hence, $$\mathcal{E}\left[\left\{1 - C_\Lambda(N, \Theta)\right\}^2\right] = \left(1 - \frac{2\omega}{N}\right)^N + O\left(\frac{1}{N}\right) = e^{-2\omega} + O\left(\frac{1}{N}\right); \tag{58}$$

while $$\left\{1 - \mathcal{E}\left[C_\Lambda(N, \Theta)\right]\right\}^2 = \left\{e^{-\omega} + O\left(\frac{1}{N^{1/k}}\right)\right\}^2 = e^{-2\omega} + O\left(\frac{1}{N^{1/k}}\right); \tag{59}$$

so that, by (53), as $N \to \infty$,

$$\mathcal{V}ar\left[C_\Lambda(N, \Theta)\right] = O\left(\frac{1}{N^{1/k}}\right) \to 0. \tag{60}$$

We now proceed exactly as in obtaining Theorem 1 to get:

THEOREM 2. *The random variable $C_\Lambda(N, \Theta)$ tends to* $1 - e^{-\omega}$ ***in quadratic mean****, and hence* ***in probability****, as* $N \to \infty$.

Proof. We have shown [see (52) and (60)] that, if

$$X_N = C_\Lambda(N, \Theta), \tag{61}$$

then
$$\mu_N = \mathcal{E}[X_N] = \mathcal{E}\big[C_\Lambda(N, \Theta)\big] \to 1 - e^{-\omega} \tag{62}$$

and
$$\sigma_N^2 = \mathcal{Var}[X_N] = \mathcal{Var}\big[C_\Lambda(N, \Theta)\big] \to 0, \tag{63}$$

as $N \to \infty$. Thus, Lemma 1 applies to $C_\Lambda(N, \Theta)$ and so the theorem holds. QED.

4. THE k-DIMENSIONAL PROBLEM, WITH ARBITRARY IDENTICAL OBJECTS

We now consider the k-dimensional problem, in which the "table", T, is still an $(a_1 \times a_2 \times \ldots \times a_k)$ hyper-rectangle, but now the "coins" are congruent, similarly-oriented objects $Q_1, Q_2, \ldots, Q_N$, of arbitrary shape '$\mathbf{Q}$', which are bounded by $(2r_1 \times 2r_2 \times \ldots \times 2r_k)$ hyper-rectangles (or "bricks") with centres at $\boldsymbol{c}_i = [c_{i1} \times c_{i2} \times \ldots c_{ik}]$ $(i = 1, 2, \ldots, N)$, as before. We now denote the k-dimensional volume of each such object of shape $\mathbf{Q}$ by $v_{\mathbf{Q}}$: so that, by (34), (35) [writing $v_{\mathbf{r}}$ more explicitly for v], and (40) – (42), we get

$$v_{\mathbf{Q}} \leq v_{\mathbf{r}} = 2^k \prod_{j=1}^{k} r_j = A\left(\frac{\omega}{N}\right). \tag{64}$$

Let $\overset{\circ}{\chi}_{\mathbf{Q}}(\boldsymbol{c}_i, \boldsymbol{x})$ denote the indicator function of the exterior of the i-th object of shape $\mathbf{Q}$; then, clearly, by (37),

$$\overset{\circ}{\chi}_{\mathbf{Q}}(\boldsymbol{c}_i : \boldsymbol{x}) \geq \bar{\chi}_{\mathbf{r}}(\boldsymbol{c}_i : \boldsymbol{x}), \tag{65}$$

in the sense that, since $\mathbf{Q}$ lies inside the brick, the function on the left can only be 0 if the function on the right is 0. The function

$$\overset{\circ}{X}_{\mathbf{Q}}(\boldsymbol{C} : \boldsymbol{x}) = \prod_{i=1}^{N} \{\overset{\circ}{\chi}_{\mathbf{Q}}(\boldsymbol{c}_i : \boldsymbol{x})\} \tag{66}$$

then equals 0, if $\boldsymbol{x}$ lies in *any* of the N objects $\mathbf{Q}$, and equals 1, otherwise. Thus, if, analogously to (7) and (43), $C_{\mathbf{Q}}(N, \Theta)$ denotes the ratio of the total volume covered by the N randomly-placed objects $\mathbf{Q}$ to the volume A of the "table", (43) immediately generalises to

$$C_{\mathbf{Q}}(N, \Theta) = \frac{1}{A}\int_0^{a_1} \mathrm{d}x_1 \int_0^{a_2} \mathrm{d}x_2 \ldots \int_0^{a_k} \mathrm{d}x_k \, [1 - \mathring{\chi}_{\mathbf{Q}}(\mathbf{C}; \boldsymbol{x})]$$

$$= 1 - \frac{1}{A}\int_0^{a_1} \mathrm{d}x_1 \int_0^{a_2} \mathrm{d}x_2 \ldots \int_0^{a_k} \mathrm{d}x_k \, \mathring{\chi}_{\mathbf{Q}}(\mathbf{C}; \boldsymbol{x}). \qquad (67)$$

We can *scale* the problem with respect to the dimensions of T, as before, using (39) - (41). If we write '$\mathbf{S}$' for the scaled shape (obtained from $\mathbf{Q}$ by dividing the j-th coordinate of every point by a_j, for $j = 1, 2, \ldots, k$) we observe that [compare (3) and (5)]

$$\mathring{\chi}_{\mathbf{S}}(\bar{\theta}; \mathbf{z}) = \mathring{\chi}_{\mathbf{Q}}(\mathbf{c}; \boldsymbol{x}), \qquad (68)$$

where
$$\bar{\theta} = [\theta_1, \theta_2, \ldots, \theta_k]. \qquad (69)$$

Similarly, if '$\mathbf{B}$' is the shape of a "brick" (or hyper-rectangle), whose projection on the j-th coordinate axis is $[z_j - \mu_j, z_j + \mu_j]$ ($j = 1, 2, \ldots, k$), and we take the shape $\mathbf{S}$ to be $\mathbf{B}$, we see that

$$\mathring{\chi}_{\mathbf{B}}(\bar{\theta}; \mathbf{z}) = \bar{\chi}_{\mathbf{r}}(\mathbf{c}; \boldsymbol{x}) = \bar{\chi}_{\bar{\mu}}(\bar{\theta}; \mathbf{z}), \qquad (70)$$

where
$$\bar{\mu} = [\mu_1, \mu_2, \ldots, \mu_k]. \qquad (71)$$

We now get, as before, that

$$C_{\mathbf{Q}}(N, \Theta) = C_{\mathbf{S}}(N, \Theta), \qquad (72)$$

and, therefore, following (44) and (45), that

$$\mathcal{E}\left[C_{\mathbf{Q}}(N, \Theta)\right] = \mathcal{E}\left[C_{\mathbf{S}}(N, \Theta)\right] = 1 - \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \ldots \int_0^1 \mathrm{d}z_k$$

$$\times \int_0^1 \mathrm{d}\theta_{11} \int_0^1 \mathrm{d}\theta_{12} \cdots \int_0^1 \mathrm{d}\theta_{1k} \int_0^1 \mathrm{d}\theta_{21} \int_0^1 \mathrm{d}\theta_{22} \cdots \int_0^1 \mathrm{d}\theta_{2k}$$

$$\times \ldots \times \int_0^1 \mathrm{d}\theta_{N1} \int_0^1 \mathrm{d}\theta_{N2} \cdots \int_0^1 \mathrm{d}\theta_{Nk} \prod_{i=1}^{N} \mathring{\chi}_{\mathbf{S}}(\bar{\theta}_i; \mathbf{z})$$

$$= 1 - \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \ldots \int_0^1 \mathrm{d}z_k \prod_{i=1}^{N} \left\{ \int_0^1 \mathrm{d}\theta_{i1} \int_0^1 \mathrm{d}\theta_{i2} \cdots \int_0^1 \mathrm{d}\theta_{ik} \, \mathring{\chi}_{\mathbf{S}}(\bar{\theta}_i; \mathbf{z}) \right\}.$$

so that

$$\mathcal{E}\left[C_{\mathbf{S}}(N,\,\Theta)\right] = 1 - \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \ldots \int_0^1 \mathrm{d}z_k \left\{ \int_0^1 \mathrm{d}\theta_1 \int_0^1 \mathrm{d}\theta_2 \ldots \int_0^1 \mathrm{d}\theta_k\ \overset{\circ}{\chi}_{\mathbf{S}}(\bar{\theta};\,\mathbf{z}) \right\}^N$$

$$= 1 - \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \ldots \int_0^1 \mathrm{d}z_k\, \overset{\circ}{F}_{\mathbf{S}}(\mathbf{z})^N, \tag{73}$$

where

$$\overset{\circ}{F}_{\mathbf{S}}(\mathbf{z}) = \int_0^1 \mathrm{d}\theta_1 \int_0^1 \mathrm{d}\theta_2 \ldots \int_0^1 \mathrm{d}\theta_k\ \overset{\circ}{\chi}_{\mathbf{S}}(\bar{\theta};\,\mathbf{z}). \tag{74}$$

Because, by (2), the function $\chi_\mu(\theta;\,z)$ is symmetric, in θ and z; we can consider it, not only as the indicator function of the exterior of the segment with centre at θ, relative to the variable z; but also as the indicator function of the exterior of the segment with centre at z, relative to the variable θ. Thus, the function $J_\mu(z)$ defined in (9) is seen to be the proportion of the θ-interval $[0,\,1]$ *not* covered by the segment $[z - \mu,\, z + \mu]$, and therefore the integrand in the second line of (45) is the N-th power of

$$\overset{\circ}{F}_{\mathbf{B}}(\mathbf{z}) = 1 - \prod_{j=1}^{k} \left[1 - J_{\mu_j}(z_j)\right]. \tag{75}$$

This $\overset{\circ}{F}_{\mathbf{B}}(\mathbf{z})$ is clearly the proportion of the "table" T *not* covered by a brick B, say, of shape $\mathbf{B}$. Similarly, we note that $\overset{\circ}{F}_{\mathbf{S}}(\mathbf{z})$ is the proportion of T *not* covered by an object S of shape $\mathbf{S}$. Thus, since

$$S \subseteq B, \tag{76}$$

we have

$$0 \le \overset{\circ}{F}_{\mathbf{B}}(\mathbf{z}) \le \overset{\circ}{F}_{\mathbf{S}}(\mathbf{z}) \le 1; \tag{77}$$

and this yields that

$$0 \le \mathcal{E}\left[C_{\mathbf{S}}(N,\,\Theta)\right] \le \mathcal{E}\left[C_{\Lambda}(N,\,\Theta)\right]. \tag{78}$$

Further, so long as B remains *inside* T, the corresponding object S which it bounds will also be inside T. Thus, just as, by (9), (35), (41), and (42), for any $\mathbf{z}$ such that $B \subseteq T$,

$$\overset{\circ}{F}_{\mathbf{B}}(\mathbf{z}) = 1 - \prod_{j=1}^{k} (2\mu_j) = 1 - \frac{v_{\mathbf{r}}}{A} = 1 - \frac{\omega}{N}; \tag{79}$$

so, if we analogously define [compare (42)]

$$\omega_{\mathbf{S}} = N\frac{v_{\mathbf{g}}}{A}, \tag{80}$$

then
$$\overset{\circ}{F}_{\mathbf{S}}(\mathbf{z}) = 1 - \frac{v_{\mathbf{g}}}{A} = 1 - \frac{\omega_{\mathbf{S}}}{N} \le 1 - \frac{\omega}{N}. \tag{81}$$

B will be inside T for $\mathbf{z}$ in a region of volume not less than, by (48),

$$\prod_{j=1}^{k} (1 - 2\mu_j) \ge 1 - \frac{1}{N^{1/k}} \sum_{j=1}^{k} \lambda_j = 1 - \frac{\tau}{N^{1/k}}, \tag{82}$$

if N is large enough; and, in the remaining region, of volume not greater than $\tau N^{-1/k}$, we know, by (77), that $\overset{\circ}{F}_{\mathbf{S}}(\mathbf{z}) \le 1$; so, by (73), (81), and (82),

$$\begin{aligned} \mathcal{E}\left[C_{\mathbf{g}}(N, \Theta)\right] &= 1 - \left(1 - \frac{\omega_{\mathbf{S}}}{N}\right)^{N} + O\left(\frac{1}{N^{1/k}}\right) \\ &= 1 - e^{-\omega_{\mathbf{S}}} + O\left(\frac{1}{N^{1/k}}\right). \end{aligned} \tag{83}$$

Turning to the variance, we see that [compare (54) and (73)]

$$\begin{aligned} \mathcal{E}\left[\left\{1 - C_{\mathbf{g}}(N, \Theta)\right\}^2\right] &= \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \ldots \int_0^1 \mathrm{d}z_k \int_0^1 \mathrm{d}z'_1 \int_0^1 \mathrm{d}z'_2 \ldots \int_0^1 \mathrm{d}z'_k \\ &\quad \times \int_0^1 \mathrm{d}\theta_{11} \int_0^1 \mathrm{d}\theta_{12} \ldots \int_0^1 \mathrm{d}\theta_{1k} \int_0^1 \mathrm{d}\theta_{21} \int_0^1 \mathrm{d}\theta_{22} \ldots \int_0^1 \mathrm{d}\theta_{2k} \\ &\quad \times \ldots \times \int_0^1 \mathrm{d}\theta_{N1} \int_0^1 \mathrm{d}\theta_{N2} \ldots \int_0^1 \mathrm{d}\theta_{Nk} \prod_{i=1}^{N} \left\{\overset{\circ}{\chi}_{\mathbf{S}}(\bar{\theta}_i; \mathbf{z})\, \overset{\circ}{\chi}_{\mathbf{S}}(\bar{\theta}_i; \mathbf{z}')\right\} \\ &= \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \ldots \int_0^1 \mathrm{d}z_k \left\{\int_0^1 \mathrm{d}\theta_1 \int_0^1 \mathrm{d}\theta_2 \ldots \int_0^1 \mathrm{d}\theta_k\, \overset{\circ}{\chi}_{\mathbf{S}}(\bar{\theta}; \mathbf{z})\, \overset{\circ}{\chi}_{\mathbf{S}}(\bar{\theta}; \mathbf{z}')\right\}^{N}. \end{aligned}$$

so that

$$\mathcal{E}\left[\left\{1 - C_{\mathbf{Q}}(N, \Theta)\right\}^2\right] = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \ldots \int_0^1 \mathrm{d}z_k \, \mathring{G}_{\mathbf{S}}(\mathbf{z}, \mathbf{z}')^N, \tag{84}$$

where

$$\mathring{G}_{\mathbf{S}}(\mathbf{z}, \mathbf{z}') = \int_0^1 \mathrm{d}\theta_1 \int_0^1 \mathrm{d}\theta_2 \ldots \int_0^1 \mathrm{d}\theta_k \; \mathring{\chi}_{\mathbf{S}}(\bar{\theta}; \mathbf{z}) \, \mathring{\chi}_{\mathbf{S}}(\bar{\theta}; \mathbf{z}'). \tag{85}$$

As is noted after (74), the function $\mathring{\chi}_{\mathbf{B}}(\bar{\theta}; \mathbf{z}) \, \mathring{\chi}_{\mathbf{B}}(\bar{\theta}; \mathbf{z}')$ may be viewed as the indicator function, relative to the variable vector $\bar{\theta}$, of the exterior of the union of two "bricks", B and B', whose centres are at $\mathbf{z}$ and $\mathbf{z}'$. Thus, the integrand in the fourth line of (54) is the N-th power of

$$\mathring{G}_{\mathbf{B}}(\mathbf{z}, \mathbf{z}') = \int_0^1 \mathrm{d}\theta_1 \int_0^1 \mathrm{d}\theta_2 \ldots \int_0^1 \mathrm{d}\theta_k \; \mathring{\chi}_{\mathbf{B}}(\bar{\theta}; \mathbf{z}) \, \mathring{\chi}_{\mathbf{B}}(\bar{\theta}; \mathbf{z}'), \tag{86}$$

which is the proportion of the "table" T *not* covered by the "bricks" B and B', whose centres are at $\mathbf{z}$ and $\mathbf{z}'$. Similarly, we see that $\mathring{G}_{\mathbf{S}}(\mathbf{z}, \mathbf{z}')$ is the proportion of T *not* covered by the objects S and S' with centres at $\boldsymbol{x}$ and $\boldsymbol{x}'$. This shows that [compare (77)]

we have
$$0 \le \mathring{G}_{\mathbf{B}}(\mathbf{z}, \mathbf{z}') \le \mathring{G}_{\mathbf{Q}}(\mathbf{z}, \mathbf{z}') \le 1. \tag{87}$$

Finally, this yields that

$$\mathcal{E}\left[\left\{1 - C_{A}(N, \Theta)\right\}^2\right] \le \mathcal{E}\left[\left\{1 - C_{\mathbf{Q}}(N, \Theta)\right\}^2\right] \le 1. \tag{88}$$

As before, so long as B remains *inside* T, S will also be inside $\mathbf{T}$; and then, just as

$$\mathring{G}_{\mathbf{B}}(\mathbf{z}, \mathbf{z}') = 1 - 2\prod_{j=1}^{k} (2\mu_j) = 1 - \frac{2\omega}{N}. \tag{89}$$

so
$$\mathring{G}_{\mathbf{S}}(\mathbf{z}, \mathbf{z}') = 1 - \frac{2v_{\mathbf{Q}}}{A} = 1 - \frac{2\omega_{\mathbf{S}}}{N}. \tag{90}$$

Now, (89) and (90) will hold for $\mathbf{z}$ and $\mathbf{z}'$ in a region of $2k$-dimensional volume not less than

$$\prod_{j=1}^{k} (1 - 4\mu_j)^2 \geq 1 - \frac{4}{N^{1/k}} \sum_{j=1}^{k} \lambda_j = 1 - \frac{4\tau}{N^{1/k}}. \tag{91}$$

if N is large enough; and, in the remaining region, of volume not greater than $4\tau N^{-1/k}$, $\overset{o}{C}_{\mathbf{Q}}(\mathbf{z}, \mathbf{z}') \leq 1$; so that

$$\mathcal{E}\left[\left\{1 - C_{\mathbf{Q}}(N, \Theta)\right\}^2\right] = \left(1 - \frac{2\omega_{\mathbf{S}}}{N}\right)^N + O\left(\frac{1}{N^{1/k}}\right)$$

$$= e^{-2\omega_{\mathbf{S}}} + O\left(\frac{1}{N^{1/k}}\right). \tag{92}$$

Now, by (53), with (83) and (92),

$$\mathcal{V}ar\left[C_{\mathbf{Q}}(N, \Theta)\right] = O\left(\frac{1}{N^{1/k}}\right) \to 0. \tag{93}$$

Proceeding again as in obtaining Theorems 1 and 2, get:

THEOREM 3. *The random variable $C_{\mathbf{Q}}(N, \Theta)$ tends to $1 - e^{-\omega_{\mathbf{S}}}$* ***in quadratic mean****, and hence* ***in probability****, as $N \to \infty$.*

Proof. We have shown [see (83) and (93)] that, if

$$X_N = C_{\mathbf{Q}}(N, \Theta), \tag{94}$$

then $$\mu_N = \mathcal{E}[X_N] = \mathcal{E}\left[C_{\mathbf{Q}}(N, \Theta)\right] \to 1 - e^{-\omega_{\mathbf{S}}} \tag{95}$$

and $$\sigma_N^2 = \mathcal{V}ar[X_N] = \mathcal{V}ar\left[C_{\mathbf{Q}}(N, \Theta)\right] \to 0, \tag{96}$$

as $N \to \infty$. Thus, Lemma 1 applies to $C_{\mathbf{Q}}(N, \Theta)$ and so the theorem holds. QED.

Now consider the problem with the objects Q of shape $\mathbf{Q}$ *randomly oriented*, as well as randomly located. It is clear that, in any orientation, such a Q will be bounded by a hypercube K of side

$$v = 2\sqrt{2}\, \max_j \mu_j. \tag{97}$$

A perusal of the derivations for the problem of similarly-oriented objects [see (64) – (96)] shows that, if we replace every μ_j by ν, defined above, add $\frac{1}{2}k\,(k-1)$ rotational parameters to the indicator function χ to specify orientation, and integrate over these to obtain expected values; we can retrace our steps for the present problem. The exact formulae will change, somewhat, but the asymptotic results, embodied in (83), (92) and (93), will remain unchanged. It follows that we have:

THEOREM 4. *The random variable* $C_{\mathbf{g}}(N, \Theta)$ *tends to* $1 - e^{-\varpi \mathbf{s}}$ ***in quadratic mean**, and hence* **in probability**, *as* $N \to \infty$, *even when the objects of shape* $\mathbf{g}$ *are allowed to take random orientations (with an arbitrary probability distribution of orientations).*

APPENDIX A

An asymptotic expansion

LEMMA. As $N \to \infty$,

$$\left\{1 + \frac{x}{N} + \frac{y}{N^2} + O\left(\frac{1}{N^3}\right)\right\}^{aN+b}$$

$$= e^{ax}\left\{1 + \frac{1}{N}\left(ay + bx - \frac{1}{2}ax^2\right) + O\left(\frac{1}{N^2}\right)\right\}. \qquad \text{(A1)}$$

Proof. If $|\xi/N| < 1$, the series expansion

$$\log\left(1 + \frac{\xi}{N}\right) = \left(\frac{\xi}{N}\right) - \frac{1}{2}\left(\frac{\xi^2}{N^2}\right) + \frac{1}{3}\left(\frac{\xi^3}{N^3}\right) - \ldots + \frac{1}{m}\left(-\frac{\xi}{N}\right)^m + \ldots \qquad \text{(A2)}$$

is absolutely convergent. Thus, asymptotically as $N \to \infty$,

$$\log\left(1 + \frac{\xi}{N}\right) = \left(\frac{\xi}{N}\right) - \frac{1}{2}\left(\frac{\xi^2}{N^2}\right) + O\left(\frac{1}{N^3}\right). \qquad \text{(A3)}$$

Therefore,

$$(aN + b)\log\left(1 + \frac{\xi}{N}\right) = a\xi + \frac{1}{N}\left(b\xi - \frac{1}{2}a\xi^2\right) + O\left(\frac{1}{N^2}\right)$$

$$= a\xi + \log\left[1 + \frac{1}{N}\left(b\xi - \frac{1}{2}a\xi^2\right)\right] + O\left(\frac{1}{N^2}\right).$$

(A4)

since, by (A3), $\qquad \log\left(1 + \frac{\eta}{N}\right) = \left(\frac{\eta}{N}\right) + O\left(\frac{1}{N^2}\right).$

and we may put $\qquad \eta = b\xi - \frac{1}{2}a\xi^2.$

Thus, exponentiating (A4), we get that

$$\left(1+\frac{\xi}{N}\right)^{aN+b} = e^{a\xi}\left[1+\frac{1}{N}\left(b\xi-\tfrac{1}{2}a\xi^2\right)\right]\exp\left[O\left(\frac{1}{N^2}\right)\right]$$

$$= e^{a\xi}\left[1+\frac{1}{N}\left(b\xi-\tfrac{1}{2}a\xi^2\right)\right]\left[1+O\left(\frac{1}{N^2}\right)\right]$$

$$= e^{a\xi}\left[1+\frac{1}{N}\left(b\xi-\tfrac{1}{2}a\xi^2\right)+O\left(\frac{1}{N^2}\right)\right]. \tag{A5}$$

Now, let
$$\xi = x+\frac{y}{N}+O\left(\frac{1}{N^2}\right); \tag{A6}$$

then (A5) becomes

$$\left\{1+\frac{x}{N}+\frac{y}{N^2}+O\left(\frac{1}{N^3}\right)\right\}^{aN+b}$$

$$= \exp\left[ax+\frac{ay}{N}+O\left(\frac{1}{N^2}\right)\right]\left[1+\frac{1}{N}\left(bx-\tfrac{1}{2}ax^2\right)+O\left(\frac{1}{N^2}\right)\right]$$

$$= e^{ax}\exp\left[\frac{ay}{N}+O\left(\frac{1}{N^2}\right)\right]\left[1+\frac{1}{N}\left(bx-\tfrac{1}{2}ax^2\right)+O\left(\frac{1}{N^2}\right)\right]$$

$$= e^{ax}\left[1+\frac{ay}{N}+O\left(\frac{1}{N^2}\right)\right]\left[1+\frac{1}{N}\left(bx-\tfrac{1}{2}ax^2\right)+O\left(\frac{1}{N^2}\right)\right]$$

$$= e^{ax}\left[1+\frac{1}{N}\left(ay+bx-\tfrac{1}{2}ax^2\right)+O\left(\frac{1}{N^2}\right)\right],$$

which is (A1). QED.

COROLLARY. *As $N\to\infty$,*

$$\left\{1-\frac{\zeta}{N}\right\}^{N+c} = e^{-\zeta}\left\{1-\frac{1}{N}\left(c\zeta+\tfrac{1}{2}\zeta^2\right)+O\left(\frac{1}{N^2}\right)\right\}. \tag{A7}$$

Proof. Put $a = 1$, $b = c$, $x = -\zeta$, and $y = 0$, in (A1). QED.

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