On The Thickness of Graphs of Given Degree

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ABSTRACT

The results presented here refer to the determination of the thickness of a graph; that is, the minimum number of planar subgraphs into which the graph can be decomposed. A useful general, preliminary result obtained is Theorem 8: that a planar graph always has a planar representation in which the nodes are placed in arbitrary given positions. It is then proved that, if we have positive integers D and T, such that any graph of degree at most D has thickness at most T:

Theorem 9: any graph of degree d has thickness at most $T\,\mathrm{roof}\{(d+1)/D\}$;

Theorem 10: any graph of degree d can always be embedded in a regular graph G° of any degree $f \geq d$;

Corollary 5: any graph of degree d has thickness at most $\mathrm{roof}(d/2)$;

Theorem 12: with D and T defined as above, we have $D \leq 4T - 2$;

Corollary 6: if $T = 2$, then $D \leq 6$.

We further conjecture that, indeed, *the thickness of any graph of degree not exceeding 6 is never more than 2.*

Since the design and fabrication of VLSI computer chips is essentially a concrete representation of the planar decomposition of a graph, all these results are of direct practical interest.

DEDICATION

This paper is humbly and affectionately dedicated to my mother, Anne Halton, whose indomitable hope and courageous perseverance in the face of difficulty have been an admirable example to me throughout my life.

---- *sine qua non...*

DEFINITIONS

Let $N = \{v_1, v_2, \ldots, v_n\}$ be a finite set of *nodes* (or *vertices*) and write

$$\mathbf{L}(N) = \{\{x, y\}: x \in N \wedge y \in N \wedge x \neq y\} \qquad (1)$$

for the set of all possible *edges* (i.e., pairs of nodes). If $E \subseteq \mathbf{L}(N)$, we call

$$G = (N, E) = (\mathbf{N}(G), \mathbf{E}(G)) \qquad (2)$$

a *graph* (more precisely, an *undirected* graph), with $n = |N|$ nodes specified by $\mathbf{N}(G) = N$ and $e = |E|$ edges specified by $\mathbf{E}(G) = E$. If

$$P \subseteq N \quad \text{and} \quad F \subseteq E \cap \mathbf{L}(P) \qquad (3)$$

then we call the graph $H = (P, F)$ a *subgraph* of G and write $H < G$; moreover, if $P = N$ and $F \subseteq E$, we call H a *spanning subgraph* of G and write $H << G$.

To any node $x \in N$ will correspond a set

$$C_x = \{y \in N: \{x, y\} \in E\} \qquad (4)$$

of *neighbors* of x in G, and the number $\delta_x = |C_x|$ of nodes in C_x is called the *valency* of the node x; while

$$d = \mathbf{d}(G) = \max\{\delta_x: x \in N\} \qquad (5)$$

is called the *degree* of the graph G. By counting edges at each node, we see that (since each edge is counted at both ends)

$$2e = \Sigma_x \delta_x \leqslant nd. \qquad (6)$$

We call a node x *maximal* if $\delta_x = d$. Write

$$M = \mathbf{M}(G) = \{x \in N: \delta_x = d\} \qquad (7)$$

for the set of maximal nodes of G. If $M = N$ (that is, if every node of G is maximal), we call the graph G a *regular* graph.

The graph

$$G^c = (N, \mathbf{L}(N) \cap E^c), \qquad (8)$$

with the same nodes as G and precisely those edges which are possible but absent from G, is called the *complement* of G.

Given a graph $G = (N, E)$, if we can find a sequence

$$x = z_0, z_1, z_2, \ldots, z_t = y, \qquad (9)$$

all different; such that every pair

$$\{z_{s-1}, z_s\} \in E \ (s = 1, 2, \ldots, t), \qquad (10)$$

we call the sequence (10) of edges of G a *path* in G, *connecting* the nodes x and y, and *passing through* the nodes $z_1, z_2, \ldots, z_{t-1}$; and we say that the nodes x and y

are *connected* in G. A graph in which *every* pair of nodes is connected is called a *connected graph*.

Let $G = (N, E)$ be any graph. If w is a *divalent* node (of valency $\delta_w = 2$), such that $\{w, x\}$ and $\{w, y\}$ are the edges at w, and there is *no* edge $\{x, y\}$, then we may eliminate the node w and its two edges, and insert the edge $\{x, y\}$ instead, performing what is called a *homeomorphic contraction*. Conversely, if G has an edge $\{x, y\}$, we may remove this edge, and create a new divalent node w and edges $\{w, x\}$ and $\{w, y\}$, performing a *homeomorphic expansion*. If two graphs G and H are such that we can transform one into the other by a (finite) sequence of homeomorphic contractions and expansions, then we say that G and H are *homeomorphic*. It is easily verified that homeomorphism is an *equivalence relation*.

Denote the Euclidean plane by $\mathbf{R}^2$ and the set of all Jordan arcs in $\mathbf{R}^2$ by $\mathbf{J}$. Suppose that we can find one-to-one mappings

$$\mathbf{f}\colon N \to \mathbf{R}^2 \quad \text{and} \quad \mathbf{g}\colon E \to \mathbf{J}, \tag{11}$$

such that, if $\{x, y\} \in E$, (12)

then $\mathbf{f}(x) \neq \mathbf{f}(y)$, (13)

and $\mathbf{g}(x, y) = \mathbf{g}(\{x, y\})$ has end-points $\mathbf{f}(x)$ and $\mathbf{f}(y)$ (14)

and contains no other points of $\mathbf{f}(N)$, and finally, no two Jordan arcs in $\mathbf{g}(E)$ have any points other than perhaps one or both of their end-points in common (i.e., they do not *cross*). If this is the case, we refer to

$$\mathbf{G} = (\mathbf{f}(N), \mathbf{g}(E)) \tag{15}$$

as a *planar representation* of G, and if a graph G possesses any such planar representation, it is said to be a *planar graph*. Of course, a planar graph will have (infinitely) many planar representations. For example, Wagner, Fåry, and Stein have independently shown that every planar graph always has planar representations in which all the Jordan arcs representing edges of the graph are straight line segments [e.g., see Ore (1967), p. 6, or Harary (1969), p. 106].

For $i = 1, 2, \ldots, t$, let $H_i = (N, F_i)$ be spanning subgraphs of G, such that

$$F_i \cap F_j = \emptyset \text{ whenever } i \neq j, \quad \text{and} \quad \bigcup_{i=1}^{t} F_i = E; \tag{16}$$

then we call the set of graphs H_i a t-fold *decomposition* (or *factorization*) of G, and write

$$G = \sum_{i=1}^{t} H_i . \tag{17}$$

A graph G may have many decompositions of the form defined in (16) and (17). Imposing the further condition that all the subgraphs H_i be *planar*, we have a *planar decomposition* of G; and we shall refer to its planar subgraphs H_i as *laminae* of this decomposition. The smallest number t of laminae for which a t-fold planar decomposition of G exists is called the *thickness* of G and is denoted by $\theta(G)$. Evidently, if a graph G has m maximal connected *component* subgraphs G_j ($j = 1, 2,$

..., m), then

$$\theta(G) = \max_j \{\theta(G_j)\}; \tag{18}$$

so that it is sufficient to restrict our consideration of thickness to *connected graphs*. Indeed, we observe that a connected graph may further be divided into 2-*connected* components, any two of which have only one same such 2-component are connected by at least two paths with only the end-points in common (this is MENGER'S THEOREM). It is clear that (18) holds when the G_j are so-defined 2-components; and therefore, it is sufficient to restrict our consideration of thickness to 2-*connected graphs*.

Four special families of graphs will be needed here. First, we define the graph C_n to be *connected* and to have n nodes, *all divalent*, the edges thus forming a simple closed ring: this is a *cycle* of n nodes. In particular, any graph G is 2-connected if and only if each two of its nodes lie on some cycle which is contained in (i.e., is a subgraph of) G. We sometimes refer to a regular divalent spanning subgraph (consisting entirely, therefore, of disjoint cycles) as a "2-*factor*".

By contrast, any graph which is *connected* and contains *no cycles* (this is referred to as being *acyclic*) is called a (*free*) *tree*. It is easily verified that a tree with n nodes has exactly $n - 1$ edges, that the removal of any edge disconnects it, and that the addition of any edge creates a cycle. There will be at least two *univalent* nodes (with valency 1), and any such node is called a *leaf*.

We define the graph K_n to have n nodes and

$$\mathbf{E}(K_n) = \mathbf{L}(N); \tag{19}$$

so that all possible edges, $n(n - 1)/2$ in number, occur: this is the *complete graph* of n nodes. We may observe that K_n can always be decomposed into any graph G with n nodes and its complement G^c:

$$K_n = \Sigma\{G,\ G^c\} = G + G^c. \tag{20}$$

The graph $K_{n,n}$ has $2n$ nodes, partitioned into two sets of n:

$$\begin{aligned} N &= N_1 \cup N_2, \\ N_1 \cap N_2 &= \emptyset, \\ |N_1| &= |N_2| = n; \end{aligned} \tag{21}$$

and every node in N_1 is connected by an edge to every node in N_2:

$$\mathbf{E}(K_{n,n}) = \{\{x,\ y\}:\ x \in N_1 \wedge y \in N_2\}. \tag{22}$$

This is the *complete symmetric bipartite graph* of $2n$ nodes.

KNOWN RESULTS

Theorem 1 (KURATOWSKI). *A graph is planar if and only if it has no subgraph homeomorphic to* K_5 *or to* $K_{3,3}$.

[See Kuratowski (1930); as well as Baylis (1985), Berge (1962), Bondy and Murthy (1976), Harary (1969), Ore (1967), and Tutte (1963), who all give proofs.] We do not give a proof here, but point out the essential character of this result. A graph is prevented from being planar if and only if it contains:

(i) a cycle;

(ii) two "bridges" formed across this cycle (i.e., trees whose leaves lie on the cycle), such that, in a plane representation such as is defined in (11) - (15), in which the cycle is mapped into a simple closed contour, they would have to cross if they were placed both inside or both outside the cycle;

and (iii) a path connecting a node in one (interior) bridge with a node in the other (exterior) bridge.

It then remains to demonstrate that this can only occur in the situations shown in Figures 1 and 2, which are homeomorphic to $K_{3,3}$ and K_5, respectively.

Corollary 1. *Any graph of degree* 2 *or less is planar* (i.e., has thickness 1).

K_5 has degree 4; $K_{3,3}$ has degree 3. Thus, any graph of degree less than 3 cannot have a subgraph homeomorphic to either of the "Kuratowski graphs" necessary for non-planarity.

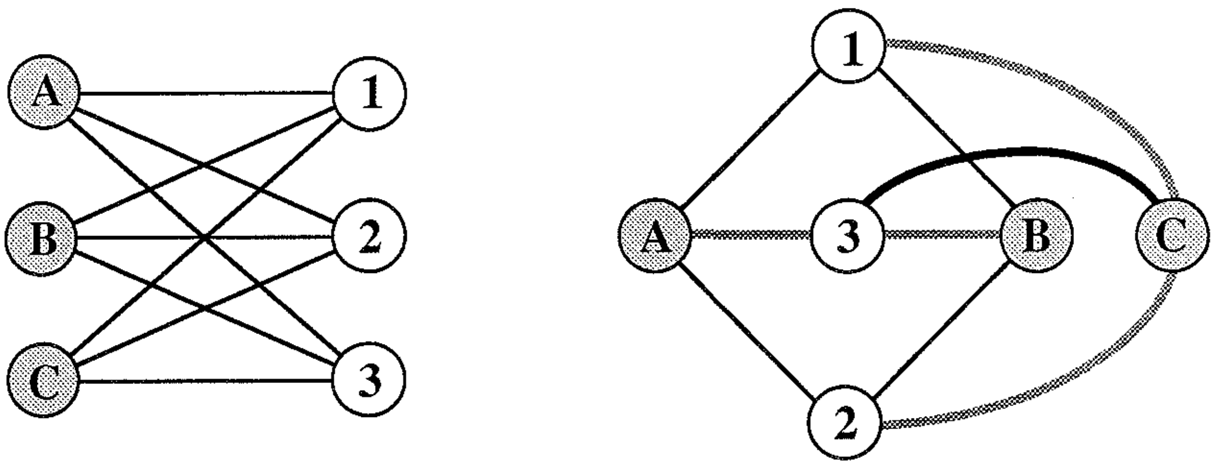
Theorem 2 (VIZING). *If a graph G has degree d, and if we seek to color its edges in such a way that no two edges incident on any given node are of the same color, then the minimum number of colors required to achieve this, the "edge chromatic number"* $c = \mathcal{C}(G)$ *of G satisfies the inequality,*

$$d \leq c \leq d + 1. \tag{23}$$

[See Vizing (1964); as well as Harary (1969), p. 133, and Ore (1967), p. 245, where a proof in English is given.]

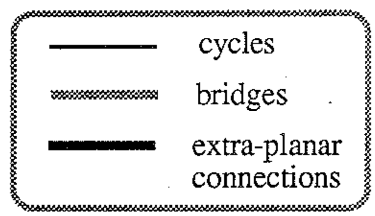
Theorem 3 (PETERSEN). *A connected graph G can be decomposed into edge-disjoint subgraphs* H_i*, each of which is entirely composed of disjoint cycles with all nodes divalent, if and only if G is regular and of even degree, say* $d = 2g$.

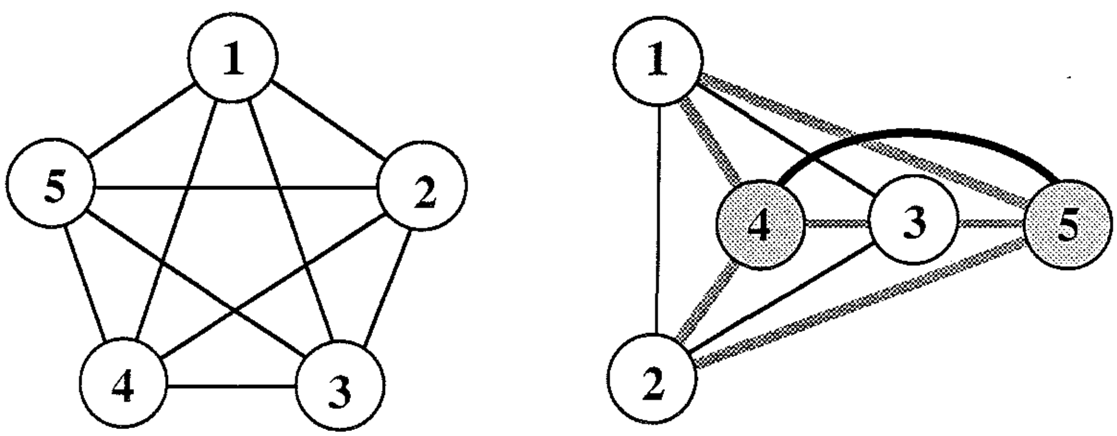
In other words, a connected graph is decomposable, as in (16) and (17), into "2-*factors*" (this is also expressed by saying that the graph is "2-*factorable*"), if and only if it is regular and of even degree. If G is *not* connected, it can be



Kuratowski graph $K_{3,3}$

Figure 1.





Kuratowski graph K_5

Figure 2.

2-factored if every connected component is regular and of even degree, but now different components may be of different even degrees. [See Petersen (1891) and Harary (1969), p. 90.]

Theorem 4 (BATTLE, HARARY, KODAMA, TUTTE). *When* K_n *is decomposed as in* (20), *with* $n \geq 9$, *then* G *and* G^c *cannot both be planar; but if* $n \leq 8$, *it is possible for both* G *and* G^c *to be planar.*

[See Battle, Harary, and Kodama (1962), and Tutte (1963a); as well as Beineke and Harary (1965), and Harary (1969), p. 120.] We have rephrased the theorem in our own terms.

Corollary 2. K_9 *has thickness* 3, *but any subgraph of* K_9 *has thickness only* 2 *or less:*

$$\text{(a) } \theta(K_9) = 3, \quad \text{(b) } \theta(K_9 - \text{edge}) = 2. \tag{24}$$

[See Harary (1969), p. 120, where these results are stated without proof.] Here, "K_9 - edge" denotes the graph obtained from K_9 by removing any one edge. The assertion (a) follows immediately from Theorem 4; but the proof of (b) is not so evident. However, the latter fact can be established by any example, and Figure 3 provides just such an example (we note that all nodes and all edges of K_n are topologically equivalent, for any n). This corollary implies that the graph K_9 is (locally) ***minimal*** for triplanarity (i.e., to have thickness 3).

Theorem 5 (BEINEKE, HARARY). *The thickness of the complete graphs is known in most cases:*

(a) *for* $n \neq 9$ *and* $n \not\equiv 4 \pmod 6$,

$$\theta(K_n) = \text{floor}[(n + 7)/6] = \text{floor}[(n + 1)/6] + 1 = \text{roof}\{(n + 2)/6\}; \tag{25}$$

(b) $\theta(K_9) = 3$;

(c) $\theta(K_4) = 1, \quad \theta(K_{10}) = 3, \quad \theta(K_{22}) = 4,$

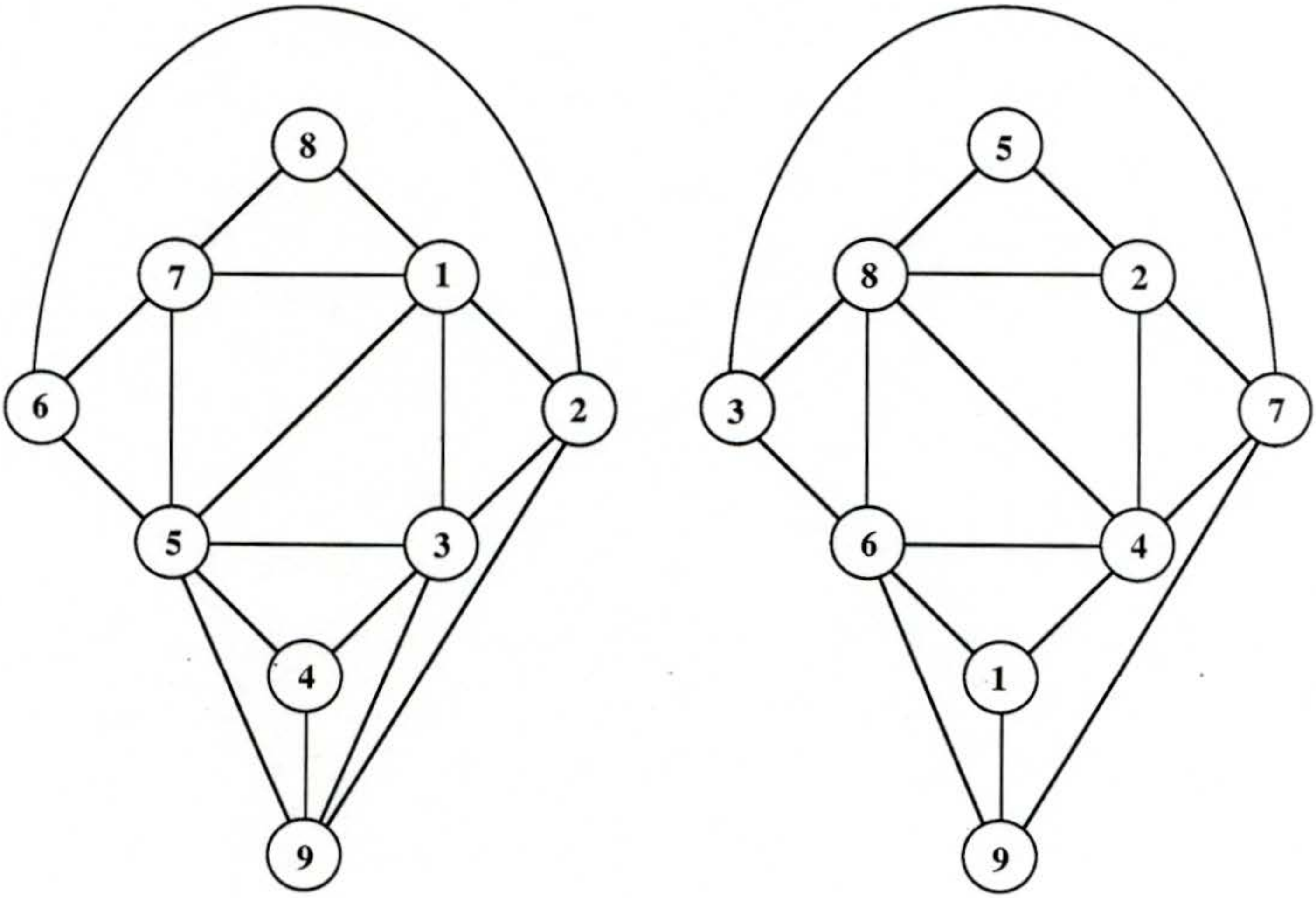
$\theta(K_{28}) = 5, \quad \theta(K_{34}) = 6, \quad \theta(K_{40}) = 7.$

[See Beineke and Harary (1965), and other references in Harary (1969), p. 120] It is conjectured in Harary (1969) that $\theta(K_{16}) = 4$, and that (25) holds for all $n > 16$. Note that (b) is just a repetition of Corollary 2 (a).

Theorem 6 (BEINEKE, HARARY, MOON). *The thickness of the complete symmetric bipartite graphs is known:*

$$\theta(K_{n,n}) = \text{floor}[(n + 5)/4] = \text{floor}[(n + 1)/4] + 1 = \text{roof}\{(n + 2)/4\}. \tag{26}$$

[See Beineke, Harary, and Moon (1964); as well as Harary (1969), p. 121; they actual give a more general result, covering most cases of $K_{m,n}$.]



Plane 1: 18 edges

Plane 2: 17 edges

Biplanar decomposition of K_9 - {8, 9}.

Figure 3.

The authors of Theorems 5 and 6 use the notation

$$[x] = \text{floor}(x), \quad \text{for the } \textit{integer infimum} \text{ of } x, \tag{27}$$

and
$$\{x\} = \text{roof}(x), \quad \text{for the } \textit{integer supremum} \text{ of } x; \tag{28}$$

we prefer the floor/roof notation as more intuitive; our use of [...] and {...} in (25) and (26) is *not* specific, but the choice was made in conformity with Beineke, Harary, and Moon's notation, so as not to confuse the reader further!] These authors state their result in the first forms of (25) and (26); their equivalence to the second forms is obvious; the last forms result from the easily-verified general result, that, if a and b are positive integers, then

$$\text{floor}(a/b) + 1 = \text{roof}\{(a+1)/b\}. \tag{29}$$

Corollary 3. $K_{7,7}$ *has thickness* 3; *but* $K_{6,6}$ *has thickness only* 2:

$$\text{(a) } \theta(K_{7,7}) = 3, \quad \text{(b) } \theta(K_{6,6}) = 2. \tag{30}$$

Theorem 7. $K_{7,7}$ *has thickness* 3, *but any subgraph of* $K_{7,7}$ *has thickness not greater than* 2:

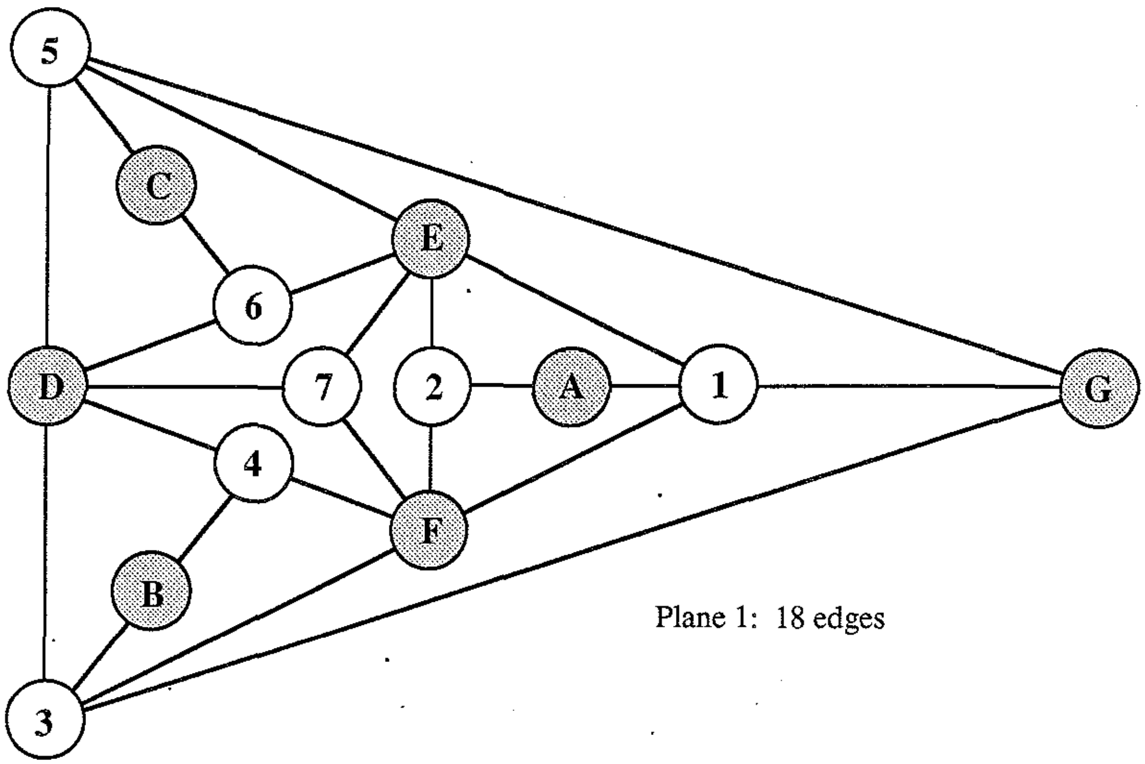
$$\text{(a) } \theta(K_{7,7}) = 3, \quad \text{(b) } \theta(K_{7,7} - \text{edge}) = 2. \tag{31}$$

Proof. Part (a) is a recapitulation of Corollary 3 (a). Since Corollary 3 (b) tells us that $\theta(K_{6,6}) = 2$, and since

$$K_{6,6} < K_{7,7} - \text{edge}, \tag{32}$$

i.e., $K_{6,6}$ is a subgraph of any graph obtained by removing one edge from $K_{7,7}$, we see that $\theta(K_{7,7} - \text{edge}) > 2$. Therefore, Part (b) can be proved simply by exhibiting a biplanar decomposition of $K_{7,7}$ - edge. This is provided by Figure 4, thus completing the theorem. *QED*

After completing this work, I informed Professor Lowell W. Beineke of some of my results, and he pointed out that the minimality of K_9 (see Corollary 2) and of $K_{7,7}$ (see Theorem 7), as well as that of $K_{5,13}$, were already demonstrated in the unpublished part of his 1965 doctoral thesis.



Biplanar decomposition of $K_{7,7}$ - {7, G}.

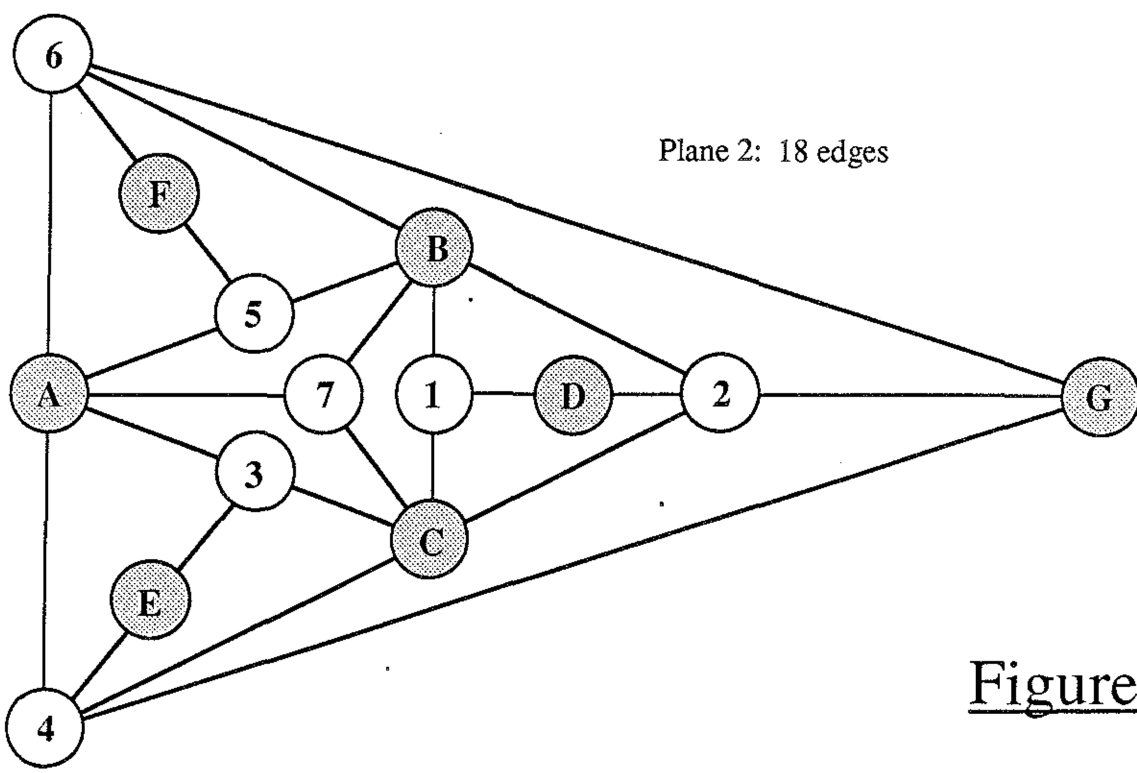


Figure 4.

NEW RESULTS

Theorem 8. *If a graph* $G = (N, E)$ *is planar, and if* $\mathbf{S}$ *is any given set of* $|\mathbf{S}| = n = |N|$ *points of* $\mathbf{R}^2$ *with a one-to-one mapping* $\mathbf{p}$: $N \to \mathbf{S}$; *then we can define a planar representation*

$$\mathbf{G}^* = (\mathbf{p}(N),\ \mathbf{q}(E)) \tag{33}$$

of G (as in (11) - (15)) *with the given mapping* $\mathbf{p}$ *for which*

$$\mathbf{p}(N) = \mathbf{S}. \tag{34}$$

Proof. Since G is planar, there exists a planar representation of the form (15). Let

$$N = \{v_1,\ v_2,\ \ldots,\ v_n\} \tag{35}$$

and $$\mathbf{f}(v_i) = \mathbf{u}_i \quad \text{and} \quad \mathbf{p}(v_i) = \mathbf{w}_i \in \mathbf{S}. \tag{36}$$

We now proceed inductively, by continuously deforming the planar representation $\mathbf{G}$ of (15), node by node, until we arrive at a planar representation $\mathbf{G}^*$ of the form (33). For $j = 1, 2, \ldots, n$, we suppose that nodes $\mathbf{u}_i$ with $0 < i < j$ have already been successfully moved to the respective required positions:

$$\mathbf{u}_i \to \mathbf{w}_i \quad \text{for} \quad 0 < i < j. \tag{37}$$

Thus, we have a planar representation of G with nodes at

$$\mathbf{w}_1,\ \ldots,\ \mathbf{w}_{j-1},\ \mathbf{u}_j,\ \ldots,\ \mathbf{u}_n, \tag{38}$$

and now seek to move

$$\mathbf{u}_j \to \mathbf{w}_j \tag{39}$$

in a continuous manner, keeping the representation of the graph planar. We use the facts that we may alter a Jordan arc (which may be thought of as the set of points in the complex plane $\mathbf{R}^2$,

$$\{\mathbf{Z}(t) = X(t) + \mathrm{i}\,Y(t):\ 0 \leqslant t \leqslant 1\}, \tag{40}$$

where X and Y are continuous functions) by:

(i) parallel translation of the entire arc:

$$\mathbf{Z}(t) \to \mathbf{Z}(t) + \mathbf{C}, \tag{41}$$

for some complex $\mathbf{C} = A + \mathrm{i}\,B$;

(ii) rigid rotation of the entire arc about any point in the plane; or of the segment parametrized by either

$$[0,\ p] = \{t:\ 0 \leqslant t \leqslant p < 1\} \tag{42}$$

or $$[p,\ 1] = \{t:\ 0 < p \leqslant t \leqslant 1\}, \tag{43}$$

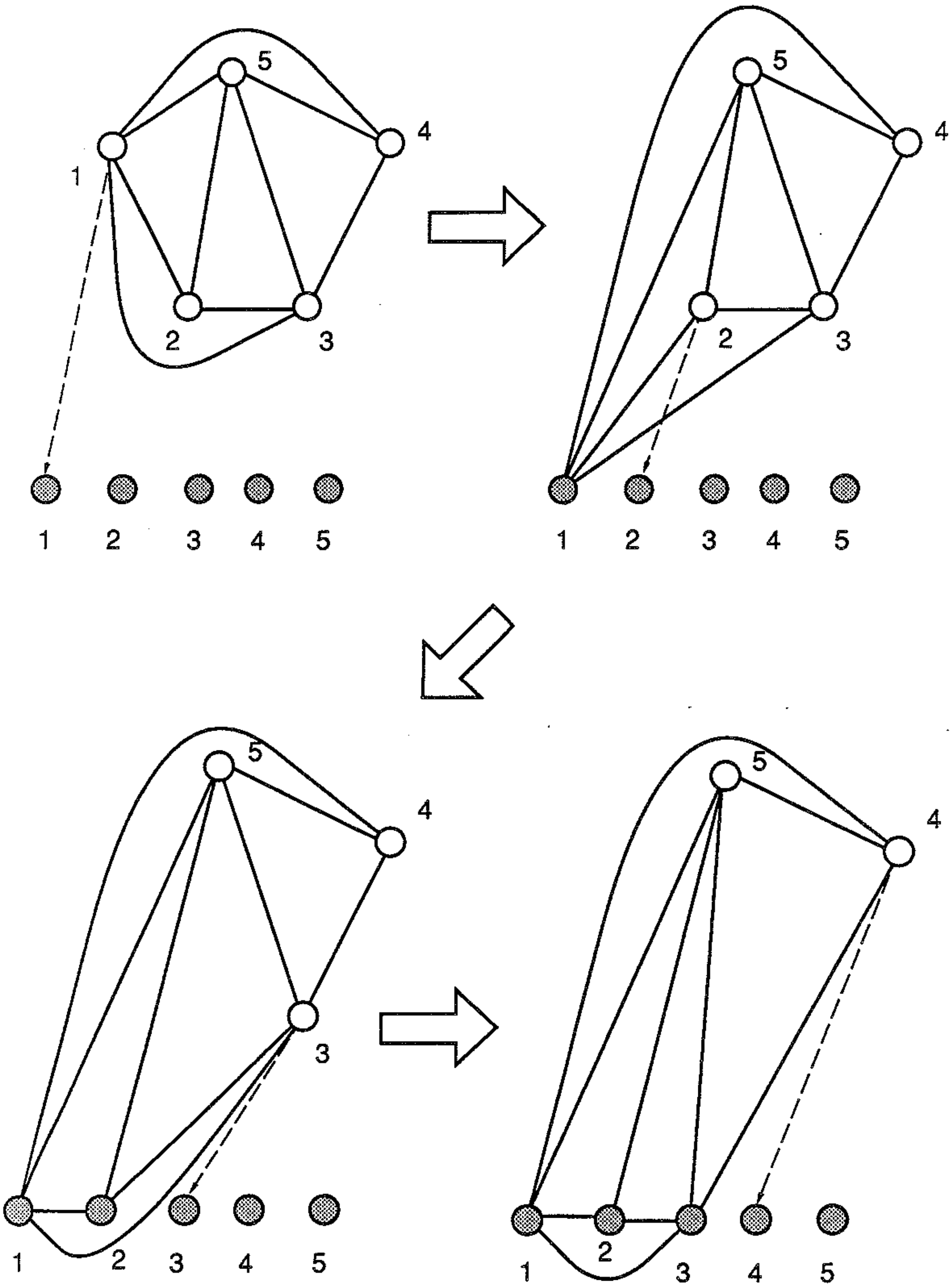


Figure 5.
(Part 1)

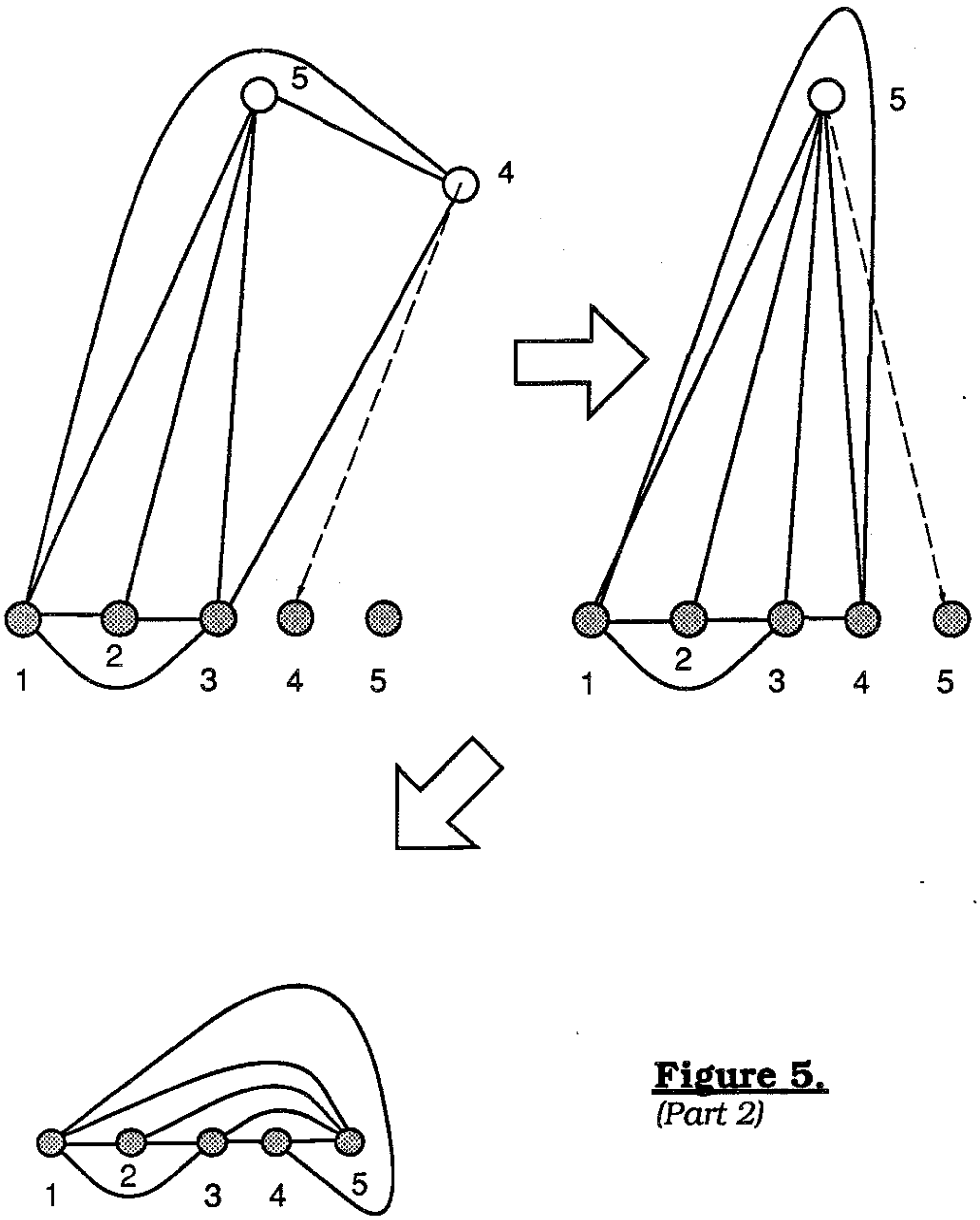


Figure 5.
(Part 2)

Figure 5 illustrates the construction used in demonstrating Theorem 8. The white nodes represent the points $\mathbf{u}_j$ (j = 1, 2, 3, 4, 5) of the initial planar representation, and the shaded nodes represent the points $\mathbf{w}_j$ (j = 1, 2, 3, 4, 5) of the given target set **S**. The dotted arrows indicate the successive translations $\mathbf{w}_j = \mathbf{p}(\mathbf{u}_j)$ of nodes from initial to target positions, $\mathbf{u}_j \to \mathbf{w}_j$.

about the end-point $\mathbf{Z}(p) = (X(p),\ Y(p))$; that is to say, if the center of rotation is $\mathbf{C}$, and the angle is s radians, then, for the appropriate range of the parameter t,

$$\mathbf{Z}(t) \to \mathbf{C} + e^{is}[\mathbf{Z}(t) - \mathbf{C}]; \tag{44}$$

(iii) "straightening" of any segment of the arc (parametrized by $[p,\ q]$), that is,

$$\begin{aligned}\mathbf{Z}(t) &\to [(q - t)\mathbf{Z}(p) + (t - p)\mathbf{Z}(q)]/(q - p)\\ &= \mathbf{Z}(p) + (t - p)[\mathbf{Z}(q) - \mathbf{Z}(p)]/(q - p),\\ &\qquad \text{for } 0 \leqslant p \leqslant t \leqslant q \leqslant 1;\end{aligned} \tag{45}$$

(iv) "stretching" of any (straight) segment of the arc by a factor $k > 0$: given that, for $t \in [p,\ q]$, $\mathbf{Z}(t)$ takes the form on the right of (45),

$$\begin{aligned}\mathbf{Z}(t) &\to \mathbf{Z}(p) + k(t - p)[\mathbf{Z}(q) - \mathbf{Z}(p)]/(q - p),\\ &\qquad \text{for } 0 \leqslant p \leqslant t \leqslant q \leqslant 1,\\ \mathbf{Z}(t) &\to \mathbf{Z}(t) + \mathbf{Z}(p) + k[\mathbf{Z}(q) - \mathbf{Z}(p)];\\ &\qquad \text{for } q \leqslant t \leqslant 1,\end{aligned} \tag{46}$$

(v) "bending" of any segment of the arc around a circular arc:

$$\begin{aligned}\mathbf{Z}(t) &\to \mathbf{C} + e^{is(t-p)/(q-p)}[\mathbf{Z}(p) - \mathbf{C}],\\ &\qquad \text{for } 0 \leqslant p \leqslant t \leqslant q \leqslant 1,\\ \mathbf{Z}(t) &\to \mathbf{Z}(t) - \mathbf{Z}(q) + \mathbf{C} + e^{is}[\mathbf{Z}(p) - \mathbf{C}],\\ &\qquad \text{for } q \leqslant t \leqslant 1,\end{aligned} \tag{47}$$

(vi) any combination of the above.

All these transformations are continuous (i.e., homeomorphic) and map Jordan arcs into Jordan arcs; so that their application retains the required properties of the planar representation, *so long as the resulting arcs do not cross*.

The nodes may be thought of as represented by rigid round pegs on an arbitrarily large flat board, and the edges by flexible, longitudinally elastic strings of fixed thickness, each attached at both ends to pegs. This may be formalized as follows. Let the minimum distance between any *non-coincident* pair of points $\mathbf{u}_i$ or $\mathbf{w}_i$ and $\mathbf{u}_j$ or $\mathbf{w}_j$ be ρ, and choose any $0 < \lambda < \rho$. The "pegs" are now taken to be circles of radius λ, centered at the representative points $\mathbf{u}_i$ or $\mathbf{w}_i$ of the corresponding nodes, and the "strings" are taken to be the envelopes of families of circles of the same radius, centered at every point (or, equivalently, at all points of rational parametric identification, if countability is relevant) of the representative Jordan arcs of the corresponding edges. It is clear that we may first use transformations (ii) - (v), without moving any pegs, to make all arcs representing edges consist only of concatenated straight segments and circular arcs. As the center of the peg (or circle) representing the j-th node moves across the board, from position $\mathbf{u}_j$ to position $\mathbf{w}_j$, in a "simple" Jordan arc (without loops; preferably a straight segment, or straight segments alternating with circular arcs), chosen so that this peg does not collide with any of the $n - 1$ stationary pegs representing all the other nodes of G, it "drags" all the strings representing its edges with it (by a combination of the transformations (i) - (v) above), clearly retaining the planarity of the representation, until some kind of a "collision" occurs. These collisions can only be of three kinds:

(a) the moving peg hits a stationary string,

(b) a moving string hits a stationary peg,

or (c) a moving string hits a stationary string.

We now deal with (a) by letting the moving peg "push" the string(s) in front of it (several strings may eventually "pile up" side-by-side in front of the moving peg, without crossing one-another). We deal with (b) by letting the moving string(s) "bend" (and stretch) around the stationary peg as they continue to be "pushed" by the moving peg. We deal with (c) by letting the moving string first bend so as to "lie parallel" to the stationary string, so long as this is possible, and then by letting it "push" the latter. Finally, it is possible that the positions of $\mathbf{w}_j$ and some $\mathbf{u}_i$ should coincide. If so, either

(d) $i < j$,

(e) $i = j$,

or (f) $i > j$.

In Cases (d) and (e), there is no problem: in (d), the i-th peg has already moved elsewhere; in (e), the j-th peg never moves at all. In Case (f), we allow the j-th peg to "push" the i-th peg (with its strings) sufficiently for the former to take its rightful place. Thus, we see that, in every case, the movement (39) may be completed in a smooth manner without relinquishing the planarity of the representation. This completes the inductive step; induction now proves the theorem. ***QED***

The importance of this theorem is in the application of the planar-decomposition theorems to practical problems, such as the design of VLSI "chips" for computer components. A chip consists of several superimposed flat layers, insulated from one-another, in which "nodes", consisting of electronic gates and other processing elements, are connected by conducting "edges". The construction is such that each edge lies entirely in one layer, no two edges may cross in a single layer, and each node may be thought of as lying in all layers at the same point (like a peg perpendicular to all layers and accessible to all of them). Now suppose that a chip design is given as a graph, and that this graph has a t-fold planar decomposition, which is to be used as the basis for physical fabrication of the chip. Each lamina of the decomposition is a planar graph, and so has a planar representation; but it is essential to the practical application that all the planar representations should have the nodes in the same positions. Theorem 8 tells us that the choice of planar representation for each lamina can indeed be made after the nodes are arbitrarily positioned (in the present situation, in the same way in all laminae).

Note, too, that this theorem has been used implicitly to enable the almost-planar representations of $K_{3,3}$ and of K_5 in Figures 1 and 2, respectively, to be made simple and evident, and to permit the biplanar representations of K_9 - edge and $K_{7,7}$ - edge in Figures 3 and 4, respectively, to be shown in a clear and highly symmetric form.

Theorem 9. *Suppose that we are given positive integers D and T, such that any graph G of degree at most D has thickness at most T; that is, such that,*

$$\text{if} \quad d = \mathbf{d}(G) \leqslant D, \quad \text{then} \quad \theta(G) \leqslant T. \tag{48}$$

Then, for any graph G of arbitrary degree d,

$$\theta(G) \leqslant T\,\mathrm{roof}\{(d+1)/D\}. \tag{49}$$

We note that, when $d < D$, $\mathrm{roof}\{(d+1)/D\} = 1$, and so (48) and (49) agree; while, for $d = D$, $\mathrm{roof}\{(d+1)/D\} = \mathrm{roof}\{(D+1)/D\} = 2$, so that (48) is stronger than (49).

Proof. We appeal to Theorem 2, which tells us that, using at most $d+1$ colors, we can certainly color the edges of G so that no two edges incident on any given node have the same color. This means that we can arbitrarily partition these colors into $m = \mathrm{roof}\{(d+1)/D\}$ sets, each of at most D colors. Now decompose G into m subgraphs H_i, in such a way that H_i has all its edges colored from the i-th set of at most D colors; then each of these subgraphs has degree no greater than the number of edge-colors used in the subgraph, and so, by our hypothesis, not greater than D; whence its thickness is at most T, by (48). The thickness of G (the minimum total number of laminae required) is then no greater than the total number, mT, of planar laminae generated by our construction; and result (49) now follows at once, completing the theorem. ***QED***

Corollary 4 (PLAISTED). *If a graph G has degree d, then*

$$\theta(G) \leqslant \mathrm{roof}\{(d+1)/2\}. \tag{50}$$

Proof. By Corollary 1, the relation (48) holds for $D = 2$ and $T = 1$. Therefore, by (49), the corollary follows. ***QED***

[Private communication; see Acknowledgements.]

Theorem 10. *Given a graph G of degree d and any integer $f \geqslant d$, we can always construct a regular graph G^0 of degree f, containing G as a subgraph.*

Proof. We seek to bring the valency of every node x of G up to a value f, not less than δ_x. For each node x, therefore, there is a number

$$\varepsilon_x = f - \delta_x \geqslant 0 \tag{51}$$

of "free valencies" to be combined. First, if $\varepsilon_x > 0$, $\varepsilon_y > 0$, and $\{x, y\}$ is *not* an edge of G, we add the edge $\{x, y\}$ to G and reduce ε_x and ε_y appropriately by one; and this can be repeated until no more such edges are possible. Call the resulting augmented graph G^+ (clearly, $G \ll G^+$, since no nodes have been added or removed and edges have only been added). We may now be left with *no* free valencies (i.e., all $\varepsilon_x = 0$), in which case we take $G^o = G^+$ and the construction is complete; or there may remain some free valencies, in which case we proceed as follows.

Let $z \in G^+$. If the adjusted value of $\varepsilon_z \leqslant 1$, do nothing. If $\varepsilon_z \geqslant 2$, construct a copy the Kuratowski complete graph K_f (which has f nodes, all of valency $f - 1$) and give each of its nodes an additional valency (for a total valency of f at each of these f new nodes). Now add this new graph to G^+, and add ε_z edges, connecting the node z to ε_z nodes of this K_f, and leaving $f - \varepsilon_z$ nodes, each with a single free valency. Repeat this procedure with every such z of degree 2 or more. We call this further-augmented graph $G^>$. Now construct a second copy of the augmented graph $G^>$, say $G^<$, and pair-off corresponding ">" and "<" free valencies (necessarily equal in number) of the two identical graphs into additional edges. The "K-graph" construction ensures that no two such new edges connect the same pair of nodes, since all nodes with free valencies are now univalent. The final result is a regular graph G^o of degree f, containing the original graph G as a subgraph. ***QED***

Corollary 5. *If a graph G has degree d, then*

$$\theta(G) \leqslant \mathrm{roof}(d/2). \tag{52}$$

Proof. Let
$$f = 2\ \mathrm{roof}(d/2); \tag{53}$$

then f is *even*, and equals either d or $d + 1$. By Theorem 10, we can always construct a regular graph G^o of degree f, which has G as a subgraph. By Theorem 3, we can then decompose G^o into (by Corollary 1, planar) 2-factors, and clearly there will be just $f/2 = \mathrm{roof}(d/2)$ such 2-factors. This is a planar decomposition of G^o, and, *ipso facto*, a planar decomposition of the subgraph G. Thus, the thickness of G cannot exceed that of G^o, which cannot itself exceed $\mathrm{roof}(d/2)$. ***QED***

Theorem 11. *If T_d denotes the supremum of the thicknesses of all graphs of degree d, then*

$$T_d \geqslant \mathrm{roof}\{(d + 2)/4\}. \tag{54}$$

Proof. By Theorem 6, equation (26), since $\mathbf{d}(K_{d,d}) = d$,

$$\theta(K_{d,d}) = \mathrm{roof}\{(d + 2)/4\}; \tag{55}$$

so that the supremum T_d must be greater than or equal to this value, and the theorem is proved. ***QED***

Theorem 12. *Let D and T be defined, as in Theorem 9, by the relation* (48); *then we have that*

$$D \leqslant 4T - 2. \tag{56}$$

Proof. By (48), if $d \leqslant D$, then $\theta \leqslant T$; so $T_d \leqslant T$. Thus, by Theorem 11,

$$(\forall d \leqslant D) \quad T \geqslant T_d \geqslant \mathrm{roof}\{(d+2)/4\}. \tag{57}$$

Therefore,

$$T \geqslant \mathrm{roof}\{(D+2)/4\}; \tag{58}$$

and it is easily verified that $D = 4T - 2$ gives the right-hand side of (58) the value $\mathrm{roof}(T) = T$, which satisfies (58) (as does, of course, any smaller value of D); while $D = 4T - 1$ gives the same term the value $\mathrm{roof}\{(4T+1)/4\} = T + 1$, which does *not* satisfy (58) (nor does, of course, any larger value of D). The theorem follows immediately. ***QED***

Corollary 6.

$$\textit{If} \quad T = 2, \quad \textit{then} \quad D \leqslant 6. \tag{59}$$

Proof. The result is immediate, by direct substitution in (56). ***QED***

CONCLUDING REMARKS

The most fundamental results presented here are Theorems 8, 9, 10, and 12. Theorem 8 is a formalization of what everyone must have been doing for years; but I have not found it in the published literature. The practical importance of Theorem 8 in facilitating VLSI chip design is indicated in the text. Of greatest practical importance are the results of Corollaries 5 and 6.

Important problems remain; notably, the tightening of the degree-thickness relation. Examination of many examples very strongly suggests the following conjectures:

Conjecture 1. *Any graph of degree not exceeding* 6 *has thickness not exceeding* 2; *that is, the bound of Corollary 6 is attained.*

Conjecture 2. *Any graph of degree not exceeding D has thickness not exceeding* roof$\{(D+2)/4\}$; *that is, the bounds of Theorems 11 and 12 are attained.*

Of great practical interest is the question of designing *efficient algorithms* to create planar decompositions of graphs, of minimal or, failing this, near-minimal thickness. [See Booth and Lueker (1976), Even and Tarjan (1976), Hopcroft and Tarjan (1974), Lempel, Even, and Cederbaum (1967), and the comprehensive review in Even (1979).] It appears that some of the linear-time techniques developed for planarity-testing could be effectively adapted to create the required algorithm.

Beyond this, there arises the knotty problem of generalizing Kuratowski's theorem to multi-planar graphs. Some progress has been made here, but not to a publishable point.

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