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MINI-COURSE ON PROBABILITY & STATISTICS

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1. SETS AND PROBABILITY SPACES

1.1. SETS

A **set** is a collection of objects, its *elements* or *members*. Usually denote sets by capital letters (A, B, C, ..., X, Y, Z) and elements by lower-case letters (a, b, c, ..., x, y, z). We say that an element x *belongs to*, or *is a member of*, or simply *is in* the set A, and we write $x \in A$. If not, we write $x \notin A$. A set may be defined by evaluation: if p, q, and r are in the set, we write it as $\{p, q, r\}$. Alternatively, it may be defined by the truth of an assertion: the set of all objects z such that a statement $\mathcal{A}(z)$ *about* z is true is written $\{z \mid \mathcal{A}(z)\}$ or $\{z: \mathcal{A}(z)\}$. [Note: *repetition* or *rearrangement* of elements does not change a set; e.g., $\{x, y, z, z\}$ is the same set as $\{y, z, x, x, x\}$.]

A set may be represented in several ways. When two such representations do denote the same set, we say they are *equal*, and write, e.g.,

$$\left.\begin{aligned} \{x, y, z, z\} &= \{y, z, x, x, x\}, \\ K = \{1, 2\} &= \{s \mid s^2 - 3s + 2 = 0\}; \end{aligned}\right\} \qquad (1)$$

in the latter example, "$s^2 - 3s + 2 = 0$" is an example of an assertion $\mathcal{K}(s)$ about s. If two sets are *not* the same, we write $A \neq B$.

If every member of a set X is also a member of another set Y, we say that X is *contained* in Y or that X is a *subset* of Y and also that Y *contains* X or that Y is a *superset* of X. We write $X \subseteq Y$ or $Y \supseteq X$. [Note: this includes the possibility that $X = Y$; if not, we write $X \subset Y$ or $Y \supset X$ and say that X is a *proper* subset of Y and that Y is a *proper* superset of X. However, very unfortunately, many people write $\subset$ and $\supset$ for what we denote by $\subseteq$ and $\supseteq$: watch for this!]
example,

$$\left.\begin{aligned} \{s, q\} &\subseteq \{p, q, r, s, t\}, \\ \{y, z, z\} &\subseteq \{w, x, y, y, z\}, \\ \{x, y, z, z\} &\supseteq \{y, z, x, x, x\} \supset \{x, y, y\}. \end{aligned}\right\} \qquad (2)$$

We note that we can always write the tautology, definition by assertion,

$$S = \{t: \ t \in S\}, \qquad (3)$$

and observe the distinction between an object q and its *singleton* set $\{q\}$: while $q \in \{p, q, r\}$, $\{q\} \subset \{p, q, r\}$.

The set which has *no* members is called the **empty** set and is denoted by $\emptyset$. For all sets X, $\emptyset \subseteq X$. The set of *all* objects under consideration is called the *universe of discourse* or the *global* set; it will be denoted by W. For all sets Y, $W \supseteq Y$.

An *assertion about* x, $\mathcal{A}(x)$, may be thought of as a *function* or *mapping* of W into the set {'true', 'false'} = {T, F} = {1, 0} (with '1' denoting 'true', and '0', 'false'; but, sometimes, the opposite convention is used!) We may then write

$$\emptyset = \{x \mid \mathrm{F}\}, \qquad W = \{x \mid \mathrm{T}\}. \tag{4}$$

We may now define, among several other ways of combining assertions, the following (defined by tabulating their truth-values).

NOT $\mathcal{A}$ = $\mathcal{A}$:

$\mathcal{A}$	$\sim\mathcal{A}$
T	F
F	T

; $\mathcal{A}$ OR $\mathcal{B}$ = $\mathcal{A} \vee \mathcal{B}$:

$\mathcal{A}$ → / $\mathcal{B}$ ↓	T	F	$\mathcal{B}$
	T	T	T
	T	F	F

; $\mathcal{A}$ AND $\mathcal{B}$ = $\mathcal{A} \wedge \mathcal{B}$:

$\mathcal{A}$	T	F	$\mathcal{B}$
	T	F	T
	F	F	F

IF $\mathcal{A}$ THEN $\mathcal{B}$ = $\mathcal{A} \Rightarrow \mathcal{B}$:

$\mathcal{A}$	T	F	$\mathcal{B}$
	T	T	T
	F	T	F

; $\mathcal{A}$ NOR $\mathcal{B}$ = $\mathcal{A} \downarrow \mathcal{B}$:

$\mathcal{A}$	T	F	$\mathcal{B}$
	F	F	T
	F	T	F

(5)

From these definitions, we obtain a whole algebraic system, called **Boolean algebra**, with many theorems; among these, the following are true for all assertions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$:

1 $\sim\mathrm{T} = \mathrm{F}$, **2** $\sim\mathrm{F} = \mathrm{T}$, **3** $\sim(\mathcal{A} \vee \mathcal{B}) = (\sim\mathcal{A}) \wedge (\sim\mathcal{B})$, **4** $\sim(\mathcal{A} \wedge \mathcal{B}) = (\sim\mathcal{A}) \vee (\sim\mathcal{B})$, **5** $\mathcal{A} \Rightarrow \mathcal{B} = (\sim\mathcal{A}) \vee \mathcal{B}$, **6** $\mathcal{A} \vee \mathcal{B} = \mathcal{B} \vee \mathcal{A}$, **7** $\mathcal{A} \wedge \mathcal{B} = \mathcal{B} \wedge \mathcal{A}$, **8** $(\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C} = \mathcal{A} \vee (\mathcal{B} \vee \mathcal{C})$, **9** $(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C} = \mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$, **10** $(\mathcal{A} \vee \mathcal{B}) \wedge \mathcal{C} = (\mathcal{A} \wedge \mathcal{C}) \vee (\mathcal{B} \wedge \mathcal{C})$, **11** $(\mathcal{A} \wedge \mathcal{B}) \vee \mathcal{C} = (\mathcal{A} \vee \mathcal{C}) \wedge (\mathcal{A} \vee \mathcal{C})$, **12** $\sim\mathcal{A} = \mathcal{A} \downarrow \mathcal{A}$, **13** $\mathcal{A} \vee \mathcal{B} = (\mathcal{A} \downarrow \mathcal{B}) \downarrow (\mathcal{A} \downarrow \mathcal{B})$, **14** $\mathcal{A} \wedge \mathcal{B} = (\mathcal{A} \downarrow \mathcal{A}) \downarrow (\mathcal{B} \downarrow \mathcal{B})$, **15** $\mathcal{A} \downarrow \mathcal{B} = (\sim\mathcal{A}) \wedge (\sim\mathcal{B})$, **16** $\sim(\mathcal{A} \downarrow \mathcal{B}) = \mathcal{A} \vee \mathcal{B}$, **17** $\sim(\mathcal{A} \Rightarrow \mathcal{B}) = \mathcal{A} \wedge (\sim\mathcal{B})$, ... (6)

These may be proved by logic, substitutional manipulation, examination of *Venn diagrams* (in which intersecting ovals represent sets: see below), or from *truth tables*; a proof of (6.**17**) is shown below.

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}\Rightarrow\mathcal{B}$	$\sim(\mathcal{A}\Rightarrow\mathcal{B})$	$\sim\mathcal{B}$	$\mathcal{A}\wedge(\sim\mathcal{B})$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

The first two columns show all possible combinations of truth-values for $\mathcal{A}$ and $\mathcal{B}$. Each side of the identity to be proved is then evaluated, using the defining tables in (5). The final values are shown in columns marked ¦...¦. The identity of these columns proves the required result.

Corresponding to the Boolean algebra of assertions (or *propositional calculus*), there is a Boolean algebra of sets. We begin with the form (3), and define the **union** of sets P and Q as the set of objects belonging to either P OR Q:

$$P \cup Q = \{u\colon (u \in P) \vee (u \in Q)\}, \tag{7}$$

and the **intersection** of the two sets as the set of objects belonging to both P AND Q:

$$P \cap Q = \{v\colon (v \in P) \wedge (v \in Q)\}, \tag{8}$$

and the **complement** of P as the set of objects NOT in P:

$$P^c = \{w\colon \sim(w \in P)\} = \{x\colon x \notin P\}. \tag{9}$$

It is customary to omit the sign for intersection $\cap$ and just write PQ for $P \cap Q$. As in arithmetic, convention allows us to give priority to ^{c} over $\cap$ and to $\cap$ over $\cup$, so as to reduce the number of parentheses needed. We observe that the assertion, "$x \in A$ is true [for all objects x]", is equivalent to $A = W$; and, similarly, the assertion, "$x \in A$ is false", is equivalent to $A = \emptyset$. Thus, formally, by (6.5), $\{x|\ (x \in P) \Rightarrow (x \in Q)\} = \{x|\ \sim(x \in P) \vee (x \in Q)\} = \{x|\ \sim(x \in P)\} \cup \{x|\ x \in Q\}$ $= P^c \cup Q$. More importantly, the assertion that $P^c \cup Q = W$ is denoted by $P \subseteq Q$! [The assertion states that *for all* x, *if* x *is in* P, *it is also in* Q.]

We may now obtain the equivalents of the identities (6) in set-notation:

$$\mathbf{1}\ W^c = \emptyset,\ \mathbf{2}\ \emptyset^c = W,\ \mathbf{3}\ (P \cup Q)^c = P^cQ^c,\ \mathbf{4}\ (PQ)^c = P^c \cup Q^c,\ \mathbf{5}\ P \subseteq Q \equiv P^c \cup Q = W,$$
$$\mathbf{6}\ P \cup Q = Q \cup P,\ \mathbf{7}\ PQ = QP,\ \mathbf{8}\ (P \cup Q) \cup R = P \cup (Q \cup R) = P \cup Q \cup R,$$
$$\mathbf{9}\ (PQ)R = P(QR) = PQR,\ \mathbf{10}\ (P \cup Q)R = PR \cup QR,\ \mathbf{11}\ PQ \cup R = (P \cup R)(Q \cup R), \tag{10}$$

and so on. In particular, we may add that $\sim(\sim\mathcal{A}) = \mathcal{A}$,

$$((\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{B} \Rightarrow \mathcal{A})) \equiv (\mathcal{A} \equiv \mathcal{B}) \quad \text{and} \quad ((\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{B} \Rightarrow \mathcal{C})) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{C}); \tag{11}$$

whence $(P^c)^c = P$,

$$((A \subseteq B) \wedge (B \subseteq A)) \equiv (A = B) \quad \text{and} \quad ((A \subseteq B) \wedge (B \subseteq C)) \Rightarrow (A \subseteq C). \tag{12}$$

We may now extend the concepts of unions and intersections (using the *commutative* and *associative* rules in (10.6 – 9)) to larger collections of sets. To do this, we use logical *quantifiers*: $(\forall x \in G)\ \mathcal{S}(x)$ is read "for *all* x in G, $\mathcal{S}(x)$ [is true]"; $(\exists r \in H)\ \mathcal{S}(r)$ is read "for *some* r in H, $\mathcal{S}(r)$ [is true]" or "there *exists* an r in H, such that $\mathcal{S}(r)$".

$$\bigcup_{i=1}^{k} A_i = \{x\colon (\exists i \in \{1, 2, \ldots, k\})\ x \in A_i\}, \tag{13}$$

$$\bigcap_{i=1}^{k} A_i = \{x\colon (\forall i \in \{1, 2, \ldots, k\})\ x \in A_i\}; \tag{14}$$

this extends to infinite countable collections of sets:

$$\bigcup_{i=1}^{\infty} A_i = \{x\colon (\exists i \in \{1, 2, \ldots\})\ x \in A_i\}, \tag{15}$$

$$\bigcap_{i=1}^{\infty} A_i = \{x\colon (\forall i \in \{1, 2, \ldots\})\ x \in A_i\}; \tag{16}$$

and then to general collections of sets:

$$\cup_{t\in J} A_t = \{x\colon (\exists t \in J)\ x \in A_t\}, \tag{17}$$

$$\cap_{t\in J} A_t = \{x\colon (\forall t \in J)\ x \in A_t\}. \tag{18}$$

In every case, we see that results such as (10.**3**,**4**,**10**,**11**) hold:

$$(\cup_{t\in J} A_t)^c = \cap_{t\in J} A_t^{\,c}, \quad (\cap_{t\in J} A_t)^c = \cup_{t\in J} A_t^{\,c}, \tag{19}$$

$$(\cup_{t\in J} A_t)B = \cup_{t\in J} (A_t B), \quad (\cap_{t\in J} A_t) \cup B = \cap_{t\in J} (A_t \cup B). \tag{20}$$

Finally, we define the **Cartesian product** of two sets X and Y as the set of all *ordered pairs* $[x, y]$ with $x \in X$ and $y \in Y$, and we write it as

$$\begin{aligned} X \times Y &= \{[x, y]\colon (x \in X) \wedge (y \in Y)\} \\ &= \{(f\colon \{1, 2\} \to W) \mid (f(1) \in X) \wedge (f(2) \in Y)\}. \end{aligned} \tag{21}$$

The latter version is easily seen to be equivalent to the former, but seems perverse; however, here again, a generalization may be made as follows.

$$\times_{t\in J} A_t = \{(f\colon J \to W) \mid (\forall t \in J)\ f(t) \in A_t\}. \tag{22}$$

[In both (21) and (22), the notation $f\colon Q \to R$ refers to a *function* or *mapping*, mapping elements of the set Q into elements of R.]

1.2. EVENTS

We may classify the sets which we shall be using in probability theory into:

(a) *Finite sets* (e.g., throw a die and get one of {1, 2, 3, 4, 5, 6}; generally, sets which can be represented by {1, 2, ..., k});

(b) *Countable sets* (e.g., unbounded but finite sequences of die-throws, yielding either x_1 or $[x_1, x_2]$ or $[x_1, x_2, x_3]$ or ..., where each $x_i \in \{1, 2, ..., 6\}$; generally, sets which can be represented by {1, 2, ...});

(c) *Uncountable sets which are compact* (e.g., the unit interval [0, 1]; the unit k-cube, $\{[x_1, x_2, ..., x_k]: (\forall i \in \{1, 2, ..., k\})\ 0 \leqslant x_i \leqslant 1\}$; sets in metric spaces which are *bounded*, such as ellipsoids or spheres in k-dimensional Euclidean space);

(d) *Uncountable, non-compact sets* (e.g., the real line, or half-line; unbounded sets in metric spaces, such as all (or half, or an orthant) of k-space.

All will be treated essentially in the same way; only, where *sums* appear in cases (a) and (b), *integrals* [predominantly *Lebesgue* and *Lebesgue-Stieltjes* integrals] must be used in cases (c) and (d); with the added complication of *infinite* range in cases (b) and (d).

A **sample space** is simply a set representing the *possible outcomes* of a statistical experiment, each outcome being an element of the set [e.g., in throwing a die, we get the sample space $D = \{1, 2, 3, 4, 5, 6\}$, each of the six elements being the number on the uppermost face of the die; if instead we throw a dart at a circle, the sample space becomes the disk $C = \{(x, y) \mid x^2 + y^2 \leqslant a^2\}$... if we may assume that the dart will hit the disk! ...; but if the circle is a dart-board, then we may wish to restrict the sample space to the possible scores, $C_D = \{1, 2, ..., 20, 25\} \cup \{2, 4, 6, ..., 40, 50\} \cup \{3, 6, 9, ..., 60\}$; a *random generator* usually gives a random value x with $0 \leqslant x \leqslant 1$; but if the generator is implemented on a digital computer, there are only a finite number of available numbers (floating-point) between 0 and 1; an error in voltage is usually assumed to be a random number on the entire real line (though with extremely low probability of being very large); the same assumption is usually made, somewhat less reasonably, for a test-score or length-measurement: the first example is in case (a), the second, (c) (or perhaps (d)), the third, (a); do the rest of the list].

An **event** is a subset of the sample space considered to be "observable" for the purposes of the experiment. Just as the sample space is not uniquely defined (e.g., we disregard the orientation of a die when tossing it to make a score; we showed how the sample space differed, according to what we did with the throw of a dart, and so on), so the class of possible events is at our disposal, to a certain extent. However, this class is always defined so that it has the following properties (taking S as the universe of discourse W) :

(1) the empty set $\emptyset$ is in the class $\boldsymbol{S}$;

(2) the entire sample space S is in the class $\boldsymbol{S}$;

(3) if a set E is in $\boldsymbol{S}$, then its *complement* E^c in S [SE^c] is in $\boldsymbol{S}$ too;

(4) if a *countable* collection of sets E_i are all in $\boldsymbol{S}$, then their *union* $\cup_{i=1}^{\infty} E_i$ is in $\boldsymbol{S}$ too.

Since $\cap_i E_i = (\cup_i E_i^c)^c$, we see that the equivalent of (4) holds for all countable *intersections* also.

A class $\boldsymbol{S}$ satisfying the axioms (1) - (4) is called a σ-**algebra** (or σ-*field*), pronounced a "*sigma-algebra*". In a σ-algebra of events, we can talk of "the event that one of a (countable) class of events occurs" or "the event that all of a (countable) class of events occurs" or even "the event that infinitely many of a (countable) class of events occur", and remain in the σ-algebra. This permits us to do the necessary mathematics. The limitation to countable classes, in (4) etc., does not cramp the practical applications, while assisting the mathematics.

In a *finite* sample space, we can take the σ-algebra of events to be simply the (finite) class of all subsets of S (the so-called *power-class* of S); if S has k members, $\boldsymbol{S}$ will then have 2^k elements (events). [When in doubt, think about this situation!]

1.3. PROBABILITY

A **probability** is a particular case of what is called a *measure*. Given a sample space S with a σ-algebra $\boldsymbol{S}$ of events, let μ be a function mapping $\boldsymbol{S}$ into the real line $\boldsymbol{R}$ [note that the elements which μ maps into real values are *events*, sets in $\boldsymbol{S}$; *not* outcomes, elements of the sample space S]. Then μ is called a measure on $\boldsymbol{S}$ iff [i.e., "if and only if"]

(1) for all E in $\mathbf{S}$, $\mu(E) \geqslant 0$; [sometimes this is relaxed]

(2) in particular, $\mu(\emptyset) = 0$;

(3) for any countable collection of **disjoint sets** E_i (for which $i \neq j$ implies $E_i E_j = \emptyset$) in $\mathbf{S}$,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i); \tag{23}$$

[this is referred to as the *disjoint additivity* property]. If, in addition, $\mu(S)$ is finite [we say that μ is *totally finite*], it is a simple matter of scaling to set

(4) in particular, $\mu(S) = 1$,

and then we call μ a probability on the sample space. We call the complete structure, consisting of $(S, \mathbf{S}, \mu)$ a **probability space**, or sometimes the term "sample space" is used for this, too.

2. RANDOM VARIABLES, DISTRIBUTIONS, INDEPENDENCE

2.1. RANDOM VARIABLES

Consider an arbitrary function φ mapping S, the sample space of a probability space $(S, \mathbf{S}, \mu)$, into the real line $\mathcal{R}$. For each outcome $s \in S$, φ yields a real number $\varphi(s)$. Let I_α denote the semi-infinite interval $(-\infty, \alpha)$, for any $\alpha \in \mathcal{R}$:

$$I_\alpha = \{x \in \mathcal{R}:\ x < \alpha\}. \tag{24}$$

Then we write $\varphi^{-1}(I_\alpha)$ for the *inverse image* under φ of the set I_α, the subset of S whose images under φ in $\mathcal{R}$ lie in I_α:

$$\varphi^{-1}(I_\alpha) = \{s \in S:\ \varphi(s) < \alpha\}. \tag{25}$$

We call φ a **random variable** iff, for all real α,

$$\varphi^{-1}(I_\alpha) \in \mathbf{S}. \tag{26}$$

It is readily verified that there is a σ-algebra of subsets of the real line $\mathcal{R}$, which is the intersection of all the σ-algebras containing all the I_α; it is called the σ-algebra of *Borel* and will be denoted by $\mathcal{B}$. We can also easily check that, for any subsets J and K_r of $\mathcal{R}$,

$$\varphi^{-1}(\emptyset) = \emptyset, \quad \varphi^{-1}(\mathcal{R}) = S, \quad \varphi^{-1}(J^c) = (\varphi^{-1}(J))^c, \quad \varphi^{-1}\left(\bigcup_{r=1}^{\infty} K_r\right) = \bigcup_{r=1}^{\infty} \varphi^{-1}(K_r); \tag{27}$$

whence we see that (by the axioms (1) - (4) of a σ-algebra), if (26) holds, then

$$(\forall K \in \mathcal{S})\ \varphi^{-1}(K) \in \boldsymbol{S}. \tag{27}$$

The structure $(\mathcal{R}, \mathcal{S}, \mu\varphi^{-1})$, where $\mathcal{R}$ is the real line (range of the function φ), $\mathcal{S}$ is the class of Borel sets defined above, and $\mu\varphi^{-1}$ is the function such that

$$(\forall K \in \mathcal{S})\ (\mu\varphi^{-1})(K) = \mu(\varphi^{-1}(K)), \tag{28}$$

is called the **distribution** of the random variable φ. It is easily seen that the distribution of a *r.v.* [random variable] is itself a probability space, with $\mu\varphi^{-1}$ the probability (satisfying the axioms (1) - (4) of a probability).

Further, since (26) suffices to generate all of (27); i.e., the $\mu\varphi^{-1}(I_\alpha)$ determine the $\mu\varphi^{-1}(K)$, so to speak; it follows that we may limit our consideration to the function

$$F_\varphi(\alpha) = \mu\varphi^{-1}(I_\alpha) = \mu(\{s \in S \mid \varphi(s) < \alpha\}). \tag{29}$$

This function, non-decreasing with increasing α (and left-continuous; if the $<$ is replaced by $\leqslant$, it becomes right-continuous), is called the **cumulative distribution function** [**c.d.f.**] of φ. [Note: where F_φ has zero slope over an interval C, $\mu\varphi^{-1}(C) = 0$; where the slope is positive, there is positive probability; and where F_φ has a (positive) jump-discontinuity, there is positive probability *at a point*; the last situation occurs when the distribution is *discrete*, e.g., the dice-score with $S = \{1, 2, 3, 4, 5, 6\}$, $\boldsymbol{S} = \{\emptyset, \{1\}, \{2\}, \ldots, \{6\}, \{1, 2\}, \{1, 3\}, \ldots, \{5, 6\}, \{1, 2, 3\}, \{1, 2, 4\}, \ldots \ldots, \{1, 2, 3, 4, 5\}, \ldots, \{2, 3, 4, 5, 6\}, S\}$ ($2^6 = 64$ sets in all), $\mu\varphi^{-1}(\{i\}) = p_i$ ($i = 1, 2, \ldots, 6$); then, if an event K is determined by $(\omega_1, \omega_2, \ldots, \omega_6)$, with $\omega_i = 1$ if i is in the set, $= 0$ otherwise, $\mu\varphi^{-1}(K) = \Sigma_{i=1}^{6}\omega_i p_i$, so that $\mu\varphi^{-1}(S) = \Sigma_{i=1}^{6} p_i = 1$ (generally, we try to make all the p_i equal, which implies that every $p_i = 1/6$ and $\mu\varphi^{-1}(K) = \Sigma_{i=1}^{6}\omega_i/6$); now F_φ takes on the appearance of a staircase, rising from height 0 to the left to height 1 to the right, with steps of height p_α at each of the points $\alpha = 1, 2, \ldots, 6$.]

While mixed distributions do occasionally occur; most commonly, we have to do with *discrete* distributions [cases (a) and (b) of §1.2] or *continuous* distributions [cases (c) and (d)]. Discrete distributions have a finite or countably infinite set of points α_i ($i = 1, 2, \ldots$) at which there is *probability* p_i, with

$$\sum_i p_i = 1 \qquad \text{and} \qquad F_\varphi(\alpha) = \sum_{i:\alpha_i<\alpha} p_i. \tag{30}$$

Continuous distributions [the corresponding probability is said to be *absolutely continuous*] have a **probability density** $\rho(\alpha)$ [called the *Radon-Nikodym derivative* of the probability $\mu\varphi^{-1}$] given by

$$\rho(\alpha) = \frac{d}{d\alpha}F_{\varphi}(\alpha), \tag{31}$$

with
$$\int_{-\infty}^{+\infty} dx\ \rho(x) = 1 \quad \text{and} \quad F_{\varphi}(\alpha) = \int_{-\infty}^{\alpha} dx\ \rho(x). \tag{32}$$

[Note: if we are in case (a) or (c), the sums or integrals in (30) and (32) become finite, instead of infinite; this is sometimes described by saying that the dsitribution has *finite support* (in case (c) or (d), the "support" is the set of values for which ρ is positive (i.e., non-zero)).]

It is easily seen that (i) a *constant* function is a r.v.; (ii) any *constant multiple* $c\varphi$ of a r.v. φ is a r.v.; (iii) any *linear combination* $\Sigma_i\, c_i\varphi_i$ of r.v. φ_i is a r.v. [To prove that $\varphi + \psi$ is a r.v., see that if $\varphi(s) + \psi(s) < \alpha$, then $\varphi(s) = \xi$, $\psi(s) = \alpha - \xi - \varepsilon$, with $\varepsilon > 0$; whence there is a rational number q, such that $\xi < q < \xi + \varepsilon$, for which $\varphi(s) < q$ and $\psi(s) < \alpha - q$; and therefore, by definition,

$$[\varphi + \psi]^{-1}(I_{\alpha}) \subseteq P_{\alpha} = \cup_{q\in Q}\ \varphi^{-1}(I_q)\psi^{-1}(I_{\alpha-q}), \tag{33}$$

where Q is the (countable) set of rationals. Now, $\varphi^{-1}(I_{\alpha})$ and $\psi^{-1}(I_{\beta})$ are in $\boldsymbol{S}$, by (26), since φ and ψ are supposed to be r.v.; so that P_{α}, being obtained from these sets by intersections and countable union, is itself in the σ-algebra $\boldsymbol{S}$. Further, if $s \in P_{\alpha}$, there is a $q \in Q$ for which $\varphi(s) < q$ and $\psi(s) < \alpha - q$, whence $\varphi(s) + \psi(s) < \alpha$, putting s in $[\varphi + \psi]^{-1}(I_{\alpha})$, which therefore equals P_{α}]; (iv) the *product* of r.v. φ_i is a r.v.; (v) the *maximum* and *minimum* of r.v. φ_i are r.v.; (vi) a *simple function* is one which takes a finite number of distinct values, each on a set in $\boldsymbol{S}$ (these sets necessarily being disjoint and forming a partition of the sample space S); any linear combination or product of simple functions is also a simple function, and any simple function is a r.v.; (vii) any *non-negative valued* r.v. is the (pointwise) limit of a nondecreasing sequence of simple r.v. [For each $p = 1, 2, \ldots$, partition the positive values in steps of 2^{-p} to 2^{p}, by $J_{p,q} = \{x\colon (q-1)2^{-p} \leqslant x < q2^{-p}\}$ for $q = 1, 2, \ldots, 2^{2p}$ and $J_{p,0} = \{x\colon x \geqslant 2^{p}\}$. Define the simple r.v. τ_p by putting

$$\tau_p(s) = (q-1)2^{-p} \quad \text{if} \quad s \in \varphi^{-1}(J_{p,q}), \quad \text{else } \tau_p(s) = 2^{p}. \tag{34}$$

Each $J_{p,q} \in \mathcal{S}$ ($J_{p,q} = I_\alpha I_\beta^{\text{c}}$ with $\alpha = q2^{-p}$ and $\beta = (q-1)2^{-p}$), so each $\varphi^{-1}(J_{p,q}) \in \boldsymbol{S}$; whence the τ_p are indeed simple r.v. Clearly, for every s, $\tau_p(s) \leqslant \varphi(s)$. Since $\varphi(s) \geqslant 0$ for all s, $\varphi(s)$ must lie in one of the $J_{p,q}$, for each choice of p. If $\varphi(s) \in J_{p,0}$, $\varphi(s) \geqslant 2^{-p} = 2 \times 2^{-p-1}$; so that $\tau_p(s) \leqslant \tau_{p+1}(s)$. If, instead, $\varphi(s) \in J_{p,q}$, then $(q-1)2^{-p} \leqslant \varphi(s) < q2^{-p}$, so that $(2q-2)2^{-p-1} \leqslant \varphi(s) < 2q2^{-p-1}$; whence $\tau_p(s) = (q-1)2^{-p}$ and $\tau_{p+1}(s) \geqslant (2q-2)2^{-p-1} = \tau_p(s)$.]; (viii) the *positive part* $\varphi_+(s) = \max\{\varphi(s), 0\}$ and the *negative part* $\varphi_-(s) = -\min\{\varphi(s), 0\}$ of any r.v. φ are r.v. and are non-negative, and $\varphi(s) = \varphi_+(s) - \varphi_-(s)$; (ix) the *supremum*, *infimum*, *lim sup*, and *lim inf* of r.v. φ_i are r.v., and so, if a (poinwise) limit exists, it is a r.v.; (x) any *continuous function* is a r.v. [In order for this to have meaning, we need to have a *topology* on S, such that all the *open sets* of the topology are in $\boldsymbol{S}$. If so, since a function is continuous iff the inverse images of open sets in $\mathcal{R}$ are open in S, it suffices to show that all open sets in $\mathcal{R}$ are in $\mathcal{S}$. Since all open sets in $\mathcal{R}$ are countable unions of open intervals, and since the open interval $(\alpha, \beta) = I_\beta(\cap_{n=1}^\infty I_{\alpha+1/n})^{\text{c}}$, we see that this is indeed the case.]

All this is to confirm that all the usual algebraic and limiting operations may be applied to r.v., leaving them as r.v., and all the simpler functions are r.v.

2.2. CONDITIONAL DISTRIBUTIONS AND INDEPENDENCE

Consider a probability space $(S, \boldsymbol{S}, \mu)$ and let $E \in \boldsymbol{S}$ be an arbitrary event. The structure $(E, \boldsymbol{S}_E, \mu_E)$ is a probability space if $\boldsymbol{S}_E$ is a σ-algebra of subsets of E and if μ_E is a probability on $\boldsymbol{S}_E$. Let

$$\boldsymbol{S}_E = \{EX \mid X \in \boldsymbol{S}\}. \tag{35}$$

Then, since $\emptyset = E\emptyset$, $E = ES$ (since $E \subseteq S$), $E(EX)^{\text{c}} = E(E^{\text{c}} \cup X^{\text{c}}) = EE^{\text{c}} \cup EX^{\text{c}} = \emptyset \cup EX^{\text{c}} = EX^{\text{c}} = E(SX^{\text{c}})$, and $\cup_{i=1}^\infty EX_i = E(\cup_{i=1}^\infty X_i)$, we see that all four axioms of a σ-algebra are satisfied. Now for all $X \in \boldsymbol{S}$, let

$$\mu_E(EX) = \mu(EX)/\mu(E). \tag{36}$$

Then all axioms of a probability are easily verified; notably, $\mu_E(E) = \mu(E)/\mu(E) = 1$ and $\mu_E(\cup_{i=1}^\infty EX_i) = \mu(\cup_{i=1}^\infty EX_i)/\mu(E) \leqslant \Sigma_{i=1}^\infty \mu(EX_i)/\mu(E) = \Sigma_{i=1}^\infty \mu_E(EX_i)$.

The probability space $(E, \mathcal{S}_E, \mu_E)$ so defined is called the space **conditional** on E. We call $\mu_E(EX)$ the *probability of* X, *given* E or *conditional on* E, and commonly write

$$\mu_E(EX) = \mu(X| E). \tag{37}$$

In a finite sample space, such as the die-throwing experiment mentioned earlier, we may, e.g., take $E = \{1, 2, 3, 4\}$, "the score is less than 5", and then $\mu(E) = 4/6 = 2/3$; so that $\mu(\{i\}| E) = \frac{1}{6}/\frac{4}{6} = \frac{1}{4}$ for every $i \in E$. This is intuitively reasonable (the relative probability of outcomes in E should not be affected by the rejection of data outside E).

Any random variable φ on the original probability space will have a *conditional distribution* with respect to E, namely $(\mathcal{R}, \mathcal{S}, \mu_E\varphi_E^{-1})$, with

$$F_\varphi(\alpha| E) = \mu_E\varphi_E^{-1}(I_\alpha) = \mu(\{e \in E\colon \varphi(e) < \alpha\})/\mu(E), \tag{38}$$

where we write φ_E^{-1} for the inverse image by φ *restricted to* E:

$$\varphi_E\colon E \to \mathcal{R}, \quad \text{with} \quad (\forall e \in E)\ \varphi_E(e) = \varphi(e). \tag{39}$$

If two sets X and Y in $\mathcal{S}$ are such that

$$\mu(XY) = \mu(X)\mu(Y), \tag{40}$$

then we say that the sets are [**statistically**] **independent**; and we note that, then

$$\mu(X| Y) = \mu(X) \quad \text{and} \quad \mu(Y| X) = \mu(Y), \tag{41}$$

by (36) and (37). Thus, the probability of X, given Y, is the same as the probability of X, given S (i.e., given no information about the occurrence or not of the event Y): this certainly satisfies the intuitive concept of independence. This may be extended to collections of sets; the sets X_i are *collectively* [statistically] *independent* iff, for any finite subcollection $\{X_{i_1}, X_{i_2}, \ldots, X_{i_s}\}$,

$$\mu\left(\bigcap_{r=1}^{s} X_{i_r}\right) = \prod_{r=1}^{s} \mu(X_{i_r}). \tag{42}$$

Let

$$S = \times_{t\in J} S_t, \tag{43}$$

where $(S_t, \mathcal{S}_t, \mu_t)$ are probability spaces for all $t \in J$. Write

$$\mathcal{S} = \bigotimes_{t\in J} \mathcal{S}_t \tag{44}$$

for the intersection (itself a σ-algebra) of all the σ-algebras containing the Cartesian product (compare (22))

$$\times_{t\in J} \boldsymbol{S}_t = \{(R\colon J \to \cup_{t\in J} \boldsymbol{S}_t) \mid (\forall t \in J)\ R(t) \in \boldsymbol{S}_t\}. \tag{45}$$

Now it is possible to prove that, if one has a probability defined on the product (45), it can be extended uniquely to a probability on the class (44).

In particular, it is possible to define a probability μ on (45) by saying that, if the function R maps infinitely many of the t into proper subsets of S_t, then $\mu(R) = 0$, while if only a finite number of them, say $t_1, t_2, \ldots, t_m$ do not map into S_t itself, then

$$\mu(R) = \prod_{h=1}^{m} \mu_{t_h}(R(t_h)), \quad \text{if} \quad R(t) \subset S_t \text{ only if } t = t_1, t_2, \ldots, t_m. \tag{46}$$

This probability, extended to $\boldsymbol{S}$ is then called the **product probability**. Now consider the event R; this corresponds to an experiment in which there are many trials (each identified by the index t in the set J, whatever it is), and in which we observe that, in the trial indexed t_h, the event $R(t_h)$ occurs, for $h = 1, 2, \ldots, m$; while no information is given for the outcome of the other trials. The adoption of the product probability then corresponds to the *statistical independence* of the trials.

If the individual probability spaces $(S_t, \boldsymbol{S}_t, \mu_t)$ are identical, being instances of the same trial, then we can write our Cartesian products as *Cartesian powers*

$$S = \underset{\cdots}{S}^{\times J}, \qquad \boldsymbol{S} = \underset{\cdots}{\boldsymbol{S}}^{\otimes J}, \qquad \mu = \underset{\cdots}{\mu}^{\otimes J}, \tag{47}$$

with the dotted underline denoting the previously-indexed, now-identical sets. This situation is referred to as *repeated independent trials* or just **Bernoulli trials**. The probability formula (46) then simplifies to

$$(\forall m)\ \mu(R) = \prod_{h=1}^{m} \underset{\cdots}{\mu}(R(t_h)), \quad \text{if} \quad R(t) \subset \underset{\cdots}{S} \text{ only if } t = t_1, t_2, \ldots, t_m. \tag{48}$$

For an example, we return to the dice-throwing experiment (either in the form of N successive throws of the same die, or of the simultaneous throw of N identical dice, so as to have no interference between their motions), and ask for the probability of getting just m scores of a 3 or a 4. The probability of this single event is $2/6 = 1/3$. Thus, by (48), the probability of at least a specified m throws yielding this event is $(\frac{1}{3})^m$; but the probability that these will be the *only* occurrences of the event is $(\frac{1}{3})^m(1 - \frac{1}{3})^{N-m} = 2^m/3^N$. Now, the m throws in question may be selected from the total of N throws in just $\binom{N}{m}$ ways, and each of the resulting events is disjoint from all the others; so the additivity property (3) of probabilities tells us that the probabilities of the $\binom{N}{m}$ alternative events is just the sum of their individual probabilities (which are all equal); whence the overall probability is $\binom{N}{m}2^m/3^N$. In relation to our previous formalism, we have $\underset{\cdots}{E} = \{3, 4\} \subset \underset{\cdots}{S}$, $\mu(\underset{\cdots}{E}) = 1/3$, $\mu(\underset{\cdots}{E}^c) = 1 - \mu(\underset{\cdots}{E}) = 2/3$ [since $\underset{\cdots}{E}$ and $\underset{\cdots}{E}^c$ are disjoint and $\underset{\cdots}{E} \cup \underset{\cdots}{E}^c = \underset{\cdots}{S}$; so $\mu(\underset{\cdots}{E}) + \mu(\underset{\cdots}{E}^c) = \mu(\underset{\cdots}{E} \cup \underset{\cdots}{E}^c) = \mu(\underset{\cdots}{S}) = 1$], with dotted underlines referring to the single-throw probability space; and $\mu(R) = (\frac{1}{3})^m$ when $R(t_h) = \underset{\cdots}{E}$ for $h = 1, 2, \ldots, m$, while $\mu(R') = (\frac{1}{3})^m(\frac{2}{3})^{N-m}$ when $R'(t_h) = \underset{\cdots}{E}$ for the same t_h and $R'(t) = \underset{\cdots}{E}^c$ for all other $t \in \{1, 2, \ldots, N\}$; finally, $R'R'' = \emptyset$ unless $t_1' = t_1''$, $t_2 = t_2''$, $\ldots$, $t_m' = t_m''$ (i.e., $R' = R''$).

This last example generalizes slightly to an important distribution, the **binomial distribution**, in which we have a sequence of N Bernoulli trials in which we pick an event $\underset{\cdots}{E}$, with probability $\mu(\underset{\cdots}{E}) = p$, and $q = 1 - p$. The same reasoning as above gives, in the most general case, that

$$\mu[\text{there are exactly } m \text{ occurrences of } \underset{\cdots}{E}] = \mu(R') = \binom{N}{m}p^m q^{N-m}, \tag{49}$$

where we write $\mu[\mathcal{G}]$ to mean "the probability *that* $\mathcal{G}$ *is true*". Note that this is a *finite discrete distribution*, with support at $m = 0, 1, 2, \ldots, N$, despite the possible infinity and continuity of the underlying product or even single-trial probability space. The fact is that, while $\underset{\cdots}{S}$ may be very large and complicated, we effectively use only the simple σ-algebra $\mathbf{S}^{*}_{\underset{\cdots}{E}} = \{\emptyset, \underset{\cdots}{E}, \underset{\cdots}{E}^c, \underset{\cdots}{S}\}$.

2.3. SPECIAL DISTRIBUTIONS.

We have just seen one special distribution, the binomial. Another which is very important, but is continuous and unbounded, is the **normal distribution**. Here, the probability density is proportional to $\exp(-\frac{1}{2}x^2)$. Since, as is readily verified,

$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{1}{2}x^2} = \sqrt{2\pi}, \tag{50}$$

we see that the probability density must be

$$\rho_{0,1}(\alpha) = e^{-\frac{1}{2}\alpha^2}/\sqrt{2\pi}. \tag{51}$$

A simple change of coordinates shows that, if

$$x = \frac{t - m}{\sqrt{v}}, \tag{52}$$

the probability density becomes

$$\rho_{m,v}(\beta) = e^{-(\beta-m)^2/2v}/\sqrt{2\pi v}. \tag{53}$$

This density-curve is centered and symmetric about the value m and tends smoothly to zero as $\beta \to \pm\infty$. The "width" of the central peak is proportional to $\sqrt{v}$.

A different change of coordinates,

$$x = \sqrt{u}, \tag{54}$$

yields the (*one-degree of freedom*) **chi-squared distribution** with density

$$\rho_{\chi,1}(\gamma) = e^{-\frac{1}{2}\gamma}/\sqrt{2\pi\gamma}. \tag{55}$$

Note that the square root now contains the variable γ, not a parameter of the distribution.

Two r.v. φ and ψ are said to be *independent* iff the sets $\varphi^{-1}(I_\alpha)$ and $\psi^{-1}(I_\beta)$ are statistically independent for all real α and β. This means that (by (29), (36), (37), (38), (40))

$$\mu[\varphi(s) < \alpha \mid \psi(s) < \beta] = \mu[\varphi(s) < \alpha], \tag{56}$$

agreeing with the notion expressed by (41). Then the *conditional c.d.f.* of φ is

$$F_\varphi(\alpha \mid \psi < \beta) = F_\varphi(\alpha \mid \psi^{-1}(I_\beta)) = \mu[\varphi(s) < \alpha]/\mu(\psi^{-1}(I_\beta))$$
$$= F_\varphi(\alpha)/F_\psi(\beta). \tag{57}$$

Also, $\mu[\varphi(s) \in (\alpha, \alpha+\delta) \wedge \psi(s) \in (\beta, \beta+\varepsilon)] = \mu[\varphi(s) \in (\alpha, \alpha+\delta)]\mu[\psi(s) \in (\beta, \beta+\varepsilon)] = [F_\varphi(\alpha+\delta) - F_\varphi(\alpha)][F_\psi(\beta+\varepsilon) - F_\psi(\beta)]$. Thus the *joint probability density* of φ and ψ is

$$\rho_{\varphi,\psi}(\alpha, \beta) = \rho_\varphi(\alpha)\rho_\psi(\beta). \tag{58}$$

The extension to several r.v. is immediate.

If we take f independent (0, 1)-normal r.v., with density (51), we get a joint probability density

$$\rho_f(\xi) = e^{-\frac{1}{2}\xi^2}/(2\pi)^{-\frac{1}{2}f}, \tag{59}$$

where ξ denotes the real vector $(\xi_1, \xi_2, \ldots, \xi_f)$, and $\xi^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_f^2$. If we now put $u = \xi^2$ (compare (54)), it can be shown that this r.v. has density

$$\rho_{\chi,f}(u) = e^{-\frac{1}{2}u}\, u^{\frac{1}{2}f-1}/2^{\frac{1}{2}f}\, (\tfrac{1}{2}f - 1)!, \tag{60}$$

where, if f is odd, $(\frac{1}{2}f - 1)! = \sqrt{\pi}\,(f-1)!/2^{f-1}\,(\frac{1}{2}(f-1))!$, a *gamma function*.

$E[\psi_+] = E[\psi_-] = \infty$ [sometimes, integrability is taken to imply finiteness, restricting the class somewhat further].

For *distributions* of random variables, we can effect what amounts to a "change of variables", yielding

$$E[\psi] = \int_S d\mu(s)\ \psi(s) = \int_{-\infty}^{+\infty} dx\ \rho(x)\ x. \tag{67}$$

For a discrete distribution, the corresponding formula becomes

$$E[\psi] = \sum_i p_i\ \alpha_i, \tag{68}$$

as in (30). Both formulae may be put in the Lebesgue-Stieltjes form

$$E[\psi] = \int_{-\infty}^{+\infty} dF_\psi(x)\ x. \tag{69}$$

Using the characteristic function of a set T, we may define the integral of ψ over the set T as

$$E_T[\psi] = E[\psi\chi_T] = \int_T d\mu\ \psi. \tag{70}$$

This is also called the **conditional expectation** of ψ, *given T*.

We observe that, if the ψ_i are r.v. and the c_i are real numbers, then

$$E[\Sigma_i\ c_i\ \psi_i] = \Sigma_i\ c_i\ E[\psi_i]; \tag{71}$$

and if φ and ψ are r.v.; then, in the discrete case, if φ takes values α_i with probability $p_i = \mu(A_i)$ and ψ takes values β_j with probability $q_j = \mu(B_j)$, then, if φ and ψ are *independent*, so that $\mu(A_iB_j) = p_iq_j$,

$$E[\varphi\psi] = \Sigma_{i,j}\ \mu(A_iB_j)\ \alpha_i\ \beta_j = \Sigma_i\ \Sigma_j\ p_i\ q_j\ \alpha_i\ \beta_j = E[\varphi]\ E[\psi]. \tag{72}$$

In the continuous case, the derivation is completely analogous, using (58) and (67).

3.2. EXPECTATIONS OF SPECIAL DISTRIBUTIONS

We begin with the binomial distribution, as given in (49). The r.v. here is m and its distribution has support $\{0, 1, 2, \ldots, N\}$ (i.e., these are the possible values of m). By (68), the expectation of m is

$$\mathrm{E}[m] = \sum_{m=0}^{N} \mu[m \text{ successes}]\ m = \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} m. \qquad (73)$$

We now use a generally very useful algebraic result: for $N \geqslant m \geqslant k$,

$$\binom{N}{m} m(m-1)(m-2)\ldots(m-k+1) = \frac{N(N-1)(N-2)\ldots(N-m+1)}{(m-k)(m-k-1)\ldots\times 3\times 2\times 1}$$

$$= N(N-1)\ldots(N-k+1)\ \frac{(N-k)(N-k-1)\ldots(N-m+1)}{(m-k)(m-k-1)\ldots\times 4\times 3\times 2\times 1}$$

$$= N(N-1)\ldots(N-k+1)\ \binom{N-k}{m-k}; \qquad (74)$$

whence, in the case of $k = 1$,

$$\mathrm{E}[m] = Np \sum_{m=1}^{N} \binom{N-1}{m-1} p^{m-1} q^{N-m}$$

$$= Np \sum_{j=0}^{N-1} \binom{N-1}{j} p^j q^{(N-1)-j} = Np\ (p+q)^{N-1}$$

$$= Np. \qquad (75)$$

[Note: in (73), the number of occurrences of the event E is called the number of "successes"; the occurrence of E^c is called a "failure".]

Now consider the normal distribution, first in the form (51). Then

$$\mathrm{E}[\alpha] = \int_{-\infty}^{+\infty} \mathrm{d}\alpha\ \mathrm{e}^{-\frac{1}{2}\alpha^2} \frac{\alpha}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\alpha\ \frac{\mathrm{d}}{\mathrm{d}\alpha}(-\mathrm{e}^{-\frac{1}{2}\alpha^2}) = \frac{1}{\sqrt{2\pi}}\ [-\mathrm{e}^{-\frac{1}{2}\alpha^2}]_{-\infty}^{+\infty} = 0. \qquad (76)$$

Now turning to the general form (53), we see, by (32), that

$$\mathrm{E}[\beta] = \int_{-\infty}^{+\infty} \mathrm{d}\beta\ \mathrm{e}^{-(\beta-m)^2/2v} \left(\frac{\beta-m}{\sqrt{2\pi v}} + \frac{m}{\sqrt{2\pi v}}\right) = \frac{\sqrt{v}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\beta\ \frac{\mathrm{d}}{\mathrm{d}\beta}(-\mathrm{e}^{-(\beta-m)^2/2v})$$

$$+ \frac{m}{\sqrt{2\pi v}} \int_{-\infty}^{+\infty} \mathrm{d}\beta\ \mathrm{e}^{-(\beta-m)^2/2v} = [-\mathrm{e}^{-(\beta-m)^2/2v}]_{-\infty}^{+\infty} + m = m. \qquad (77)$$

This shows that the parameter m is the *mean* or expectation of the normal r.v. β.

For the chi-squared distribution, we consider at once the general form (60) with f degrees of freedom [*d.f.*] Then

$$\mathrm{E}[u] = \int_{-\infty}^{+\infty} \mathrm{d}u\ \mathrm{e}^{-\frac{1}{2}u}\ u^{\frac{1}{2}f-1}\ u/2^{\frac{1}{2}f}\ (\tfrac{1}{2}f-1)! = f\int_{-\infty}^{+\infty} \mathrm{d}u\ \mathrm{e}^{-\frac{1}{2}u}\ u^{(\frac{1}{2}f+1)-1}/2^{\frac{1}{2}f+1}((\tfrac{1}{2}f+1)-1)!$$

where we note that $2^{\frac{1}{2}f+1}((\frac{1}{2}f+1)-1)! = 2^{\frac{1}{2}f}(\frac{1}{2}f - 1)!\, f$; and so, by (32) applied to (60) with $f + 2$ d.f., we get that

$$\mathrm{E}[u] = f. \tag{78}$$

Finally, for the Poisson distribution, given in (62), by (32),

$$\mathrm{E}[m] = \sum_{m=0}^{\infty} p_{m,\lambda}\, m = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} e^{-\lambda}\, m = \sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!} e^{-\lambda}$$

$$= \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} e^{-\lambda} = \lambda. \tag{79}$$

3.3. HIGHER MOMENTS; VARIANCES, COVARIANCES

The mean value of a r.v. ψ is the expectation of its first power. It is also possible to ask for the expectation of its k-th power, and this is the **k-th moment**.

$$\mathrm{m}_k[\psi] = \mathrm{E}[\psi^k]. \tag{80}$$

Since products of r.v. are r.v., this generally exists (unless $\mathrm{E}[(\psi^k)_+] = \mathrm{E}[(\psi^k)_-] = \infty$). More interesting is the **k-th central moment**, defined by

$$\mathrm{m}_k^*[\psi] = \mathrm{E}[(\psi - \mathrm{E}[\psi])^k]. \tag{81}$$

In connection with these, we have the *moment generating function*,

$$\mathrm{E}[e^{z\psi}] = 1 + z\mathrm{E}[\psi] + \frac{1}{2}z^2\mathrm{m}_2[\psi] + \ldots + \frac{z^k}{k!}\mathrm{m}_k[\psi] + \ldots, \tag{82}$$

and the *characteristic function*,

$$\mathrm{E}[e^{it\psi}] = 1 + it\mathrm{E}[\psi] - \frac{1}{2}t^2\mathrm{m}_2[\psi] - \ldots + \frac{(it)^k}{k!}\mathrm{m}_k[\psi] + \ldots \tag{83}$$

The particular case of the second central moment is of surpassing importance, since it yields a simple measure of the "spread" of the distribution. This moment is called the **variance** of ψ, written

$$\mathrm{m}_2^*[\psi] = \mathrm{E}[(\psi - \mathrm{E}[\psi])^2] = \mathrm{var}[\psi] = \mathrm{E}[\psi^2] - (\mathrm{E}[\psi])^2. \tag{84}$$

This quantity is of the dimension of the square of ψ; its square root is of the same dimension as that of ψ and it is called the **standard deviation** [**s.d.**] of ψ.

We now review our special distributions. First, the variance of the binomial distribution is (by (49), (74), and (75))

$$\mathrm{var}[m] = \mathrm{E}[m^2] - N^2p^2 = \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m}\, m^2 - N^2p^2 = \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} m(m-1)$$

$$+ \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m}\, m - N^2p^2 = N(N-1)p^2 \sum_{m=2}^{N} \binom{N-2}{m-2} p^{m-2} q^{N-m}$$

$$+ \mathrm{E}[m] - N^2p^2 = N(N-1)p^2 \sum_{j=0}^{N-2} \binom{N-2}{j} p^j q^{(N-2)-j} + Np - N^2p^2$$

$$= N(N-1)p^2(p+q)^{N-2} + Np - N^2p^2 = N(N-1)p^2 + Np - N^2p^2$$

$$= Np - Np^2 = Np(1-p) = Npq. \tag{85}$$

Secondly, for the normal distribution, first as (51), then as (53), we get

$$\mathrm{var}[\alpha] = \mathrm{E}[\alpha^2] - 0 = \int_{-\infty}^{+\infty} d\alpha\ e^{-\frac{1}{2}\alpha^2} \frac{\alpha^2}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\alpha\ \alpha\ \frac{d}{d\alpha}(-e^{-\frac{1}{2}\alpha^2})$$

$$= \frac{1}{\sqrt{2\pi}} [-\alpha\ e^{-\frac{1}{2}\alpha^2}]_{-\infty}^{+\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\alpha\ e^{-\frac{1}{2}\alpha^2} = 0 + 1 = 1, \tag{86}$$

by (50), (76), and noting that $\alpha e^{-\frac{1}{2}\alpha^2} \to 0$ as $\alpha \to \pm\infty$; and

$$\mathrm{var}[\beta] = \mathrm{E}[\beta^2] - m^2 = \int_{-\infty}^{+\infty} d\beta\ e^{-(\beta-m)^2/2v} \left(\frac{(\beta-m)^2}{\sqrt{2\pi v}} + \frac{2m\beta}{\sqrt{2\pi v}} - \frac{m^2}{\sqrt{2\pi v}}\right) - m^2$$

$$= \int_{-\infty}^{+\infty} d\alpha\sqrt{v}\ e^{-\frac{1}{2}\alpha^2} \frac{\alpha^2 v}{\sqrt{2\pi v}} + 2m \times m - m^2 - m^2 = v, \tag{87}$$

by substitution of $\beta = m + \alpha\sqrt{v}$, and (77) and (86). This shows that the parameter v is the *variance* of the normal r.v. β. Thirdly, for the chi-squared (often written χ^2) distribution with f d.f.,

$$\mathrm{var}[u] = \int_{-\infty}^{+\infty} du\ e^{-\frac{1}{2}u}\ u^{\frac{1}{2}f-1}\ u^2/2^{\frac{1}{2}f}\ (\tfrac{1}{2}f-1)! - f^2 = f(f+2) - f^2$$

$$= 2f, \tag{88}$$

much as we obtained (78), noting that $2^{\frac{1}{2}f+2}((\frac{1}{2}f+2)-1)! = 2^{\frac{1}{2}f}(\frac{1}{2}f-1)!\ 4(\frac{1}{2}f)(\frac{1}{2}f+1)$, and using (32) on (60) with $f + 4$ d.f. Finally, for the Poisson distribution,

$$\mathrm{var}[m] = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} e^{-\lambda} m^2 - \lambda^2 = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} e^{-\lambda} m(m-1) + E[m] - \lambda^2$$

$$= \lambda^2 \sum_{m=2}^{\infty} \frac{\lambda^{m-2}}{(m-2)!} e^{-\lambda} + \lambda - \lambda^2 = \lambda^2 + \lambda - \lambda^2 = \lambda. \tag{89}$$

If we have several r.v. ψ_i, we may compute a matrix, called the **variance-covariance matrix** $\mathbf{V}(\boldsymbol{\psi})$, where $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots, \psi_n)$, whose component

$$[\mathbf{V}(\boldsymbol{\psi})]_{ij} = E[(\psi_i - E[\psi_i])(\psi_j - E[\psi_j])]. \tag{90}$$

The diagonal elements of the matrix are, by (84),

$$[\mathbf{V}(\boldsymbol{\psi})]_{ii} = \mathrm{var}[\psi_i], \tag{91}$$

the *variances* of the ψ_i. The off-diagonal element is called the **covariance.** The element (90) is written

$$[\mathbf{V}(\boldsymbol{\psi})]_{ij} = \mathrm{cov}[\psi_i, \psi_j]. \tag{92}$$

This quantity may be rendered dimensionless in the form of the *correlation coefficient*:

$$r_{\varphi\psi} = \frac{\mathrm{cov}[\varphi, \psi]}{\sqrt{(\mathrm{var}[\varphi]\ \mathrm{var}[\psi])}}\ ; \tag{93}$$

and the Cauchy-Schwartz-Bunyakovsky inequality shows that

$$(\mathrm{cov}[\varphi, \psi])^2 \leqslant \mathrm{var}[\varphi]\ \mathrm{var}[\psi] \quad \text{and} \quad |r_{\varphi\psi}| \leqslant 1. \tag{94}$$

[To prove this, observe that, if $E[\varphi] = \overline{\varphi}$ and $E[\psi] = \overline{\psi}$, $\{(\varphi - \overline{\varphi}) + \lambda(\psi - \overline{\psi})\}^2 \geqslant 0$, so $E[\{\ldots\}^2] \geqslant 0$, and this expression equals $\mathrm{var}[\varphi] + 2\lambda\ \mathrm{cov}[\varphi, \psi] + \lambda^2\ \mathrm{var}[\psi]$. This is non-negative for all real λ iff there are no real solutions to the corresponding quadratic equation, i.e., iff the first inequality in (94) holds.]

3.4. LIMITING DISTRIBUTIONS

The binomial distribution leads to both the Poisson and the normal distributions as limiting forms, as the number of trials $N \to \infty$. First, we remind ourselves of a basic result of analysis: for any real x, y, and z,

$$\left(1 + \frac{x}{N}\right)^{Ny+z} \to e^{xy} \quad \text{as} \quad N \to \infty. \tag{95}$$

[A proof is given in Appendix A to these notes.] Now let $p = \frac{\lambda}{N}$ as $N \to \infty$ in (49);

then $\quad \mu[m \text{ successes}] = \frac{N(N-1)(N-2)\ldots(N-m+1)}{m!} \left(\frac{\lambda}{N}\right)^m \left(1 - \frac{\lambda}{N}\right)^{N-m}$

$$= \frac{\lambda^m}{m!} \underbrace{\left(1 - \frac{1}{N}\right)\left(1 - \frac{2}{N}\right)\ldots\left(1 - \frac{m-1}{N}\right)}_{\alpha} \underbrace{\left(1 - \frac{\lambda}{N}\right)^{N-m}}_{\beta}. \qquad (96)$$

Now, as $N \to \infty$, the factor α satisfies $\left(1 - \frac{1}{N}\right)^{m-1} \geqslant \alpha \geqslant \left(1 - \frac{m-1}{N}\right)^{m-1}$, and both bounds tend to 1 [case $y = 0$ of (95), with $x = 1$ or $m - 1$, $z = m - 1$]; and the factor β tends to $e^{-\lambda}$ [case $x = -\lambda$, $y = 1$, $z = -m$ of (95)]. Thus, we recover (62), the *Poisson distribution*.

To understand the Poisson distribution, we consider a point x uniformly randomly located in the interval $\left(-\frac{N}{2\lambda}, +\frac{N}{2\lambda}\right)$, of length $\frac{N}{\lambda}$. Consider a fixed unit interval, say (0, 1). The probability density of x is $\frac{\lambda}{N}$ in the larger interval [we assume that $N > 2\lambda$], and the probability that x lies in the unit subinterval is then $\frac{\lambda}{N} = p$. Now take N points, independently uniformly distributed in the larger interval; then the total density of points is λ, regardless of the value of N, and the probability that m of the N points fall in the unit subinterval is then $\mu[m]$ as in (96). If we let $N \to \infty$, we create a situation in which infinitely many points are independently uniformly distributed in the entire real line, with fixed density λ (per unit length), and the probability that m fall in any unit subinterval is then $p_{m,\lambda}$.

Now, let us consider a situation in which m and N both tend to infinity, in such a way that $\xi = (m - Np)/\sqrt{Npq}$ remains finite, and write $X = m - Np$. Then the probability density of the r.v. ξ is, asymptotically,

$$\rho(\xi) = \lim_{N\to\infty} \frac{\mu[(m - \frac{1}{2} - Np)/\sqrt{Npq} \leqslant \xi < (m + \frac{1}{2} - Np)/\sqrt{Npq}]}{1/\sqrt{Npq}}$$

$$= \lim_{N\to\infty} \sqrt{Npq}\ \mu(m) = \lim_{N\to\infty} \sqrt{Npq} \binom{N}{m} p^m q^{N-m}$$

$$= \lim_{N\to\infty} \sqrt{Npq} \frac{N!}{(Np + X)!(Nq - X)!} p^{Np+X} q^{Nq-X}$$

$$= \lim_{N\to\infty} \sqrt{Npq} \frac{\sqrt{2\pi}N^{N+\frac{1}{2}}e^{-N}}{\sqrt{2\pi}(Np+X)^{Np+X+\frac{1}{2}}e^{-Np-X}\sqrt{2\pi}(Nq-X)^{Nq-X+\frac{1}{2}}e^{-Nq+X}} p^{Np+X} q^{Nq-X}$$

$$= \lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \left(1 + \frac{X}{Np}\right)^{-Np-X-\frac{1}{2}} \left(1 - \frac{X}{Nq}\right)^{-Nq+X-\frac{1}{2}}, \qquad (97)$$

where we have used the famous asymptotic formula of Stirling,

$$S! \sim \sqrt{2\pi}\ S^{S+\frac{1}{2}}\ e^{-S}. \qquad (98)$$

Now let us proceed much as in deriving (A1) of Appendix A.

$$\log(1 + \frac{X}{Np}) = \log(1 + \frac{\xi\sqrt{pq}}{p\sqrt{N}}) = \frac{\xi\sqrt{pq}}{p\sqrt{N}} - \frac{1}{2}\frac{\xi^2 pq}{p^2 N} + \frac{1}{3}(\frac{\xi\sqrt{pq}}{p\sqrt{N}})^3 - \ldots$$

$$= \frac{\xi\sqrt{pq}}{p\sqrt{N}} - \frac{1}{2}\frac{\xi^2 pq}{p^2 N} + O(\frac{1}{N^{3/2}}); \qquad (99)$$

so, $$\log[(1 + \frac{X}{Np})^{Np+X+\frac{1}{2}}] = (Np + \xi\sqrt{Npq} + \frac{1}{2})[\frac{\xi\sqrt{pq}}{p\sqrt{N}} - \frac{1}{2}\frac{\xi^2 pq}{p^2 N} + O(\frac{1}{N^{3/2}})]$$

$$= \xi\sqrt{Npq} + \frac{1}{2}\xi^2 q + O(\frac{1}{\sqrt{N}}), \qquad (100)$$

and, similarly, when we change the sign of ξ and X, and interchange p and q,

$$\log[(1 - \frac{X}{Nq})^{Nq-X+\frac{1}{2}}] = -\xi\sqrt{Npq} + \frac{1}{2}\xi^2 p + O(\frac{1}{\sqrt{N}}). \qquad (101)$$

Now return to (97), using the exponential of the sum of (100) and (101), with the leading terms cancelling, to yield that

$$\rho(\xi) = \frac{1}{\sqrt{2\pi}}\lim_{N\to\infty}\exp[-\frac{1}{2}\xi^2(p + q) + O(\frac{1}{\sqrt{N}})] = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\xi^2} = \rho_{0,1}(\xi). \qquad (102)$$

Thus, we see that, for *any choice* of finite p and $q = 1 - p$, when $N \to \infty$ and we look at $\xi = (m - Np)/\sqrt{Npq}$ remaining fixed and finite, we see that its distribution approaches the normal distribution with mean 0 and variance 1. This result is called the *Demoivre-Laplace Limit Theorem*. This is a particular example of the *Central Limit Theorem*, described later.

3.5. MISCELLANEOUS RESULTS

(i) *BAYES' THEOREM*: Let $(S, \boldsymbol{S}, \mu)$ be a probability space and let events $E_i \in \boldsymbol{S}$ form a partition of S (i.e., $\cup_i E_i = S$ and $(\forall j \neq i)\ E_i E_j = \emptyset$); then, for any event X,

$$\mu(X) = \Sigma_j\ \mu(X|\ E_j)\ \mu(E_j) \qquad (103)$$

and $$\mu(E_i|\ X) = \mu(X|\ E_i)\ \mu(E_i)\ /\ \Sigma_j\ \mu(X|\ E_j)\ \mu(E_j). \qquad (104)$$

[By (36) and (37), $\mu(X|\ E) = \mu(XE)/\mu(E)$ and $\mu(E|\ X) = \mu(XE)/\mu(X)$; and by the additive property (23) of the probability-measure, since the sets XE_i are disjoint if the E_i are disjoint,

$$\mu(X) = \mu(XS) = \mu(X \cup_i E_i) = \mu(\cup_i XE_i) = \Sigma_i\ \mu(XE_i); \qquad (105)$$

whence (103) follows. Since $\mu(E_i|\ X) = \mu(XE_i)/\mu(X) = \mu(X|\ E_i)\mu(E_i)/\mu(X)$, we see that (104) follows from (103).] If we know the probability of X, conditional on each E_j, and if we observe that X occurs, this theorem tells us the probability that the event E_i has occurred.

(ii) *CHEBYSHEV'S INEQUALITY*: Let ψ be a r.v. with mean m and variance v. Let $\varepsilon > 0$; then

$$\mu[|\psi - m| \geqslant \sqrt{(v/\varepsilon)}] \leqslant \varepsilon; \tag{106}$$

or

$$\mu[m - \delta \leqslant \psi \leqslant m + \delta] \geqslant 1 - v/\delta^2, \tag{107}$$

where we have written $\delta = \sqrt{(v/\varepsilon)}$, and have taken the complementary event. [By (84), $v = \text{var}[\psi] = \text{E}[(\psi - \text{E}[\psi])^2] = \int_{-\infty}^{+\infty}\text{d}F_\psi(x)\ (x - m)^2$. Let $A = \{x\colon |x - m| \geqslant \sqrt{(v/\varepsilon)}\}$; then $v \geqslant \int_A \text{d}F_\psi(x)\ (x - m)^2 \geqslant (v/\varepsilon) \int_A \text{d}F_\psi(x) = (v/\varepsilon)\ \mu(A)$; whence $\mu(A) \leqslant \varepsilon$, which is (106). Now (107) follows immediately by substitution.]

We note that this bound is true for any r.v. with a finite mean and variance. In the case of the normal distribution, with $\delta = 3\sqrt{v}$, say, standard tables tell us that $\mu(A) \approx 0.0026998$, while Chebyshev's inequality only yields $\mu(A) \leqslant 1/9 = 0.111...$, a very conservative bound! However, Chebyshev does apply in virtually all cases. As another example, consider a coin-tossing experiment in which we score '1' for a head and '0' for a tail; then $m = \frac{1}{2}$ and $v = \frac{1}{4}$. Thus $|\psi - m| = \frac{1}{2}$ for all outcomes and so $\mu(A) = 1$ if $\sqrt{(v/\varepsilon)} = \frac{1}{2}/\sqrt{\varepsilon} \leqslant \frac{1}{2}$; i.e., $\varepsilon \geqslant 1$; otherwise, $\mu(A) = 0$. Here, the inequality (106) yields that $\mu(A) \leqslant \varepsilon$; so that any $\varepsilon \geqslant 1$ does not bound $\mu(A)$ at all, and any $\varepsilon < 1$ gives $\mu(A) \leqslant \varepsilon$, when $\mu(A) = 0$ in fact.

(iii) *LIM SUP and LIM INF*: When sets E_i ($i = 1, 2, \ldots$) are given, we can define

$$\limsup_{i\to\infty} E_i = \bigcap_{i=1}^{\infty} \bigcup_{j=i+1}^{\infty} E_j = U, \tag{108}$$

and

$$\liminf_{i\to\infty} E_i = \bigcup_{i=1}^{\infty} \bigcap_{j=i+1}^{\infty} E_j = V. \tag{109}$$

Suppose that $x \in V$. Then, $(\exists i)\ x \in \bigcap_{j=i+1}^{\infty} E_j$; so that, for some k, $x \in E_j$ for every $j > k$. That is to say, $x \in E_j$ for *all but a finite nimber of the* j (all these odd values being no greater than k). Then $V \subseteq U$; since, for every i, we can find a value of j greater than i (indeed, any value of j greater than this i and the special k previously found will do), such that $x \in E_j$. In general, we see that, $x \in U$ iff $x \in E_j$ *for infinitely many values of* j.

(iv) *BOREL-CANTELLI LEMMAS*:

LEMMA 1. Define the E_i and U as in (iii) above. Then

$$\sum_{i=1}^{\infty} \mu(E_i) < \infty \tag{110}$$

implies

$$\mu(U) = 0. \tag{111}$$

[By (108), $U \subseteq \cup_{j=i+1} E_j$ for any i; so that $\mu(U) \leqslant \mu(\cup_{j=i+1} E_j) \leqslant \Sigma_{j=i+1} \mu(E_j)$. Thus, if (110) holds (i.e., the infinite series converges to a finite limit), we may let $i \to \infty$ and see that $\mu(U) \leqslant 0$; whence (111) follows immediately.]

LEMMA 2. With the assumptions of Lemma 1, if the E_i are independent, then the negation of (110),

$$\sum_{i=1}^{\infty} \mu(E_i) = \infty \tag{112}$$

implies

$$\mu(U) = 1. \tag{113}$$

[For any i and any $k > i$, $\mu(\cap_{j=i+1}^{\infty} E_j^{\mathrm{c}}) \leqslant \mu(\cap_{j=i+1}^{k} E_j^{\mathrm{c}}) = \Pi_{j=i+1}^{k} \mu(E_j^{\mathrm{c}}) = \Pi_{j=i+1}^{k}[1 - \mu(E_j)] \leqslant \Pi_{j=i+1}^{k} \exp[-\mu(E_j)] = \exp[-\Sigma_{j=i+1}^{k} \mu(E_j)]$; since, for $z \geqslant 0$, $(1 - z)e^{z} = 1 - \frac{1}{2}z^2 - \ldots - \left(\frac{1}{(k-1)!} - \frac{1}{k!}\right)z^k - \ldots \leqslant 1$; i.e., $1 - z \leqslant e^{-z}$. Letting $j \to \infty$, we see that therefore $\mu(\cap_{j=i+1}^{\infty} E_j^{\mathrm{c}}) \leqslant \exp[-\Sigma_{j=i+1}^{\infty}\mu(E_j)] = 0$, by (112); and so $\mu(U^{\mathrm{c}}) = 1 - \mu(U) \leqslant \Sigma_{i=1}^{\infty} \mu(\cap_{j=i+1} E_j^{\mathrm{c}}) = 0$, whence (113) follows.]

This useful pair of tools exhibits an important property. If the E_i are independent events, then either (111) or (113) holds, according to whether (110) or (112) holds (and just one of these alternatives must be true). This is called a *zero-one law*.

4. LAWS OF LARGE NUMBERS

4.1. CONVERGENCE IN QUADRATIC MEAN

We now deal with a series of increasingly strong results, for Bernoulli trials. We assume in all cases that we have a countably infinite product space of identical independent experiments $(\underset{\dots}{S}, \underset{\dots}{\boldsymbol{S}}, \underset{\dots}{\mu})$, yielding a total space $(S, \boldsymbol{S}, \mu)$, with

$$S = \underset{\dots}{S}^{\times\infty}, \qquad \boldsymbol{S} = \underset{\dots}{\boldsymbol{S}}^{\otimes\infty}, \qquad \mu = \underset{\dots}{\mu}^{\otimes\infty}, \tag{114}$$

as in (47). Let ψ be a r.v. defined for a single experiment. Then $\psi(s_i)$ may be evaluated in each experiment S_i, yielding independent r.v., and it is easily seen that, for the average of the first k values

$$\Psi_k(s_1, s_2, \ldots, s_k) = \frac{1}{k} \sum_{i=1}^{k} \psi(s_i), \tag{115}$$

we have, by (71), that

$$E[\Psi_k] = \frac{1}{k} \sum_{i=1}^{k} E[\psi] = E[\psi]; \tag{116}$$

and, by (72), that

$$\begin{aligned} \text{var}[\Psi_k] &= \frac{1}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} E[\psi(s_i)\psi(s_j)] - (E[\psi])^2 \\ &= \frac{2}{k^2} \sum_{i=1}^{k} \sum_{j=i+1}^{k} E[\psi(s_i)]E[\psi(s_j)] + \frac{1}{k^2} \sum_{i=1}^{k} E[\psi^2] - (E[\psi])^2 \\ &= \frac{2}{k^2} \frac{k(k-1)}{2} (E[\psi])^2 - (E[\psi])^2 + \frac{1}{k} E[\psi^2] \\ &= \frac{1}{k}\left(E[\psi^2] - (E[\psi])^2\right) = \frac{1}{k} \text{var}[\psi]. \end{aligned} \tag{117}$$

Thus, by Chebyshev's inequality in the form (107), with $\psi = \Psi_k$, $m = E[\psi]$, and $v = \text{var}[\Psi_k] = \text{var}[\psi]/k$, we have

$$\mu[|\Psi_k - E[\psi]| \leqslant \delta] \geqslant 1 - \frac{\text{var}[\psi]}{k\delta^2}. \tag{118}$$

That is to say, the probability that Ψ_k differs from its expectation $E[\psi]$ by less than any $\delta > 0$ tends to 1 as $k \to \infty$, being less than 1 by less than $\text{var}[\psi]/k\delta^2$. This explains the importance of the expectation (and the use of the name!)

When a sequence of r.v. $\varphi_1, \varphi_2, \ldots$ is such that $E[(\varphi_k - \Phi)^2] \to 0$ as $k \to \infty$, where Φ is a suitable function, then we can always have Φ a r.v. also, and we say that the φ_k **converge in quadratic mean** to Φ. Thus, by (117), the Ψ_k converge in quadratic mean to the constant $E[\psi]$, so long as $E[\psi]$ and $\text{var}[\psi]$ are finite.

4.2. CONVERGENCE IN PROBABILITY; WEAK LAW OF LARGE NUMBERS

If the sequence of r.v. $[\varphi_k]_{k=1}^{\infty}$ is such that $\mu[|\varphi_k - \Phi| \geqslant \delta] \to 0$ as $k \to \infty$, we say that the φ_k **converge in probability** to Φ. Thus, by (118), the Ψ_k converge in probability to the constant r.v. $E[\psi]$. The argument above, using Chebyshev's inequality, shows that *convergence in quadratic mean implies convergence in probability.*

By a more elaborate argument, it can be shown that

$$\text{if} \quad E[\psi] < \infty, \qquad \text{then} \quad \Psi_k \to E[\psi] \ (\text{p.}) \quad \text{as} \quad k \to \infty; \tag{119}$$

where we write '... → ... (p.)' for convergence *in probability*. This result is known as the *Weak Law of Large Numbers*. Note that the finiteness of the variance var[ψ] is *not* required.

4.3. ALMOST-SURE CONVERGENCE; STRONG LAW OF LARGE NUMBERS

If the sequence of r.v. $[\varphi_k]_{k=1}^{\infty}$ is such that the set

$$C = \{\tau: \ \varphi_k(\tau) \to \Phi(\tau) \ \text{as} \ k \to \infty\}, \tag{120}$$

where Φ is also a r.v., has probability

$$\mu(C) = 1; \tag{121}$$

then we say that the φ_k **converge almost surely** [a.s.] (or **with probability one**) to Φ. In general, a set defined as in (120) is called the *convergence-set* of the φ_k to Φ. The convergence referred to in (120) is ordinary convergence in the value (or image) set of the φ_k, i.e., the real line $\mathcal{R}$; so that the convergence theorem of Cauchy shows that any convergent sequence has a real limit. Indeed, it can be shown that any sequence which is convergent in probability or almost surely has a r.v. as limit. The situation defined by (120) and (121) is written

$$\varphi_k \to \Phi \ (\text{a.s.}) \quad \text{as} \quad k \to \infty. \tag{122}$$

The following result is due to *Kolmogorov* and is known as the *Strong Law of Large Numbers*:

$$\text{if} \quad E[\psi] < \infty, \qquad \text{then} \quad \Psi_k \to E[\psi] \ (\text{a.s.}) \quad \text{as} \quad k \to \infty. \tag{123}$$

It can be shown that *almost sure convergence implies convergence in probability* and that the converse is not generally true (see Appendix B).

4.4. CENTRAL LIMIT THEOREM

This theorem is the general form of the formula (102) giving the limit of a binomial distribution. With ψ a r.v. with finite mean E[ψ] and variance var[ψ], and Ψ_k defined as in (115),

$$\mu\left[\frac{\Psi_k - E[\psi]}{\sqrt{\text{var}[\psi]/k}} \leqslant z\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} dx \ e^{-\frac{1}{2}x^2} \quad \text{as} \quad k \to \infty. \tag{124}$$

We may also write (124) in the form

$$\mu[k\mathrm{E}[\psi] - z\sqrt{k\,\mathrm{var}[\psi]} \leqslant \Sigma_{i=1}^{k}\psi_i \leqslant k\mathrm{E}[\psi] + z\sqrt{k\,\mathrm{var}[\psi]}] \to \frac{1}{\sqrt{2\pi}}\int_{-z}^{+z} dx\ e^{-\frac{1}{2}x^2}, \quad (125)$$

where we write ψ_i for $\psi(s_i)$ (for brevity) and let $k \to \infty$.

Note that the speed with which the limit in (124) or (125) is approached will depend on the distribution of the variable ψ.

It is the property of the normal distribution as a common limit of averages of almost any well-behaved r.v. that makes it so important. It means that we need not know the distribution of a r.v., so long as we take sufficiently large samples of it and average them; this estimate will then be nearly normally distributed!

5. SAMPLE STATISTICS: MEAN, VARIANCE, COVARIANCE

5.1. THE SAMPLE MEAN

Statistics very often deals with the following problem. A certain *statistical* (*stochastic*, *random*) *experiment* has an unknown representation as a probability space ($\underset{\sim}{S}$, $\underset{\sim}{\mathcal{S}}$, $\underset{\sim}{\mu}$), and the experiment may be repeated independently any number of times, with the representation of these Bernoulli trials given in (114), also unknown. A r.v. ψ is defined for the experiment and yields *sample values* ψ_1, ψ_2, ..., ψ_k in k repetitions of the experiment (a *sample of size* k). What can we say about the distribution ($\mathcal{R}$, $\mathcal{S}$, $\underset{\sim}{\mu}\psi^{-1}$) of ψ, given these sample values? In particular, can we estimate the moments of the distribution?

The simplest moment is the first, the mean value or expectation $\mathrm{E}[\psi]$. We now see from the various laws of large numbers described above that the **sample mean** Ψ_k *converges to* $\mathrm{E}[\psi]$, as we would wish. We further see from (116) that its expectation is $\mathrm{E}[\psi]$: this is described by saying that Ψ_k is an **unbiased estimator** for (or of) $\mathrm{E}[\psi]$. Observe that, for example, $\Psi_k + 1/k$ still converges to $\mathrm{E}[\psi]$, but is a *biased* estimator, with mean value $\mathrm{E}[\psi] + 1/k$, not equal to the required value $\mathrm{E}[\psi]$.

The variance of a r.v. is, intuitively, a measure of its variability about the mean. The standard deviation (s.d.) of the r.v. has the same dimension as the r.v. itself and can be expected to be "proportional", in some qualitative sense, to the likely error in using the sample mean as an estimate of the true mean. Thus, (117) suggests, and (118) and (125) quantify the fact that, to get n times the accuracy (i.e., $1/n$ times the likely error), we must sample n^2 times as many sample values.

This may be seen more vividly by observing that, to get one more decimal digit of probable accuracy, it is necessary to multiply the sample-size by 100.

If data are collected until sufficient accuracy is believed to have been achieved, it is clear that what must be accumulated are the values of

$$k = \text{size of sample} \qquad \text{and} \qquad S_{\psi k} = \Sigma_{i=1}^{k}\psi_i; \tag{126}$$

as each new value $\psi_{k+1} = \psi(s_{k+1})$ is obtained, we update the parameters by

$$\text{new } k = (\text{old } k) + 1, \qquad S_{\psi k} = S_{\psi(k-1)} + \psi_k. \tag{127}$$

If several r.v. are being recorded, a parameter $S_{\psi k}$ must be maintained for each ψ.

Clearly, from (115) - (117),

$$\Psi_k = \frac{S_{\psi k}}{k}, \qquad \text{E}[S_{\psi k}] = k\ \text{E}[\psi], \qquad \text{var}[S_{\psi k}] = k\ \text{var}[\psi]. \tag{128}$$

5.2. SAMPLE VARIANCES AND COVARIANCES.

The variance formula (84) suggests that we should estimate the variance of ψ by accumulating the sum-of-squares parameter

$$P_{\psi k} = \Sigma_{i=1}^{k}\psi_i^2. \tag{129}$$

We would then expect to use the estimator

$$\Phi_k(\psi,\ \psi) = \Phi_k(\psi,\ \psi;\ s_1,\ s_2,\ \ldots,\ s_k) = \frac{1}{k}[P_{\psi k} - \frac{1}{k}\ S_{\psi k}^2] = \frac{1}{k}[\sum_{i=1}^{k}\psi_i^2 - \frac{1}{k}\sum_{i=1}^{k}\ \sum_{j=1}^{k}\psi_i\psi_j] \tag{130}$$

for the variance. However, we find that, by (72),

$$\text{E}[\Phi_k(\psi,\ \psi)] = \frac{1}{k}\sum_{i=1}^{k}\text{E}[\psi_i^2] - \frac{1}{k^2}\sum_{i=1}^{k}\left\{\sum_{\substack{j=1\\(j\neq i)}}^{k}\text{E}[\psi_i\psi_j] + \text{E}[\psi_i^2]\right\} = \frac{1}{k}(k\ \text{E}[\psi^2]) - \frac{1}{k^2}\left\{k(k-1) \times (\text{E}[\psi])^2 + k\ \text{E}[\psi^2]\right\}$$

$$= \frac{k-1}{k}\{\text{E}[\psi^2] - (\text{E}[\psi])^2\} = \frac{k-1}{k}\ \text{var}[\psi]. \tag{131}$$

Thus we must use the estimator

$$\text{T}_k(\psi,\ \psi) = \text{T}_k(\psi,\ \psi;\ s_1,\ s_2,\ \ldots,\ s_k) = \frac{1}{k-1}[P_{\psi k} - \frac{1}{k}\ S_{\psi k}^2]$$

$$= \frac{1}{k-1}[\sum_{i=1}^{k}\psi_i^2 - \frac{1}{k}\sum_{i=1}^{k}\ \sum_{j=1}^{k}\psi_i\psi_j], \tag{132}$$

which is *unbiased.* Now, we observe that, if we literally accumulate the sums $P_{\psi k}$ and $S_{\psi k}$ to compute T_k, we find that we are forming two large numbers and subtracting one from the other to obtain a relatively small difference; this is computationally very inaccurate. Therefore, when the s.d. of the r.v. is small, compared with its mean, it is advisable to compute, instead of $P_{\psi k}$, the parameter

$$T_{\psi\psi k} = \sum_{j=2}^{k} \frac{j}{j-1}[\psi_i - \frac{1}{j} S_{\psi j}]^2; \tag{133}$$

which is updated by

$$T_{\psi\psi k} = T_{\psi\psi(k-1)} + \frac{k}{k-1}[\psi_k - \frac{1}{k} S_{\psi k}]^2. \tag{134}$$

with $T_{\psi\psi 1} = 0$. We now see that

$$T_{\psi\psi k} - T_{\psi\psi(k-1)} = \frac{k}{k-1}[\psi_k - \frac{1}{k}\sum_{i=1}^{k}\psi_i]^2 = \frac{k}{k-1}[\frac{k-1}{k}\psi_k - \frac{1}{k}\sum_{i=1}^{k-1}\psi_i]^2$$

$$= \frac{k-1}{k}\psi_k^2 - \frac{2}{k}\psi_k\sum_{i=1}^{k-1}\psi_i + \frac{1}{k(k-1)}(\sum_{i=1}^{k-1}\psi_i)^2$$

$$= \psi_k^2 - \frac{1}{k}(\psi_k + \sum_{i=1}^{k-1}\psi_i)^2 + \frac{1}{k-1}(\sum_{i=1}^{k-1}\psi_i)^2; \tag{135}$$

whence $$T_{\psi\psi m} = P_{\psi m} - \frac{1}{m} S_{\psi m}^2, \tag{136}$$

by summing (135) for $k = 2, 3, \ldots, m$, with 'telescoping' of the sum. [We actually get $T_{\psi\psi m} - T_{\psi\psi 1} = \Sigma_{k=2}^{m}\psi_k^2 - \frac{1}{m} S_{\psi m}^2 + \psi_1^2$.] Thus,

$$T_k(\psi, \psi) = \frac{T_{\psi\psi k}}{k-1}, \quad E[T_{\psi\psi k}] = (k-1)\,\mathrm{var}[\psi]. \tag{137}$$

Now, $$\mathrm{var}[T_k(\psi, \psi)] = \frac{1}{(k-1)^2} E[(\sum_{i=1}^{k}\psi_i^2 - \frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{k}\psi_i\psi_j)^2] - (\mathrm{var}[\psi])^2$$

$$= \frac{1}{(k-1)^2} E[\sum_{i=1}^{k}\sum_{j=1}^{k}\psi_i^2\psi_j^2 - \frac{2}{k}\sum_{h=1}^{k}\sum_{i=1}^{k}\sum_{j=1}^{k}\psi_h^2\psi_i\psi_j$$

$$+ \frac{1}{k^2}\sum_{g=1}^{k}\sum_{h=1}^{k}\sum_{i=1}^{k}\sum_{j=1}^{k}\psi_g\psi_h\psi_i\psi_j] - (\mathrm{var}[\psi])^2$$

and, arguing as before, we collect terms with all possible arrangements of like and unlike indices (remembering that the ψ_i are all independently and identically distributed) to obtain that

$$\mathrm{var}[T_k(\psi,\ \psi)] = \frac{1}{k}\,E[\psi^4] - \frac{4}{k}\,E[\psi^3]\,E[\psi] + \frac{k^2 - 2k + 3}{k(k-1)}(E[\psi^2])^2 - \frac{(k-2)(k-3)}{k(k-1)}$$
$$\times\ \{2E[\psi^2](E[\psi])^2 - (E[\psi])^4\} - (\mathrm{var}[\psi])^2. \qquad (138)$$

But we know that, by (81),

$$m_3^*[\psi] = E[(\psi - E[\psi])^3] = E[\psi^3] - 3E[\psi^2]E[\psi] + 2(E[\psi])^3$$
$$= E[\psi^3] - 3E[\psi]\mathrm{var}[\psi] - (E[\psi])^3, \qquad (139)$$

and $$m_4^*[\psi] = E[(\psi - E[\psi])^4] = E[\psi^4] - 4E[\psi^3]E[\psi] + 6E[\psi^2](E[\psi])^2$$
$$- 3(E[\psi])^4 = E[\psi^4] - 4m_3^*[\psi]E[\psi] - 6\mathrm{var}[\psi](E[\psi])^2 - (E[\psi])^4, \quad (140)$$

by substitution for $E[\psi^3]$ from (139) and for $E[\psi^2]$ from (84); whence

$$\mathrm{var}[T_k(\psi,\ \psi)] = \frac{1}{k}\left\{m_4^*[\psi] + 4m_3^*[\psi]E[\psi] + 6\mathrm{var}[\psi](E[\psi])^2 + (E[\psi])^4\right\}$$
$$- \frac{4}{k}\left\{m_3^*[\psi]E[\psi] + 3\mathrm{var}[\psi](E[\psi])^2 + (E[\psi])^4\right\}$$
$$+ \frac{k^2 - 2k + 3}{k(k-1)}\left\{(\mathrm{var}[\psi])^2 + 2\mathrm{var}[\psi](E[\psi])^2 + (E[\psi])^4\right\}$$
$$- \frac{(k-2)(k-3)}{k(k-1)}\left\{2\mathrm{var}[\psi](E[\psi])^2 + (E[\psi])^4\right\} - (\mathrm{var}[\psi])^2$$
$$= \frac{1}{k}\,m_4^*[\psi] - \frac{k-3}{k(k-1)}(\mathrm{var}[\psi])^2. \qquad (141)$$

Note: this derivation is given in full mainly to show the reader that it can be done; messy algebra, if carefully executed (with much intermediate checking) can yield worth-while and rather simple results!

From (141),

$$\mathrm{var}[T_{\psi\psi k}] = \frac{k-1}{k}\{(k-1)\ m_4^*[\psi] - (k-3)(\mathrm{var}[\psi])^2\}. \qquad (142)$$

Since $m_4^*[\psi]$ and $\mathrm{var}[\psi]$ are constants, we see that, like $\mathrm{var}[\Psi_k]$, $\mathrm{var}[T_k(\psi,\ \psi)]$ tends to 0 proportionally to $1/k$ as $k \to \infty$.

Now suppose that we have several r.v., ψ_r; then we can write $\psi_{ri} = \psi_r(s_i)$, for the sampled value of ψ_r in the i-th experiment. For each r, the expectation $E[\psi_r]$ and the variance $\mathrm{var}[\psi_r]$ may respectively be estimated by the unbiased estimators

$$\Psi_{rk} = \Psi_{rk}(s_1,\ s_2,\ \ldots,\ s_k) = \frac{1}{k}\sum_{i=1}^{k} \psi_{ri} = \frac{1}{k}\sum_{i=1}^{k} \psi_r(s_i) \qquad (143)$$

and

$$T_{rrk} = T_k(\psi_r,\ \psi_r) = T_k(\psi_r,\ \psi_r;\ s_1,\ s_2,\ \ldots,\ s_k)$$
$$= \frac{1}{k-1}\left[\sum_{i=1}^{k} \psi_{ri}^{\ 2} - \frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{k} \psi_{ri}\ \psi_{rj}\right]. \qquad (144)$$

Ψ_{rk} is obtained by accumulating the sample size k and the sum $S_{rk} = \Sigma_{i=1}^{k}\psi_{ri}$ (for each r), and T_{rrk} is obtained by accumulating the sum T_{rrk} defined by $T_{rr1} = 0$ and

$$T_{rrk} = T_{rr(k-1)} + \frac{k}{k-1}[\psi_{rk} - \frac{1}{k} S_{rk}]^2. \qquad (145)$$

All this is exactly as before. But now, we can define the *variance-covariance matrix* $\boldsymbol{V}(\psi)$ as in (90) - (92), and we see that $[\boldsymbol{V}(\psi)]_{rs}$ can be estimated without bias by

$$\begin{aligned} \mathrm{T}_{rsk} &= \mathrm{T}_k(\psi_r, \psi_s) = \mathrm{T}_k(\psi_r, \psi_s; s_1, s_2, \ldots, s_k) \\ &= \frac{1}{k-1}[\sum_{i=1}^{k} \psi_{ri}\, \psi_{si} - \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{k} \psi_{ri}\, \psi_{sj}]. \end{aligned} \qquad (146)$$

The similarity to (144) and to (132) before it is obvious, and all the derivations for this parameter are virtually the same as those presented earlier. Thus, we get T_{rsk} by accumulating the sum $Q_{rsk} = \Sigma_{i=1}^{k}\psi_{ri}\psi_{si}$, or, better, the sum T_{rsk}, defined by $T_{rs1} = 0$ and

$$T_{rsk} = T_{rs(k-1)} + \frac{k}{k-1}[\psi_{rk} - \frac{1}{k} S_{rk}][\psi_{sk} - \frac{1}{k} S_{sk}]; \qquad (147)$$

and, if $r = s$, we get exactly what we had before, for the variances; while, if $r \neq s$, we get

$$\mathrm{E}[\mathrm{T}_{rsk}] = \mathrm{cov}[\psi_r, \psi_s], \qquad \mathrm{E}[T_{rsk}] = (k-1)\ \mathrm{cov}[\psi_r, \psi_s]. \qquad (148)$$

The interested reader is urged to compute var$[\mathrm{T}_{rsk}]$, var$[T_{rsk}]$, and even corresponding covariances, much as was done above for var$[\mathrm{T}_{rrk}]$ and var$[T_{rrk}]$.

5.3. REGRESSION AND CORRELATION

Suppose that we have two r.v. (the generalization to several r.v. is straightforward but tedious) ϕ and ψ, and that we believe that some linear relationship exists between them, of the form (known as a **regression** of ψ on ϕ)

$$\psi - \mathrm{E}[\psi] \approx \alpha + \beta\ (\phi - \mathrm{E}[\phi]). \qquad (149)$$

In general, the linear dependence will not be strictly true, and there will be a residual fluctuation, measurable by

$$\Lambda_{\alpha\beta}(\psi, \phi) = \mathrm{E}[\{\psi - \mathrm{E}[\psi] - \alpha - \beta\ (\phi - \mathrm{E}[\phi])\}^2] \qquad (150)$$

We may minimize (150) by putting $-\tfrac{1}{2}\partial\Lambda_{\alpha\beta}/\partial\alpha = 0 = -\tfrac{1}{2}\partial\Lambda_{\alpha\beta}/\partial\beta$, differentiating before taking the expectation, which yields

$$\alpha = 0, \quad \text{and} \quad \beta\ \mathrm{E}[(\phi - \mathrm{E}[\phi])^2] = \mathrm{E}[(\phi - \mathrm{E}[\phi])(\psi - \mathrm{E}[\psi])], \qquad (151)$$

or $$\alpha = 0 \quad \text{and} \quad \beta = \frac{\text{cov}[\phi, \psi]}{\text{var}[\phi]} = \beta_{\psi\phi}. \tag{152}$$

Then the optimized regression line is

$$\psi - E[\psi] \approx \alpha + \beta_{\psi\phi}(\phi - E[\phi]) \tag{153}$$

and the minimized residue is

$$\overline{\Lambda}(\phi, \psi) = \text{var}[\psi] - 2\beta_{\psi\phi}\,\text{cov}[\phi, \psi] + \beta_{\psi\phi}{}^2\text{var}[\phi]. \tag{154}$$

Similarly, there is a regression of ϕ on ψ; so that we may define the two **regression coefficients** $\beta_{\psi\phi}$ and $\beta_{\phi\psi}$; and now we see that, by (93),

$$\beta_{\psi\phi}\,\beta_{\phi\psi} = r_{\phi\psi}{}^2. \tag{155}$$

Now $$\overline{\Lambda}(\phi, \psi) = \frac{\text{var}[\phi]\text{var}[\psi] - (\text{cov}[\phi, \psi])^2}{\text{var}[\phi]} = \text{var}[\psi](1 - r_{\phi\psi}{}^2). \tag{156}$$

In order to estimate the regression coefficients, all we need to do is to use the estimators T_{11k}, T_{12k}, and T_{22k} defined in (144), (145), and (147), with $\psi_1 \equiv \phi$ and $\psi_2 \equiv \psi$. Then, since $T_{rsk}/(k - 1) \to [\boldsymbol{V}(\boldsymbol{\psi})]_{rs}$ (a.s.) [By (123), $\Psi_{rk} \to E[\psi_r]$ (a.s.), and similarly, $\frac{1}{k}\Sigma_{i=1}^{k}\psi_{ri}\psi_{si} \to E[\psi_r\psi_s]$ (a.s.).], we can use

$$b_{\psi\phi k} = T_{12k}/T_{11k} \tag{157}$$

to estimate $\beta_{\psi\phi}$, and we have

$$T_{rsk}/(k - 1) \to [\boldsymbol{V}(\boldsymbol{\psi})]_{rs} \quad \text{and} \quad b_{\psi\phi k} \to \beta_{\psi\phi}, \text{ (a.s.)} \tag{158}$$

We observe, by (152) - (156), that

$$\begin{aligned} E[\{\psi - E[\psi]\}^2] = \text{var}[\psi] &= E[\{\psi - E[\psi] - \beta_{\psi\phi}(\phi - E[\phi])\}^2] + \beta_{\psi\phi}{}^2E[\{\phi - E[\phi]\}^2] \\ &= \overline{\Lambda}(\phi, \psi) + r_{\phi\psi}{}^2\text{var}[\psi] \\ &= \textit{mean squared deviation } \textbf{\textit{from}} \textit{ regression line} \\ &\quad + \textit{mean squared deviation } \textbf{\textit{of}} \textit{ regression line}; \end{aligned} \tag{159}$$

with the cross-term vanishing identically [since $E[\{\psi - E[\psi] - \beta_{\psi\phi}(\phi - E[\phi])\}(\phi - E[\phi])] = \text{cov}[\psi, \phi] - \beta_{\psi\phi}\,\text{var}[\phi] = 0$]. Similarly,

$$\begin{aligned} \Sigma_{i=1}^{k}\{\psi_i - E[\psi] - \beta_{\psi\phi}(\phi_i - E[\phi]\}^2 &= \Sigma_{i=1}^{k}\{\psi_i - \Psi_{\psi k} - b_{\psi\phi k}(\phi_i - \Psi_{\phi k})\}^2 \\ &+ \Sigma_{i=1}^{k}\{\Psi_{\psi k} - E[\psi] - \beta_{\psi\phi}(\Psi_{\phi k} - E[\phi]) + (b_{\psi\phi k} - \beta_{\psi\phi})(\phi_i - \Psi_{\phi k})\}^2; \end{aligned} \tag{160}$$

where, again, the cross-term vanishes: $\Sigma_{i=1}^{k}\{\psi_i - \Psi_{\psi k} - b_{\psi\phi k}(\phi_i - \Psi_{\phi k})\}\{\Psi_{\psi k} - E[\psi] - \beta_{\psi\phi}(\Psi_{\phi k} - E[\phi]) + (b_{\psi\phi k} - \beta_{\psi\phi})(\phi_i - \Psi_{\phi k})\} = \{\Psi_{\psi k} - E[\psi] - \beta_{\psi\phi}(\Psi_{\phi k} - E[\phi])\}\ \Sigma_{i=1}^{k}\{\psi_i - \Psi_{\psi k} - b_{\psi\phi k}(\phi_i - \Psi_{\phi k})\} + (b_{\psi\phi k} - \beta_{\psi\phi})\ \Sigma_{i=1}^{k}\{\psi_i - \Psi_{\psi k} - b_{\psi\phi k}(\phi_i - \Psi_{\phi k})\}(\phi_i - \Psi_{\phi k})$ $= 0 + (b_{\psi\phi k} - \beta_{\psi\phi})(T_{12k} - b_{\psi\phi k}T_{11k}) = 0$, by (157), since

$$\Sigma_{i=1}^{k}(\phi_i - \Psi_{\phi k}) = \Sigma_{i=1}^{k}(\psi_i - \Psi_{\psi k}) = 0 \text{ [by (115), (126), (128)]}, \qquad (161)$$

$$\Sigma_{i=1}^{k}(\phi_i - \Psi_{\phi k})^2 = T_{11k} \text{ [by (136), with (129), (134), (145)]}, \qquad (162)$$

and
$$\Sigma_{i=1}^{k}(\phi_i - \Psi_{\phi k})(\psi_i - \Psi_{\psi k}) = T_{12k} \text{ [similarly, by (146), (147)]}. \qquad (163)$$

6. STATISTICAL PROBLEMS

6.1. ESTIMATION

We have seen how a sample mean Ψ_k (see (115)) converges almost surely to the mean of the distribution from which it is drawn (see (123)). Similarly, the sample variance $\frac{T_{rsk}}{k-1}$ converges almost surely to the corresponding population variance (see (158)). Thus, these are effective estimators of the corresponding population parameters. Given the corresponding variances of the sample parameters (see, e.g., (128), (141)), we could estimate the extent to which the estimates are likely to be in error, using Chebyshev's inequality (see (106), (107)); or, better, when the sample size k is large, we may use the sharper bound afforded by the Central Limit Theorem (see (125)).

Much of statistical estimation theory relies on the assumption, often warranted in practice (*but not always*), that ***the samples are drawn from normal distributions***. A major theorem states that, if a sample $\{\psi_1, \psi_2, \ldots, \psi_k\}$ is drawn from the normal distribution with mean m and variance v (see (53)), then the two quantities

$$A = \frac{k}{v}(\Psi_{\psi k} - m)^2 \qquad \text{and} \qquad B = \frac{k-1}{v}T_k(\psi, \psi) \qquad (164)$$

are each distributed as chi-squared, with 1 and $k - 1$ d.f., respectively, and A and B are statistically independent. This implies that

$$\Psi_{\psi k} \quad \textit{and} \quad T_k(\psi, \psi) \quad \textit{are independent.} \qquad (165)$$

That is to say, our estimators of the mean and variance of the distribution are statistically independent.

Further, W.S.Gossett (writing under the pseudonym of "Student") pointed out that the quantity

$$t = (\frac{A}{B})^{\frac{1}{2}} = (\frac{k}{k-1})^{\frac{1}{2}} \frac{\Psi_{\psi k} - m}{\sqrt{T_k(\psi, \psi)}} \tag{166}$$

is distributed with probability density

$$\rho_{k-1}^{(G)}(t) = \frac{1}{\sqrt{\pi(k-1)}} \frac{((k-2)/2)!}{((k-3)/2)!} \frac{1}{[1 + t^2/(k-1)]^{k/2}}, \tag{167}$$

known, of course, as *"Student's" t-distribution with* $k - 1$ *d. f.* The parameter t depends only on m and the observed values $\psi_i = \psi(s_i)$ as represented in $\Psi_{\psi k}$ and $T_k(\psi, \psi)$ [see (128) and (132)]. Tabulations of the distribution of this r.v. [see, for example, the tables given at the end of this book, on pages -22-43-] enable us readily to calculate, for any given m, the probability that the observed values of t will deviate from 0 by more than any chosen amount.

For instance, suppose that $k = 16$ observations of a r.v. ψ yield sample mean $\Psi_{\psi k} = \Psi_{\psi,16} = 3.2$ and sample variance $T_k(\psi, \psi) = T_{16}(\psi, \psi) = 5.6$. This corresponds to a value of $t = (3.2 - m)/\sqrt{5.6 \times 15 / 16} = (3.2 - m) / 2.29$ with 15 d.f. From the table on page -40-, under 15 d.f., we see that the *critical point* for Q = "1.0e-02" = 1.0×10^{-2} = 1.0% is 2.60248030. As explained on pages -1-2-, this means that the value of ξ for which the "tail" of the distribution,

$$T_{G,f}(\xi) = \int_{\xi}^{+\infty} dx\ \rho_f^{(G)}(x), \tag{168}$$

with $f = k - 1 = 15$, equals $Q = 0.01$ is $\xi = 2.60...$; and this, in turn, means that the probability of obtaining a value of t, with 15 d.f., exceeding 2.60... is 1%. [Alternatively, we might note, on page -34-, that, with 15 d.f., the tail probability for ξ = "2E0" = 2×10^0 = 2.00 is 0.03197... and for "3E0" = 3.00 is 0.004486...] The probability of having t less than -2.60... is also 1%. The former inequality, $t > 2.60...$, corresponds to $3.2 - m > 2.60... \times 2.29... = 5.963031485$, or $m < -2.76...$ The latter, $t < -2.60...$, corresponds to $3.2 - m < -5.96...$, or $m > 9.16...$

The useful conclusion that we may draw from this result is that, with a probability of 98% (100% minus 1% for each tail), the expectation m of the r.v. ψ (*already supposed to have a normal distribution with mean m and unknown variance v*) will lie in the **98% confidence interval** [-2.76..., 9.16...] The 90% confidence interval will similarly lie between 3.2 ± 4.0167..., since the table entry for Q = "5.0e-02" = 5% is 1.75305036. By contrast, if we use a normal approximation with the assumption that the actual population variance equals the sample variance, 5.6, we observe that $(3.2 - m) / \sqrt{5.6/16} = (3.2 - m) / 0.59\ldots$ is normally distributed with mean 0 and variance 1. From the table on page -10-, we see that the 1% critical point for this distribution is 2.326347874 and the 5% critical point is 1.644853627. These values yield 98% and 90% confidence intervals for the population mean m of $3.2 \pm 1.376\ldots$ and $3.2 \pm 0.973\ldots$ Note that these approximate confidence intervals are *very considerably too narrow*, indeed by a factor of about 4, in this case; the t-test, which makes no assumptions about the variance of the normal population, is quite a bit more realistic.

As another application, consider two independent normal samples, taken from populations with the same (unknown) variance and possibly different means; e.g., let $k = 10$ for each sample, with $\Psi_{\phi k} = \Psi_{\phi,10} = 5.7$ and $\Psi_{\psi k} = \Psi_{\psi,10} = 6.0$, and $T_{\phi\phi k} = 0.24$ and $T_{\psi\psi k} = 0.64$, respectively. If $A = \frac{k}{2v}[(\Psi_{\psi k} - \Psi_{\phi k}) - (n - m)]^2$ and $B = (T_{\phi\phi k} + T_{\psi\psi k})/2v(k - 1)$, then these are independent chi-squared-distributed r.v. with 1 and $2k - 2$ d.f., respectively; here, $m = \mathrm{E}[\phi]$ and $n = \mathrm{E}[\psi]$, and we know that ϕ and ψ are completely independent, and that the Ψ and T variables are independent. Thus,

$$t = \left(\frac{A}{B}\right)^{\frac{1}{2}} = \sqrt{9 \times 10}\,\frac{(6.0 - 5.7) - (n - m)}{\sqrt{0.24 + 0.64}} = 10.113(0.3 - \Delta)$$

is a t-variable with $2k - 2 = 18$ d.f. Here, we have written $\Delta = n - m$. It follows that, if the two samples were to have come from the same distribution (i.e., $\Delta = 0$), a value of t as large as $10.113 \times 0.3 = 3.0339$ or larger would have probability (see page -40-) between ¼% and ½% [on page -34-, we see that the tail of the distribution from $t = 3.0$ has probability 0.003842706; so the probability for our value of 3.0339 is somewhat less than this]. From our earlier point of view, above, the 98% confidence interval for Δ is $0.3 \pm 2.552\ldots / 10.113 = 0.3 \pm 0.252\ldots = [0.048, 0.552]$.

Another extremely useful result is that, if two r.v. A and B are independently distributed as chi-squared with f and g d.f., respectively, then the quantity

$$F = \left(\frac{A}{f}\right) / \left(\frac{B}{g}\right) = \frac{g}{f}\frac{A}{B} \qquad (169)$$

is distributed with the probability density

$$\rho^{(F)}_{f,g}(F) = \frac{((f+g-2)/2)!}{((f-2)/2)!((g-2)/2)!}\left(\frac{f}{g}\right)^{f/2} \frac{F^{(f-2)/2}}{\left(1 + \frac{f}{g}F\right)^{(f+g)/2}} \qquad (170)$$

which we call the *variance-ratio* or *F-distribution* (with f and g d.f.) This is again tabulated and enables us to test the equality of variances with given sample estimates. (See tables on pages -44-59-.)

6.2. ANALYSIS OF VARIANCE; CHI-SQUARED TEST

When we have a sample from an unknown normal distribution (the normality assumption is essential, and is often justified), whose variance may be attributed to several independent effects, so that the sample sum-of-squares $T_{\psi\psi k}$ may be separated into several chi-squared-distributed terms [as, for example, we did in (159) and (160)], the relative importance of these influences may be estimated by using the F-test (in terms of the ratios of pairs of partial variances). This procedure is called **analysis of variance** [or **ANOVA**], or **analysis of covariance** if regression effects are in question. The procedure and its analysis (i.e., justification) may be quite elaborate; but a simple example will suffice to illustrate the idea.

Let an experiment involve three factors, A, B, and C, which may be expected to play a part in the outcomes [one could quantify these factors as r.v. α, β, and γ on the underlying probability space]. Let ψ be a r.v. whose value is suspected of being affected by these three factors. Let the values of α be indexed $a = 1, 2, \ldots, k$, those of β be indexed $b = 1, 2, \ldots, l$, and of γ, $c = 1, 2, \ldots, m$; and let a sample be taken with n values of ψ in each combined category; denote these by ψ_{abcd}, with $d = 1, 2, \ldots, n$. [Indeed, α, β, and γ may not be random variables at all, but rather parameters of the population from which sampling is done, the question being whether the distribution of ψ depends on the values of these parameters.] Let us define conditional sample averages by

$$\left.\begin{aligned} &S_{abc\blacktriangle} = \frac{1}{n}\Sigma_d \psi_{abcd}, \quad S_{\blacktriangle bc\blacktriangle} = \frac{1}{k}\Sigma_a S_{abc\blacktriangle}, \quad S_{a\blacktriangle c\blacktriangle} = \frac{1}{l}\Sigma_b S_{abc\blacktriangle}, \quad S_{ab\blacktriangle\blacktriangle} = \frac{1}{m}\Sigma_c S_{abc\blacktriangle}, \\ &S_{a\blacktriangle\blacktriangle\blacktriangle} = \frac{1}{l}\Sigma_b S_{ab\blacktriangle\blacktriangle} = \frac{1}{m}\Sigma_c S_{a\blacktriangle c\blacktriangle}, \quad S_{\blacktriangle b\blacktriangle\blacktriangle} = \frac{1}{k}\Sigma_a S_{ab\blacktriangle\blacktriangle} = \frac{1}{m}\Sigma_c S_{\blacktriangle bc\blacktriangle}, \\ &S_{\blacktriangle\blacktriangle c\blacktriangle} = \frac{1}{k}\Sigma_a S_{a\blacktriangle c\blacktriangle} = \frac{1}{l}\Sigma_b S_{\blacktriangle bc\blacktriangle}, \quad S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle} = \frac{1}{k}\Sigma_a S_{a\blacktriangle\blacktriangle\blacktriangle} = \frac{1}{l}\Sigma_b S_{\blacktriangle b\blacktriangle\blacktriangle} = \frac{1}{m}\Sigma_c S_{\blacktriangle\blacktriangle c\blacktriangle}, \end{aligned}\right\} \quad (171)$$

where ▲ in any index-position indicates summation over the index normally in that position. We then see that

$$\begin{aligned}
\Sigma_{a=1}^{k}\Sigma_{b=1}^{l}\Sigma_{c=1}^{m}\Sigma_{d=1}^{n}(\psi_{abcd} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})^2 &= \Sigma_{a,b,c,d}(\psi_{abcd} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})^2 \\
&= \Sigma_{a,b,c,d}[(S_{a\blacktriangle\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}) + (S_{\blacktriangle b\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}) + (S_{\blacktriangle\blacktriangle c\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}) \\
&\quad + (S_{\blacktriangle bc\blacktriangle} - S_{\blacktriangle b\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle c\blacktriangle} + S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}) + (S_{a\blacktriangle c\blacktriangle} - S_{a\blacktriangle\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle c\blacktriangle} + S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}) \\
&\quad + (S_{ab\blacktriangle\blacktriangle} - S_{a\blacktriangle\blacktriangle\blacktriangle} - S_{\blacktriangle b\blacktriangle\blacktriangle} + S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}) + (\psi_{abcd} - S_{abc\blacktriangle}) \\
&\quad + (S_{abc\blacktriangle} - S_{\blacktriangle bc\blacktriangle} - S_{a\blacktriangle c\blacktriangle} - S_{ab\blacktriangle\blacktriangle} + S_{a\blacktriangle\blacktriangle\blacktriangle} + S_{\blacktriangle b\blacktriangle\blacktriangle} + S_{\blacktriangle\blacktriangle c\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})]^2 \\
&= lmn\Sigma_a(S_{a\blacktriangle\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})^2 + kmn\Sigma_b(S_{\blacktriangle b\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})^2 + kln\Sigma_c(S_{\blacktriangle\blacktriangle c\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})^2 \\
&\quad + kn\Sigma_{b,c}(S_{\blacktriangle bc\blacktriangle} - S_{\blacktriangle b\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle c\blacktriangle} + S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})^2 \\
&\quad + ln\Sigma_{a,c}(S_{a\blacktriangle c\blacktriangle} - S_{a\blacktriangle\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle c\blacktriangle} + S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})^2 \\
&\quad + mn\Sigma_{a,b}(S_{ab\blacktriangle\blacktriangle} - S_{a\blacktriangle\blacktriangle\blacktriangle} - S_{\blacktriangle b\blacktriangle\blacktriangle} + S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})^2 \\
&\quad + \Sigma_{a,b,c,d}(\psi_{abcd} - S_{abc\blacktriangle})^2 \\
&\quad + n\Sigma_{a,b,c}(S_{abc\blacktriangle} - S_{\blacktriangle bc\blacktriangle} - S_{a\blacktriangle c\blacktriangle} - S_{ab\blacktriangle\blacktriangle} + S_{a\blacktriangle\blacktriangle\blacktriangle} + S_{\blacktriangle b\blacktriangle\blacktriangle} + S_{\blacktriangle\blacktriangle c\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})^2,
\end{aligned} \tag{172}$$

so long as the sums of products of the expressions in parentheses all vanish. If we note that (by (171)), e.g., $n\Sigma_{a,b,c}S_{abc\blacktriangle}S_{ab\blacktriangle\blacktriangle} = mn\Sigma_{a,b}S_{ab\blacktriangle\blacktriangle}^2$, $n\Sigma_{a,b,c}S_{ab\blacktriangle\blacktriangle}S_{\blacktriangle bc\blacktriangle} = mn\Sigma_{a,b}S_{ab\blacktriangle\blacktriangle}S_{\blacktriangle b\blacktriangle\blacktriangle} = kmn\Sigma_b S_{\blacktriangle b\blacktriangle\blacktriangle}^2$, and, in general, only repeated indices remain, after summation over the others; it is tedious but not hard to verify that indeed all the sums of products above vanish. For example,

$$\begin{aligned}
\Sigma_{a,b,c,d}(S_{a\blacktriangle\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})(S_{\blacktriangle b\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}) &= mn\Sigma_{a,b}(S_{a\blacktriangle\blacktriangle\blacktriangle}S_{\blacktriangle b\blacktriangle\blacktriangle} - S_{a\blacktriangle\blacktriangle\blacktriangle}S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle} \\
&\quad - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}S_{\blacktriangle b\blacktriangle\blacktriangle} + S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}^2) = mn(\Sigma_a S_{a\blacktriangle\blacktriangle\blacktriangle})(\Sigma_b S_{\blacktriangle b\blacktriangle\blacktriangle}) - lmnS_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}(\Sigma_a S_{a\blacktriangle\blacktriangle\blacktriangle}) \\
&\quad - kmnS_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}(\Sigma_b S_{\blacktriangle b\blacktriangle\blacktriangle}) + klmnS_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}^2 = 0,
\end{aligned} \tag{173}$$

and $\Sigma_{a,b,c,d}(S_{ab\blacktriangle\blacktriangle} - S_{a\blacktriangle\blacktriangle\blacktriangle} - S_{\blacktriangle b\blacktriangle\blacktriangle} + S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle})(S_{abc\blacktriangle} - S_{\blacktriangle bc\blacktriangle} - S_{a\blacktriangle c\blacktriangle} - S_{ab\blacktriangle\blacktriangle}$

$$+ S_{a\blacktriangle\blacktriangle\blacktriangle} + S_{\blacktriangle b\blacktriangle\blacktriangle} + S_{\blacktriangle\blacktriangle c\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}) = n\Sigma_{a,b,c}(S_{abc\blacktriangle}S_{ab\blacktriangle\blacktriangle} - S_{ab\blacktriangle\blacktriangle}S_{\blacktriangle bc\blacktriangle}$$

$$- S_{ab\blacktriangle\blacktriangle}S_{a\blacktriangle c\blacktriangle} - S_{ab\blacktriangle\blacktriangle}^2 + 2S_{ab\blacktriangle\blacktriangle}S_{a\blacktriangle\blacktriangle\blacktriangle} + 2S_{ab\blacktriangle\blacktriangle}S_{\blacktriangle b\blacktriangle\blacktriangle} + S_{ab\blacktriangle\blacktriangle}S_{\blacktriangle\blacktriangle c\blacktriangle}$$

$$+ S_{\blacktriangle bc\blacktriangle}S_{a\blacktriangle\blacktriangle\blacktriangle} + S_{\blacktriangle bc\blacktriangle}S_{\blacktriangle b\blacktriangle\blacktriangle} + S_{a\blacktriangle c\blacktriangle}S_{a\blacktriangle\blacktriangle\blacktriangle} + S_{a\blacktriangle c\blacktriangle}S_{\blacktriangle b\blacktriangle\blacktriangle} - S_{abc\blacktriangle}S_{a\blacktriangle\blacktriangle\blacktriangle}$$

$$- S_{abc\blacktriangle}S_{\blacktriangle b\blacktriangle\blacktriangle} + S_{abc\blacktriangle}S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle} - 2S_{ab\blacktriangle\blacktriangle}S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle} - S_{\blacktriangle bc\blacktriangle}S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle} - S_{a\blacktriangle c\blacktriangle}S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}$$

$$- S_{a\blacktriangle\blacktriangle\blacktriangle}^2 - 2S_{a\blacktriangle\blacktriangle\blacktriangle}S_{\blacktriangle b\blacktriangle\blacktriangle} - S_{\blacktriangle b\blacktriangle\blacktriangle}^2 - S_{a\blacktriangle\blacktriangle\blacktriangle}S_{\blacktriangle\blacktriangle c\blacktriangle} - S_{\blacktriangle b\blacktriangle\blacktriangle}S_{\blacktriangle\blacktriangle c\blacktriangle}$$

$$+ 2S_{a\blacktriangle\blacktriangle\blacktriangle}S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle} + 2S_{\blacktriangle b\blacktriangle\blacktriangle}S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle} + S_{\blacktriangle\blacktriangle c\blacktriangle}S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle} - S_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}^2) = mn\Sigma_{a,b}S_{ab\blacktriangle\blacktriangle}^2$$

$$\times (1 - 1) - lmn\Sigma_a S_{a\blacktriangle\blacktriangle\blacktriangle}^2(1 - 2 - 1 + 1 + 1) - kmn\Sigma_b S_{\blacktriangle b\blacktriangle\blacktriangle}^2(1 - 2$$

$$- 1 + 1 + 1) + klmnS_{\blacktriangle\blacktriangle\blacktriangle\blacktriangle}^2(1 + 1 + 1 + 1 - 2 - 1 - 1 - 2 - 1$$

$$- 1 + 2 + 2 + 1 - 1) = 0, \qquad (174)$$

These eight terms (in (172)), when divided by v, are all independently chi-squared-distributed, with $k - 1$, $l - 1$, $m - 1$, $(l - 1)(m - 1)$, $(k - 1)(m - 1)$, $(k - 1)(l - 1)$, $klm(n - 1)$, and $(k - 1)(l - 1)(m - 1)$ d.f., respectively; and their significance may be compared by means of the F-test, applied pairwise. The first three terms measure the significance of the effect of A, B, and C, respectively, acting singly; the next three measure the effect of B and C, A and C, and A and B, acting together; and the eighth term measures the triple effect of A, B, and C, together; the seventh term measures the residual random effect, when the effects of A, B, and C have been accounted for.

As an example, we consider a somewhat simpler situation, in which there are only two factors at work. A factory needs new machines to manufacture a certain part, and tries out four different types of machines (factor B; b = 1, 2, 3, 4 = l), by having them run by each of five workers (factor A; a = 1, 2, 3, 4, 5 = k). For each machine, each worker does one production-run (n = 1) and returns the number ψ_{ab} of parts manufactured in one day. The results are shown below.

		MACHINE 1	2	3	4	Sum	$S_{a\blacktriangle}$	$S_{a\blacktriangle} - S_{\blacktriangle\blacktriangle}$
	1	53	47	57	45	202	50.50	-0.30
M	2	56	50	63	52	221	55.25	4.45
A	3	45	47	54	42	188	47.00	-3.80
N	4	52	47	57	41	197	49.25	-1.55
	5	49	53	58	48	208	52.00	1.20
	Sum	255	244	289	228	1016	254.00	0
	$S_{\blacktriangle b}$	51.0	48.8	57.8	45.6	203.2	50.80	0
	$S_{\blacktriangle b} - S_{\blacktriangle\blacktriangle}$	0.2	-2.0	7.0	-5.2	0	0	

(175)

We may partition the sum of squares, much as in (172), but more simply:

$$\Sigma_{a,b}(\psi_{ab} - S_{\blacktriangle\blacktriangle})^2 = 4\Sigma_a(S_{a\blacktriangle} - S_{\blacktriangle\blacktriangle})^2 + 5\Sigma_b(S_{\blacktriangle b} - S_{\blacktriangle\blacktriangle})^2 + \Sigma_{a,b}(\psi_{ab} - S_{a\blacktriangle} - S_{\blacktriangle b} + S_{\blacktriangle\blacktriangle})^2 \tag{176}$$

The three terms above have the chi-squared distribution, when divided by the unknown variance v, with 4, 3, and $12 = 4 \times 3$ d.f., respectively. Their values are

$$4\Sigma_a(S_{a\blacktriangle} - S_{\blacktriangle\blacktriangle})^2 = 4 \times (0.09 + 19.8025 + 14.44 + 2.4025 + 1.44) = 152.7,$$

$$5\Sigma_b(S_{\blacktriangle b} - S_{\blacktriangle\blacktriangle})^2 = 5 \times (0.04 + 4 + 49 + 27.04) = 400.4,$$

$$\begin{aligned}\Sigma_{a,b}(\psi_{ab} - S_{a\blacktriangle} - S_{\blacktriangle b} + S_{\blacktriangle\blacktriangle})^2 &= 2.3^2 + (-1.5)^2 + (-0.5)^2 + (-0.3)^2 \\ &+ 0.55^2 + (-3.25)^2 + 0.75^2 + 1.95^2 + (-2.2)^2 + 2^2 + 0 \\ &+ 0.2^2 + 2.55^2 + (-0.25)^2 + 0.75^2 + (-3.05)^2 + (-3.2)^2 \\ &+ 3^2 + (-1)^2 + 1.2^2 = 70.1;\end{aligned} \tag{177}$$

and so the variances to be compared are:

$$\left.\begin{array}{l}\text{variation due to A (different men)} = 38.175, \\ \text{variation due to B (different machines)} = 133.4667, \\ \text{residual variation} = 5.841667.\end{array}\right\} \tag{178}$$

The F-values are $38.175/5.841667 = 6.53495$ (with 4 and 12 d.f.) and $133.4667/5.841667 = 22.84736$ (with 3 and 12 d.f.); the first is significant at less than 1% level, the second, at less than $\frac{1}{100}$% level. This is to say that F-values exceeding these (with the appropriate d.f.) would only occur in less than 1% and $\frac{1}{100}$% of cases, respectively.

[The critical points for these d.f. are on pages -47- and -46-, respectively. For 4 and 12 d.f.; we note that $F = 6.53\ldots$ is between 1% and ½% ($a = 4$, $b = 10$ d.f.), and between ¼% and $^1/_{10}$% ($a = 4$, $b = 20$ d.f.); and since 12 is much closer to 10 than to 20, the former range is probably more realistic; in any case 'less than 1%' must be correct. For 3 and 12 d.f.; we note that $F = 22.84\ldots$ is below $^1/_{100}$% (10^{-4}) ($a = 3$, $b = 10$ d.f.), and farther below it for $a = 3$, $b = 20$ d.f.; again, we can assert that 'less than $^1/_{100}$%' is correct.]

In interpreting these results, we must observe that there is considerable variability in output between different workers, but that this is not relevant to the selection of a new machine. The difference between machines is even more marked and suggests that the choice of Machine No. 3 (the one with the highest production overall) will prove realistically advantageous.

As a final example of the application of the chi-squared distribution, we consider the testing of **goodness of fit** in *contingency tables*. We suppose that a sample of size n is taken from a probability space by Bernoulli trials, and the values of three r.v., α, β, and γ, are classified into k ranges of α, indexed by a, l ranges of β, indexed by b, and m of γ, indexed by c, much as in the analysis-of-variance example considered before. For each triple (a, b, c), we record the number n_{abc} of outcomes having the corresponding values of α, β, and γ, respectively. It is possible to analyze the situation exactly; but this is somewhat laborious. Instead, we use the Central Limit Theorem, to tell us that the ratios n_{abc}/n are approximately normally distributed, and a further analysis shows that the quantity

$$\chi^2 = \sum_{a=1}^{k} \sum_{b=1}^{l} \sum_{c=1}^{m} \frac{\left[n_{abc} - \frac{n_{a\blacktriangle\blacktriangle}}{n} \frac{n_{\blacktriangle b\blacktriangle}}{n} \frac{n_{\blacktriangle\blacktriangle c}}{n} n\right]^2}{\frac{n_{a\blacktriangle\blacktriangle}}{n} \frac{n_{\blacktriangle b\blacktriangle}}{n} \frac{n_{\blacktriangle\blacktriangle c}}{n} n} \tag{179}$$

is distributed as a chi-squared with f d.f. [Indeed, this is the way in which Karl Pearson first introduced the chi-squared distribution.] Here, $f = (k-1)(l-1)(m-1)$; and

$$n_{a\blacktriangle\blacktriangle} = \Sigma_{b,c} n_{abc}, \quad n_{\blacktriangle b\blacktriangle} = \Sigma_{a,c} n_{abc}, \quad n_{\blacktriangle\blacktriangle c} = \Sigma_{a,b} n_{abc}, \tag{180}$$

are the *marginal totals* of the contingency table, so that, e.g., $n_{\blacktriangle b\blacktriangle}/n$ estimates the expected probability of β taking the value b; and the expected number of entries in cell (a, b, c) is given by the denominator in (179), if we suppose the factors A, B, and C (represented by the r.v. α, β, and γ, respectively) to be independent.

In general, we may postulate a *null hypothesis*, in which a value a of a r.v. α (or factor A) is given the probability p_a, and then, if a sample of size n yields n_a entries in cell (a), we use

$$\chi^2 = \sum_{a=1}^{k} \frac{[n_a - n\, p_a]^2}{n\, p_a} \tag{181}$$

as our chi-squared with $k - 1$ d.f.

To illustrate a straightforward application of the chi-squared test to a contingency table, we consider the effect of a fertilizer on the yield of a certain crop:

	Y I E L D					
	POOR	MEDIUM	FAIR	GOOD	Sum	Prob_a
WITH	7	13	6	5	31	0.242
WITHOUT	31	56	7	3	97	0.758
Sum	38	69	13	8	128	1.000
Prob_b	0.2969	0.5390	0.1016	0.0625	1.0000	

Table entries denote number of test-plots in stated categories, by class of yield and whether or not fertilized.

The table of expectations $n \times \text{Prob}_a \times \text{Prob}_b$ on the basis of the null hypothesis, that the fertilizer has no significant effect, then takes the form:

	POOR	MEDIUM	FAIR	GOOD
WITH	9.203	16.711	3.148	1.938
WITHOUT	28.797	52.289	9.852	6.062

Now we get that

$$\chi^2 = (7 - 9.203)^2/9.203 + (13 - 16.711)^2/16.711 + (6 - 3.148)^2/3.148 + \ldots$$
$$+ (3 - 6.062)^2/6.062 = 11.577\ldots, \tag{182}$$

with $(2 - 1)(4 - 1) = 3$ d.f. A glance at page -18- shows that the 1% point of the χ^2 distribution with 3 d.f. is 11.34486673. Clearly, our value of 11.577 is *statistically significant* at the 1% level; that is, the probability of getting a chi-squared value as large as or larger than that observed here is just less than 0.01. The fertilizer appears to be working advantageously.

APPENDIX A

The following result is generally important. We write $f(N) = O(g(N))$, when there is a C, such that $\left|\frac{f(N)}{g(N)}\right| < C$, for all N. Thus, as $N \to \infty$, $|f(N)/g(N)|$ remains less than the bound C. The required result is that, as $N \to \infty$,

$$\left\{1 + \frac{x}{N} + \frac{y}{N^2} + O(\frac{1}{N^3})\right\}^{aN+b} = e^{ax}\left\{1 + \frac{1}{N}(ay + bx - \frac{1}{2}ax^2) + O(\frac{1}{N^2})\right\}. \quad \text{(A1)}$$

Proof. If $|\xi/N| < 1$, the series expansion

$$\log(1 + \frac{\xi}{N}) = \frac{\xi}{N} - \frac{1}{2}\frac{\xi^2}{N^2} + \frac{1}{3}\frac{\xi^3}{N^3} - \ldots + \frac{1}{k}(-\frac{\xi}{N})^k + \ldots \quad \text{(A2)}$$

is absolutely convergent. Thus, asymptotically as $N \to \infty$,

$$\log(1 + \frac{\xi}{N}) = \frac{\xi}{N} - \frac{1}{2}\frac{\xi^2}{N^2} + O(\frac{1}{N^3}). \quad \text{(A3)}$$

Therefore,

$$(aN + b)\log(1 + \frac{\xi}{N}) = a\xi + \frac{1}{N}(b\xi - \frac{1}{2}a\,\xi^2) + O(\frac{1}{N^2})$$

$$= a\xi + \log[1 + \frac{1}{N}(b\xi - \frac{1}{2}a\,\xi^2)] + O(\frac{1}{N^2}); \quad \text{(A4)}$$

since, by (A3), $\log(1 + \frac{\eta}{N}) = \frac{\eta}{N} + O(\frac{1}{N^2})$, and we may put $\eta = b\xi - \frac{1}{2}a\,\xi^2$. Thus,

$$(1 + \frac{\xi}{N})^{aN+b} = e^{a\xi}\,[1 + \frac{1}{N}(b\xi - \frac{1}{2}a\,\xi^2)]\,\exp[O(\frac{1}{N^2})]$$

$$= e^{a\xi}\,[1 + \frac{1}{N}(b\xi - \frac{1}{2}a\,\xi^2)]\,[1 + O(\frac{1}{N^2})]$$

$$= e^{a\xi}\,[1 + \frac{1}{N}(b\xi - \frac{1}{2}a\,\xi^2) + O(\frac{1}{N^2})]. \quad \text{(A5)}$$

Now, let

$$\xi = x + \frac{y}{N} + O(\frac{1}{N^2}); \quad \text{(A6)}$$

then (A5) becomes

$$\left\{1 + \frac{x}{N} + \frac{y}{N^2} + O(\frac{1}{N^3})\right\}^{aN+b} = \exp[ax + \frac{ay}{N} + O(\frac{1}{N^2})]\,[1 + \frac{1}{N}(bx - \frac{1}{2}a\,x^2) + O(\frac{1}{N^2})]$$

$$= e^{ax}\exp[\frac{ay}{N} + O(\frac{1}{N^2})]\,[1 + \frac{1}{N}(bx - \frac{1}{2}a\,x^2) + O(\frac{1}{N^2})]$$

$$= e^{ax}\,[1 + \frac{ay}{N} + O(\frac{1}{N^2})]\,[1 + \frac{1}{N}(bx - \frac{1}{2}a\,x^2) + O(\frac{1}{N^2})],$$

which reduces to (A1). Q.E.D.

APPENDIX B

This appendix presents some results in the theory of probability measures.

(i) Following from the axioms (1) - (4) of probability measures, we have a number of fairly immediate consequences. First take sets E and F in $\boldsymbol{S}$; then

$$\text{if} \quad E \subseteq F, \qquad \mu(E) \leqslant \mu(F). \tag{A7}$$

[Note that $E(FE^c) = F(EE^c) = \emptyset$; so that E and FE^c are disjoint; whence, by (23), $\mu(E) + \mu(FE^c) = \mu(E \cup FE^c) = \mu(F)$; and by axiom (1), $\mu(FE^c) \geqslant 0$; whence (A7).]

(ii) Next, let sets F_i be in $\boldsymbol{S}$, and define the sets (also in $\boldsymbol{S}$) $E_1 = F_1$, $E_2 = F_2F_1{}^c$, $E_3 = F_3F_1{}^cF_2{}^c$, ..., $E_i = F_i \cap_{j=1}^{i-1} F_j{}^c$. Then, every $E_i \subseteq F_i$; and if $x \in \cup_{j=1}^{i} E_j$, then $(\exists h \leqslant i)\ x \in E_h$; whence $x \in F_h \subseteq \cup_{j=1}^{i} F_j$; so that $\cup_{j=1}^{i} E_j \subseteq \cup_{j=1}^{i} F_j$. Conversely, if $x \in \cup_{j=1}^{i} F_j$, then $(\exists h \leqslant i)\ [x \in F_h \wedge (\forall j < h)\ x \notin F_j]$, i.e., $x \in E_h$; whence, as before, $x \in \cup_{j=1}^{i} E_j$, so that $\cup_{j=1}^{i} F_j \subseteq \cup_{j=1}^{i} E_j$. It follows that (for both finite and infinite i)

$$\bigcup_{j=1}^{i} E_j = \bigcup_{j=1}^{i} F_j. \tag{A8}$$

Now observe that the E_i are all disjoint; so that axiom (3) holds and

$$\mu(\bigcup_{j=1}^{i} F_j) = \mu(\bigcup_{j=1}^{i} E_i) = \sum_{j=1}^{i} \mu(F_j) \leqslant \sum_{j=1}^{i} \mu(F_j), \tag{A9}$$

so that, letting $i \to \infty$, we get the *general sub-additivity property*,

$$\mu(\bigcup_{j=1}^{\infty} F_j) \leqslant \sum_{j=1}^{\infty} \mu(F_j). \tag{A10}$$

(iii) If we have sets F_i in $\boldsymbol{S}$, with $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$ (a *monotone* sequence of sets), then, for the sets E_i defined as in (ii), we now have $E_i = F_iF_{i-1}{}^c$. Now, $\mu(\cup_{i=1}^{\infty} F_i) = \mu(\cup_{i=1}^{\infty} E_i) = \Sigma_{i=1}^{\infty} \mu(E_i) = \lim_{k\to\infty} \Sigma_{i=1}^{k} \mu(E_i) = \lim_{k\to\infty} \mu(\cup_{i=1}^{k} E_i) = \lim_{k\to\infty} \mu(\cup_{i=1}^{k} F_i)$ $= \lim_{k\to\infty} \mu(F_k)$, since the F_i are monotone. Thus we have that

$$\mu(\bigcup_{i=1}^{\infty} F_i) = \lim_{k\to\infty} \mu(F_k). \tag{A11}$$

If, instead, we have that sets G_i are in $\boldsymbol{S}$, with $G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots$, then $G_1{}^c \subseteq G_2{}^c \subseteq G_3{}^c \subseteq \ldots$; so that (A11) applies and $\mu(\cup_{i=1}^{\infty} G_i{}^c) = \lim_{k\to\infty} \mu(G_k{}^c)$. Now, since E and E^c are disjoint and $E \cup E^c = S$, and since $\mu(S) = 1$, we have

$$\mu(E^c) = 1 - \mu(E). \tag{A12}$$

Thus, $$\mu(\bigcap_{i=1}^{\infty} G_i) = 1 - \mu(\bigcup_{i=1}^{\infty} G_i^{c}) = 1 - \lim_{k\to\infty} \mu(G_k^{c})$$

$$= 1 - [1 - \lim_{k\to\infty} \mu(G_k)] = \lim_{k\to\infty} \mu(G_k). \qquad \text{(A13)}$$

(iv) We now turn to the definitions of <u>lim sup</u> and <u>lim inf</u> [§3.5 (iii)]. It was shown there that

$$\lim\inf_{i\to\infty} E_i \subseteq \lim\sup_{i\to\infty} E_i. \qquad \text{(A14)}$$

We note that

$$\left.\begin{aligned} &\lim\inf_{i\to\infty} \xi_i = \lim_{i\to\infty} \inf\{\xi_j\colon j > i\}, \\ \leqslant\ &\lim\sup_{i\to\infty} \xi_i = \lim_{i\to\infty} \sup\{\xi_j\colon j > i\}, \end{aligned}\right\} \qquad \text{(A15)}$$

for real numbers ξ_i. First consider the sets F_i defined above. If we put $F_i = \cap_{j=i+1}^{\infty} E_i$, we see that the sets are monotone-increasing as required; so that (A11) holds. Thus,

$$\mu(\lim\inf_{i\to\infty} E_i) = \lim_{i\to\infty} \mu(\cap_{j=i+1}^{\infty} E_j) \leqslant \lim_{i\to\infty} \inf\{\mu(E_j)\colon j > i\}$$

$$= \lim\inf_{i\to\infty} \mu(E_i), \qquad \text{(A16)}$$

by (A15), since $\mu(\cap_{j=i+1}^{\infty} E_j) \leqslant \mu(E_j)$ for any $j > i$. Similarly, if we put $G_i = \cup_{j=i+1}^{\infty} E_j$, we see that the G_i are monotone-decreasing, as required for (A13); so that

$$\mu(\lim\sup_{i\to\infty} E_i) = \lim_{i\to\infty} \mu(\cup_{j=i+1}^{\infty} E_j) \geqslant \lim\sup_{i\to\infty} \mu(E_i), \qquad \text{(A17)}$$

by (A15) and an argument analogous to that for (A16). We use the fact that

$$\cap_i E_i \subseteq E_j \subseteq \cup_i E_i, \qquad \text{(A18)}$$

for all j in the set over which i ranges in the union and intersection.

(v) Turning to <u>convergence</u> (p. and a.s.) [§4.2 and 4.3] let us write

$$A_k(\varepsilon) = \{\tau \in S\colon |\varphi_k(\tau) - \Phi(\tau)| < \varepsilon\}. \qquad \text{(A19)}$$

Then $\varphi_k \to \Phi$ (p.) iff $(\forall\delta > 0)\ \mu(A_k(\delta)^{c}) \to 0$ as $k \to \infty$. The assertion that $\varphi_k(\tau) \to \Phi(\tau)$ as $k \to \infty$ is equivalent to

$$(\forall\varepsilon > 0)(\exists k_0 = k_0(\tau, \varepsilon))(\forall k > k_0)\ |\varphi_k(\tau) - \Phi(\tau)| < \varepsilon,$$

or $$(\forall\varepsilon > 0)(\exists k_0 = k_0(\tau, \varepsilon))(\forall k > k_0)\ \tau \in A_k(\varepsilon),$$

or $$(\forall\varepsilon > 0)(\exists k_0 = k_0(\tau, \varepsilon))\ \tau \in \cap_{k=k_0+1}^{\infty} A_k(\varepsilon),$$

or $$(\forall\varepsilon > 0)\ \tau \in \lim\inf_{k\to\infty} A_k(\varepsilon). \qquad \text{(A20)}$$

Thus, our assertion is equivalent to

$$\tau \in \bigcap_{n=1}^{\infty} \liminf_{k\to\infty} A_k(2^{-n}) = C. \qquad \text{(A21)}$$

[To prove the equality implied in (A21), note that, if $\tau \in C$, then (A20) holds, and therefore $\tau \in \bigcap_{\text{all } \varepsilon>0} \liminf_{k\to\infty} A_k(\varepsilon) \subseteq \bigcap_{n=1}^{\infty} \liminf_{k\to\infty} A_k(2^{-n})$. On the other hand, if τ is in the last-mentioned countable intersection, then, for any choice of $\varepsilon > 0$, we can find $n \geqslant -\log_2\varepsilon$, so that $2^{-n} \leqslant \varepsilon$, so that $(\forall k)\ A_k(2^{-n}) \subseteq A_k(\varepsilon)$, whence $\liminf_{k\to\infty} A_k(2^{-n}) \subseteq \liminf_{k\to\infty} A_k(\varepsilon)$; and therefore $\tau \in C$.]

Note that, in the last proof, we have used the fact that, if $P_i \subseteq Q_i$, then $\liminf_{i\to\infty} P_i \subseteq \liminf_{i\to\infty} Q_i$. This follows from the obvious results that

$$\text{if} \quad P_i \subseteq Q_i, \qquad \cap_i P_i \subseteq \cap_i Q_i \quad \text{and} \quad \cup_i P_i \subseteq \cup_i Q_i. \qquad \text{(A22)}$$

Thus, if $\varphi_k \to \Phi$ (a.s.), (121) holds, or (by (A21)) $\mu(\bigcap_{n=1}^{\infty}\liminf_{k\to\infty} A_k(2^{-n})) = 1$, whence $(\forall n)\ \mu(\liminf_{k\to\infty} A_k(2^{-n})) = 1$; whence, by (A14), $\mu(\limsup_{k\to\infty} A_k(2^{-n})) = 1$, also. By (A15) - (A17), we see that $\lim_{k\to\infty} \mu(A_k(2^{-n}))$ exists and equals 1, for any value of n. Since we can always find, for any $\delta > 0$, n such that $2^{-n} \leqslant \delta < 2^{-n-1}$, we get $(\forall k)\ A_k(2^{-n}) \subseteq A_k(\delta) \subseteq A_k(2^{-n-1})$, so that $\mu(A_k(2^{-n})) \leqslant \mu(A_k(\delta)) \leqslant \mu(A_k(2^{-n-1}))$, whence $1 = \lim_{k\to\infty} \mu(A_k(2^{-n})) \leqslant \lim_{k\to\infty} \mu(A_k(\delta)) \leqslant \lim_{k\to\infty} \mu(A_k(2^{-n-1})) = 1$. This proves that $\varphi_k \to \Phi$ (p.) That is,

$$\text{if} \quad \varphi_k \to \Phi \text{ (a.s.)} \quad \text{then} \quad \varphi_k \to \Phi \text{ (p.)} \qquad \text{(A23)}$$

(vi) To prove that the converse of (A23) does not generally hold, we give a counter-example. Let us consider functions defined on $[0, 1)$. Let $n = 1, 2, \ldots$, and let $2^p \leqslant n < 2^{p+1}$ with $q = n - 2^p$; then p is unique for any n, and so is q. Define

$$\psi_n(x) = 1 \quad \text{if} \quad \frac{q}{2^p} \leqslant x < \frac{q+1}{2^p}, \quad \psi_n(x) = 0 \quad \text{otherwise.} \qquad \text{(A24)}$$

We take the uniform unit probability density over the interval $[0, 1)$ and consider $\Phi(x) = 0$ for all x. Then, when $\delta > 1$, $A_n(\delta) = [0, 1)$ and $\mu(A_n(\delta)) = 1$. When $\delta \leqslant 1$, $A_n(\delta)^c = [q2^{-p}, (q+1)2^{-p})$ and $\mu(A_n(\delta)^c) = 2^{-p}$. As $n \to \infty$, $p \to \infty$ too, so $\mu(A_n(\delta)^c) \to 0$; whence $\psi_n \to 0$ (p.) Consider any $x \in [0, 1)$. Let $p = 0, 1, 2, \ldots$; then $\psi_n(x) = 1$ whenever n is such that $q \leqslant 2^p x < q + 1$; i.e., $n = \lfloor 2^p(1 + x) \rfloor$, and this will happen infinitely often. Thus, $C = \emptyset$; and clearly ψ_n does not converge a.s. to 0. No other limit is possible for a.s. convergence, for then this limit would be the limit in probability, and we know already that this is 0.

STATISTICAL TABLES

1. Normal Distribution

The *normal (0, 1)* probability density function is given by

$$\rho_{0,1}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2}, \tag{A}$$

and this has mean 0 and variance (and standard deviation) 1. The first table has the values of ρ tabulated for

$$\xi = 0.00(0.01)6.00; \tag{B}$$

which means values from 0.00 through 6.00, at intervals of 0.01 (i.e., 0.00, 0.01, 0.02, ..., 5.98, 5.99, 6.00).

The second table gives the corresponding inverse cumulative distribution function (sometimes called the upper "tail" of the distribution)

$$T_{0,1}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\xi}^{+\infty} dx\, e^{-\frac{1}{2}x^2}, \tag{C}$$

tabulated for the same values (B).

The third table gives the critical points of the same distribution; that is, the values of ξ for which the i.c.d.f. (C) takes values

$$Q = \{5.0,\ 2.5,\ 1.0\} \times 10^{-\{1(1)7\}} \tag{D}$$

[excepting the very first (and largest) value $5.0 \times 10^{-1} = 0.5$, for which it is clear that the critical point is $\xi = 0$, by the symmetry of (A)]; i.e., the values of Q are 2.5×10^{-1}, 1.0×10^{-1}, 5.0×10^{-2}, 2.5×10^{-2}, ..., 1.0×10^{-6}, 5.0×10^{-7}, 2.5×10^{-7}, 1.0×10^{-7}.

2. Chi-Squared Distribution

The *chi-squared* p.d.f. with f *degrees of freedom* is given by

$$\rho_{\chi,f}(\xi) = \frac{1}{2^{\frac{1}{2}f}\ (\frac{1}{2}f - 1)!} e^{-\frac{1}{2}\xi}\ \xi^{\frac{1}{2}f-1}, \tag{E}$$

where, if f is odd,

$$(\tfrac{1}{2}f - 1)! = \frac{\sqrt{\pi}\ \ (f-1)!}{2^{f-1}\ \ (\frac{1}{2}(f-1))!}\ . \tag{F}$$

The fourth table gives the corresponding i.c.d.f. ("tail")

$$T_{\chi,f}(\xi) = \frac{1}{2^{\frac{1}{2}f}\,(\frac{1}{2}f - 1)!}\int_{\xi}^{+\infty} dx\; e^{-\frac{1}{2}x}\, x^{\frac{1}{2}f-1}, \tag{G}$$

tabulated for

$$f = 1(1)40 \tag{H}$$

and

$$\xi = \{1.0(0.5)9.5\} \times 10^{\{-4(1)2\}}, \tag{I}$$

omitting extreme values for which T is too close to 0 or 1.

The fifth table gives the critical points of the same distribution, for

$$f = 1(1)60(5)160, \tag{J}$$

i.e., f = 1, 2, 3, ..., 58, 59, 60, 65, 70, 75, ..., 150, 155, 160; and for

$$T_{\chi,f}(\xi) = Q \tag{K}$$

with

$$Q = \{5.0,\ 2.5,\ 1.0\} \times 10^{-\{2(1)5\}}. \tag{L}$$

3. Student's '*t*' Distribution

W. S. Gossett's ("Student's") t p.d.f. with f *d.f.* is given by

$$\rho_f^{(G)}(\xi) = \rho_{G,f}(\xi) = \frac{1}{\sqrt{\pi f}}\,\frac{(\frac{1}{2}(f-1))!}{(\frac{1}{2}(f-2))!}\,\frac{1}{[1+\xi^2/f]^{\frac{1}{2}(f+1)}}, \tag{M}$$

with mean 0 and variance $\frac{f}{f-2}$ (as is rather easy to verify). (Note that this variance is *infinite* for $f = 1$: the so-called Cauchy distribution.) The sixth table has this function tabulated for

$$f = 1(1)50(5)200 \tag{N}$$

and

$$\xi = \{1(1)10\} \times 10^{\{-1(1)5\}}, \tag{O}$$

again omitting extreme parameter values for which ρ is excessively small.

The seventh table gives the corresponding i.c.d.f., tabulated similarly for values (N) and (O), again truncated.

The eighth table gives the critical points of this distribution for the same d.f. values (N) and Q-values as in (L).

4. Variance-Ratio: *F*-Distribution

The ninth table (the last) gives the critical points of R. A. Fisher's *Variance-Ratio* or *F-Distribution* with a and b *d.f.* The corresponding p.d.f. is given by

$$\rho^{(\mathrm{F})}_{a,b}(\xi) = \rho_{\mathrm{F},a,b}(\xi) = \frac{(\frac{1}{2}(a+b)-1)!}{(\frac{1}{2}a-1)!(\frac{1}{2}b-1)!}\,\frac{a}{b}\,(\frac{a}{b}\,\xi)^{\frac{1}{2}a-1}\,(1+\frac{a}{b}\,\xi)^{-\frac{1}{2}(a+b)}, \qquad \text{(P)}$$

with $$1 = \int_0^{+\infty} \mathrm{d}x\ \rho_{\mathrm{F},a,b}(x) = \frac{(\frac{1}{2}(a+b)-1)!}{(\frac{1}{2}a-1)!(\frac{1}{2}b-1)!}\int_0^{+\infty}\mathrm{d}y\ y^{\frac{1}{2}a-1}\,(1+y)^{-\frac{1}{2}(a+b)}; \qquad \text{(Q)}$$

and hence mean

$$\int_0^{+\infty}\mathrm{d}x\ x\ \rho_{\mathrm{F},a,b}(x) = \frac{(\frac{1}{2}(a+b)-1)!}{(\frac{1}{2}a-1)!(\frac{1}{2}b-1)!}\,\frac{b}{a}\int_0^{+\infty}\mathrm{d}y\ y^{\frac{1}{2}a}\,(1+y)^{\frac{1}{2}(a+b)}$$

$$= \frac{(\frac{1}{2}(a+b)-1)!}{(\frac{1}{2}a-1)!(\frac{1}{2}b-1)!}\,\frac{b}{a}\,\frac{(\frac{1}{2}a)!\ (\frac{1}{2}b-2)!}{(\frac{1}{2}(a+b)-1)!}$$

$$= \frac{b}{a}\,\frac{\frac{1}{2}a}{\frac{1}{2}b-1} = \frac{b}{b-2} \qquad \text{(R)}$$

and variance $$\frac{2b^2(a+b-2)}{a(b-2)^2(b-4)}, \qquad \text{(S)}$$

by an entirely analogous argument. The tail of the distribution is given by

$$T_{\mathrm{F},a,b}(\xi) = \frac{(\frac{1}{2}(a+b)-1)!}{(\frac{1}{2}a-1)!\ (\frac{1}{2}b-1)!}\int_0^{1/(1+\frac{a}{b}\xi)}\mathrm{d}z\ (1-z)^{\frac{1}{2}a-1}\ z^{\frac{1}{2}b-1} \qquad \text{(T)}$$

where we have put $z = 1/(1+y)$ and $y = \frac{a}{b}\,x$, as in (Q) and (R). The critical points are tabulated for

$$\left.\begin{aligned} a &= 1(1)6(2)12(4)24 \text{ and } 30(10)60, \\ b &= 1(1)10(10)60(20)160, \\ \text{and}\quad T_{\mathrm{F},a,b}(\xi) &= Q = \{5.0,\ 2.5,\ 1.0\}\times 10^{-\{1(1)4\}}. \end{aligned}\right\} \qquad \text{(U)}$$

NORMAL(0, 1) PROBABILITY DENSITY FUNCTION

x add:	.00	.01	.02	.03	.04
0.00	0.398942280	0.398922334	0.398862500	0.398762797	0.398623254
0.05	0.398443914	0.398224830	0.397966068	0.397667706	0.397329832
0.10	0.396952547	0.396535966	0.396080212	0.395585421	0.395051741
0.15	0.394479331	0.393868362	0.393219015	0.392531483	0.391805971
0.20	0.391042694	0.390241878	0.389403759	0.388528585	0.387616615
0.25	0.386668117	0.385683369	0.384662661	0.383606292	0.382514571
0.30	0.381387815	0.380226355	0.379030526	0.377800677	0.376537162
0.35	0.375240347	0.373910605	0.372548319	0.371153879	0.369727684
0.40	0.368270140	0.366781662	0.365262673	0.363713600	0.362134882
0.45	0.360526962	0.358890291	0.357225325	0.355532529	0.353812370
0.50	0.352065327	0.350291879	0.348492513	0.346667721	0.344818001
0.55	0.342943855	0.341045789	0.339124313	0.337179944	0.335213199
0.60	0.333224603	0.331214680	0.329183961	0.327132977	0.325062264
0.65	0.322972360	0.320863804	0.318737138	0.316592908	0.314431657
0.70	0.312253933	0.310060285	0.307851260	0.305627410	0.303389284
0.75	0.301137432	0.298872406	0.296594755	0.294305030	0.292003780
0.80	0.289691553	0.287368897	0.285036358	0.282694482	0.280343811
0.85	0.277984886	0.275618247	0.273244431	0.270863972	0.268477402
0.90	0.266085250	0.263688042	0.261286301	0.258880547	0.256471294
0.95	0.254059056	0.251644341	0.249227652	0.246809491	0.244390351
1.00	0.241970725	0.239551098	0.237131952	0.234713764	0.232297005
1.05	0.229882141	0.227469632	0.225059935	0.222653499	0.220250767
1.10	0.217852177	0.215458162	0.213069147	0.210685552	0.208307790
1.15	0.205936269	0.203571388	0.201213543	0.198863119	0.196520499
1.20	0.194186055	0.191860155	0.189543158	0.187235418	0.184937281
1.25	0.182649085	0.180371163	0.178103839	0.175847430	0.173602247
1.30	0.171368592	0.169146761	0.166937042	0.164739715	0.162555055
1.35	0.160383327	0.158224790	0.156079696	0.153948287	0.151830800
1.40	0.149727466	0.147638504	0.145564130	0.143504551	0.141459965
1.45	0.139430566	0.137416539	0.135418062	0.133435304	0.131468430
1.50	0.129517596	0.127582951	0.125664637	0.123762790	0.121877537
1.55	0.120009001	0.118157295	0.116322528	0.114504800	0.112704207
1.60	0.110920835	0.109154766	0.107406075	0.105674831	0.103961095
1.65	0.102264925	0.100586368	0.098925471	0.097282269	0.095656796
1.70	0.094049077	0.092459133	0.090886979	0.089332623	0.087796071
1.75	0.086277319	0.084776361	0.083293186	0.081827776	0.080380109
1.80	0.078950158	0.077537892	0.076143274	0.074766262	0.073406813
1.85	0.072064874	0.070740393	0.069433312	0.068143566	0.066871091
1.90	0.065615815	0.064377664	0.063156561	0.061952425	0.060765169
1.95	0.059594706	0.058440944	0.057303789	0.056183142	0.055078902
2.00	0.053990967	0.052919228	0.051863577	0.050823901	0.049800088
2.05	0.048792019	0.047799575	0.046822635	0.045861076	0.044914772
2.10	0.043983596	0.043067418	0.042166107	0.041279530	0.040407554
2.15	0.039550042	0.038706856	0.037877859	0.037062910	0.036261869
2.20	0.035474593	0.034700939	0.033940763	0.033193921	0.032460266
2.25	0.031739652	0.031031932	0.030336959	0.029654585	0.028984661
2.30	0.028327038	0.027681567	0.027048100	0.026426485	0.025816575
2.35	0.025218220	0.024631269	0.024055574	0.023490985	0.022937354
2.40	0.022394530	0.021862367	0.021340715	0.020829427	0.020328356
2.45	0.019837354	0.019356277	0.018884977	0.018423311	0.017971133

NORMAL(0, 1) PROBABILITY DENSITY FUNCTION

x add:	.00	.01	.02	.03	.04
2.50	0.017528300	0.017094670	0.016670101	0.016254450	0.015847579
2.55	0.015449347	0.015059616	0.014678249	0.014305109	0.013940061
2.60	0.013582969	0.013233702	0.012892126	0.012558111	0.012231526
2.65	0.011912244	0.011600135	0.011295075	0.010996937	0.010705598
2.70	0.010420935	0.010142827	0.009871154	0.009605797	0.009346638
2.75	0.009093563	0.008846454	0.008605201	0.008369689	0.008139809
2.80	0.007915452	0.007696508	0.007482873	0.007274439	0.007071105
2.85	0.006872767	0.006679324	0.006490676	0.006306726	0.006127377
2.90	0.005952532	0.005782099	0.005615984	0.005454095	0.005296344
2.95	0.005142641	0.004992899	0.004847033	0.004704958	0.004566590
3.00	0.004431848	0.004300652	0.004172923	0.004048582	0.003927554
3.05	0.003809762	0.003695134	0.003583596	0.003475077	0.003369508
3.10	0.003266819	0.003166943	0.003069813	0.002975365	0.002883534
3.15	0.002794258	0.002707476	0.002623126	0.002541150	0.002461490
3.20	0.002384088	0.002308890	0.002235839	0.002164884	0.002095971
3.25	0.002029048	0.001964066	0.001900975	0.001839726	0.001780273
3.30	0.001722569	0.001666569	0.001612227	0.001559502	0.001508351
3.35	0.001458731	0.001410602	0.001363925	0.001318661	0.001274771
3.40	0.001232219	0.001190968	0.001150983	0.001112230	0.001074673
3.45	0.001038281	0.001003021	0.000968862	0.000935772	0.000903722
3.50	0.000872683	0.000842625	0.000813521	0.000785344	0.000758067
3.55	0.000731664	0.000706111	0.000681381	0.000657452	0.000634300
3.60	0.000611902	0.000590236	0.000569280	0.000549013	0.000529415
3.65	0.000510465	0.000492144	0.000474434	0.000457315	0.000440769
3.70	0.000424780	0.000409330	0.000394403	0.000379981	0.000366051
3.75	0.000352596	0.000339601	0.000327053	0.000314937	0.000303239
3.80	0.000291947	0.000281047	0.000270527	0.000260375	0.000250578
3.85	0.000241127	0.000232008	0.000223212	0.000214728	0.000206546
3.90	0.000198655	0.000191047	0.000183712	0.000176641	0.000169826
3.95	0.000163256	0.000156926	0.000150825	0.000144948	0.000139285
4.00	0.000133830	0.000128576	0.000123516	0.000118643	0.000113951
4.05	0.000109434	0.000105085	0.000100899	9.68702e-05	9.29928e-05
4.10	8.92617e-05	8.56717e-05	8.22178e-05	7.88953e-05	7.56995e-05
4.15	7.26259e-05	6.96702e-05	6.68280e-05	6.40954e-05	6.14683e-05
4.20	5.89431e-05	5.65159e-05	5.41833e-05	5.19417e-05	4.97879e-05
4.25	4.77186e-05	4.57308e-05	4.38214e-05	4.19875e-05	4.02263e-05
4.30	3.85352e-05	3.69115e-05	3.53526e-05	3.38562e-05	3.24199e-05
4.35	3.10414e-05	2.97186e-05	2.84493e-05	2.72314e-05	2.60631e-05
4.40	2.49425e-05	2.38676e-05	2.28368e-05	2.18483e-05	2.09005e-05
4.45	1.99918e-05	1.91207e-05	1.82857e-05	1.74855e-05	1.67186e-05
4.50	1.59837e-05	1.52797e-05	1.46051e-05	1.39590e-05	1.33401e-05
4.55	1.27473e-05	1.21797e-05	1.16362e-05	1.11159e-05	1.06177e-05
4.60	1.01409e-05	9.68446e-06	9.24767e-06	8.82971e-06	8.42979e-06
4.65	8.04718e-06	7.68117e-06	7.33107e-06	6.99623e-06	6.67602e-06
4.70	6.36983e-06	6.07707e-06	5.79718e-06	5.52964e-06	5.27391e-06
4.75	5.02951e-06	4.79595e-06	4.57278e-06	4.35956e-06	4.15587e-06
4.80	3.96130e-06	3.77546e-06	3.59798e-06	3.42850e-06	3.26667e-06
4.85	3.11218e-06	2.96469e-06	2.82391e-06	2.68954e-06	2.56132e-06
4.90	2.43896e-06	2.32222e-06	2.21084e-06	2.10459e-06	2.00325e-06
4.95	1.90660e-06	1.81443e-06	1.72654e-06	1.64275e-06	1.56287e-06

NORMAL(0, 1) PROBABILITY DENSITY FUNCTION

x add:	.00	.01	.02	.03	.04
5.00	1.48672e-06	1.41414e-06	1.34497e-06	1.27906e-06	1.21625e-06
5.05	1.15641e-06	1.09941e-06	1.04511e-06	9.93394e-07	9.44143e-07
5.10	8.97244e-07	8.52589e-07	8.10075e-07	7.69605e-07	7.31083e-07
5.15	6.94420e-07	6.59530e-07	6.26330e-07	5.94742e-07	5.64690e-07
5.20	5.36104e-07	5.08913e-07	4.83053e-07	4.58462e-07	4.35079e-07
5.25	4.12847e-07	3.91712e-07	3.71622e-07	3.52527e-07	3.34380e-07
5.30	3.17135e-07	3.00749e-07	2.85182e-07	2.70393e-07	2.56346e-07
5.35	2.43004e-07	2.30333e-07	2.18302e-07	2.06878e-07	1.96032e-07
5.40	1.85736e-07	1.75964e-07	1.66689e-07	1.57887e-07	1.49535e-07
5.45	1.41610e-07	1.34092e-07	1.26961e-07	1.20196e-07	1.13781e-07
5.50	1.07698e-07	1.01929e-07	9.64599e-08	9.12750e-08	8.63602e-08
5.55	8.17019e-08	7.72871e-08	7.31036e-08	6.91396e-08	6.53840e-08
5.60	6.18262e-08	5.84562e-08	5.52643e-08	5.22415e-08	4.93791e-08
5.65	4.66689e-08	4.41030e-08	4.16740e-08	3.93749e-08	3.71988e-08
5.70	3.51396e-08	3.31910e-08	3.13473e-08	2.96030e-08	2.79531e-08
5.75	2.63924e-08	2.49164e-08	2.35206e-08	2.22008e-08	2.09529e-08
5.80	1.97732e-08	1.86580e-08	1.76040e-08	1.66079e-08	1.56665e-08
5.85	1.47771e-08	1.39367e-08	1.31428e-08	1.23929e-08	1.16847e-08
5.90	1.10158e-08	1.03841e-08	9.78771e-09	9.22463e-09	8.69308e-09
5.95	8.19134e-09	7.71778e-09	7.27088e-09	6.84917e-09	6.45127e-09
6.00	6.07588e-09				

NORMAL(0, 1) CUMULATIVE DISTRIBUTION FUNCTION

x add:	.00	.01	.02	.03	.04
0.00	0.500000000	0.496010644	0.492021686	0.488033527	0.484046563
0.05	0.480061194	0.476077817	0.472096830	0.468118628	0.464143607
0.10	0.460172163	0.456204687	0.452241574	0.448283213	0.444329995
0.15	0.440382308	0.436440537	0.432505068	0.428576284	0.424654565
0.20	0.420740291	0.416833837	0.412935577	0.409045885	0.405165128
0.25	0.401293674	0.397431887	0.393580127	0.389738752	0.385908119
0.30	0.382088578	0.378280478	0.374484165	0.370699981	0.366928264
0.35	0.363169349	0.359423567	0.355691245	0.351972708	0.348268273
0.40	0.344578258	0.340902974	0.337242727	0.333597821	0.329968554
0.45	0.326355220	0.322758110	0.319177509	0.315613697	0.312066949
0.50	0.308537539	0.305025731	0.301531788	0.298055965	0.294598516
0.55	0.291159687	0.287739719	0.284338849	0.280957309	0.277595325
0.60	0.274253118	0.270930904	0.267628893	0.264347292	0.261086300
0.65	0.257846111	0.254626915	0.251428895	0.248252230	0.245097094
0.70	0.241963652	0.238852068	0.235762498	0.232695092	0.229649997
0.75	0.226627352	0.223627292	0.220649946	0.217695438	0.214763884
0.80	0.211855399	0.208970088	0.206108054	0.203269392	0.200454193
0.85	0.197662543	0.194894521	0.192150202	0.189429655	0.186732943
0.90	0.184060125	0.181411255	0.178786380	0.176185542	0.173608780
0.95	0.171056126	0.168527607	0.166023246	0.163543059	0.161087060
1.00	0.158655254	0.156247645	0.153864230	0.151505003	0.149169950
1.05	0.146859056	0.144572300	0.142309654	0.140071090	0.137856572
1.10	0.135666061	0.133499513	0.131356881	0.129238112	0.127143151
1.15	0.125071936	0.123024403	0.121000484	0.119000107	0.117023196
1.20	0.115069670	0.113139446	0.111232437	0.109348552	0.107487697
1.25	0.105649774	0.103834681	0.102042315	0.100272568	0.098525329
1.30	0.096800485	0.095097918	0.093417509	0.091759136	0.090122672
1.35	0.088507991	0.086914962	0.085343451	0.083793322	0.082264439
1.40	0.080756659	0.079269841	0.077803841	0.076358510	0.074933700
1.45	0.073529260	0.072145037	0.070780877	0.069436623	0.068112118
1.50	0.066807201	0.065521712	0.064255488	0.063008364	0.061780177
1.55	0.060570758	0.059379941	0.058207556	0.057053433	0.055917403
1.60	0.054799292	0.053698928	0.052616138	0.051550748	0.050502583
1.65	0.049471468	0.048457226	0.047459682	0.046478658	0.045513977
1.70	0.044565463	0.043632937	0.042716221	0.041815138	0.040929509
1.75	0.040059157	0.039203903	0.038363570	0.037537980	0.036726956
1.80	0.035930319	0.035147894	0.034379502	0.033624969	0.032884119
1.85	0.032156775	0.031442763	0.030741909	0.030054039	0.029378980
1.90	0.028716560	0.028066607	0.027428950	0.026803419	0.026189845
1.95	0.025588060	0.024997895	0.024419185	0.023851764	0.023295468
2.00	0.022750132	0.022215594	0.021691694	0.021178270	0.020675163
2.05	0.020182215	0.019699270	0.019226172	0.018762766	0.018308900
2.10	0.017864421	0.017429178	0.017003023	0.016585807	0.016177383
2.15	0.015777607	0.015386335	0.015003423	0.014628731	0.014262118
2.20	0.013903448	0.013552581	0.013209384	0.012873721	0.012545461
2.25	0.012224473	0.011910625	0.011603792	0.011303844	0.011010658
2.30	0.010724110	0.010444077	0.010170439	0.009903076	0.009641870
2.35	0.009386706	0.009137468	0.008894043	0.008656319	0.008424186
2.40	0.008197536	0.007976260	0.007760254	0.007549411	0.007343631
2.45	0.007142811	0.006946851	0.006755653	0.006569119	0.006387155

NORMAL(0, 1) CUMULATIVE DISTRIBUTION FUNCTION

x add:	.00	.01	.02	.03	.04
2.50	0.006209665	0.006036558	0.005867742	0.005703126	0.005542623
2.55	0.005386146	0.005233608	0.005084926	0.004940016	0.004798797
2.60	0.004661188	0.004527111	0.004396488	0.004269243	0.004145301
2.65	0.004024589	0.003907033	0.003792562	0.003681108	0.003572601
2.70	0.003466974	0.003364160	0.003264096	0.003166716	0.003071959
2.75	0.002979763	0.002890068	0.002802815	0.002717945	0.002635402
2.80	0.002555130	0.002477075	0.002401182	0.002327400	0.002255677
2.85	0.002185961	0.002118205	0.002052359	0.001988376	0.001926209
2.90	0.001865813	0.001807144	0.001750157	0.001694810	0.001641061
2.95	0.001588870	0.001538195	0.001488999	0.001441242	0.001394887
3.00	0.001349898	0.001306238	0.001263873	0.001222769	0.001182891
3.05	0.001144207	0.001106685	0.001070294	0.001035003	0.001000782
3.10	0.000967603	0.000935437	0.000904255	0.000874032	0.000844739
3.15	0.000816352	0.000788846	0.000762195	0.000736375	0.000711364
3.20	0.000687138	0.000663675	0.000640953	0.000618951	0.000597648
3.25	0.000577025	0.000557061	0.000537737	0.000519035	0.000500937
3.30	0.000483424	0.000466480	0.000450087	0.000434230	0.000418892
3.35	0.000404058	0.000389712	0.000375841	0.000362429	0.000349463
3.40	0.000336929	0.000324814	0.000313106	0.000301791	0.000290857
3.45	0.000280293	0.000270088	0.000260229	0.000250707	0.000241510
3.50	0.000232629	0.000224053	0.000215773	0.000207780	0.000200064
3.55	0.000192616	0.000185427	0.000178491	0.000171797	0.000165339
3.60	0.000159109	0.000153099	0.000147302	0.000141711	0.000136319
3.65	0.000131120	0.000126108	0.000121275	0.000116617	0.000112127
3.70	0.000107800	0.000103630	9.96114e-05	9.57399e-05	9.20101e-05
3.75	8.84173e-05	8.49567e-05	8.16238e-05	7.84142e-05	7.53236e-05
3.80	7.23480e-05	6.94834e-05	6.67258e-05	6.40716e-05	6.15172e-05
3.85	5.90589e-05	5.66935e-05	5.44177e-05	5.22282e-05	5.01221e-05
3.90	4.80963e-05	4.61481e-05	4.42745e-05	4.24729e-05	4.07408e-05
3.95	3.90756e-05	3.74749e-05	3.59363e-05	3.44576e-05	3.30366e-05
4.00	3.16712e-05	3.03594e-05	2.90991e-05	2.78884e-05	2.67256e-05
4.05	2.56088e-05	2.45364e-05	2.35066e-05	2.25179e-05	2.15687e-05
4.10	2.06575e-05	1.97830e-05	1.89436e-05	1.81382e-05	1.73653e-05
4.15	1.66238e-05	1.59124e-05	1.52300e-05	1.45755e-05	1.39477e-05
4.20	1.33457e-05	1.27685e-05	1.22151e-05	1.16846e-05	1.11760e-05
4.25	1.06885e-05	1.02213e-05	9.77365e-06	9.34467e-06	8.93366e-06
4.30	8.53991e-06	8.16273e-06	7.80146e-06	7.45547e-06	7.12414e-06
4.35	6.80688e-06	6.50312e-06	6.21233e-06	5.93397e-06	5.66753e-06
4.40	5.41254e-06	5.16853e-06	4.93505e-06	4.71165e-06	4.49794e-06
4.45	4.29351e-06	4.09798e-06	3.91098e-06	3.73215e-06	3.56116e-06
4.50	3.39767e-06	3.24138e-06	3.09198e-06	2.94918e-06	2.81271e-06
4.55	2.68230e-06	2.55768e-06	2.43862e-06	2.32488e-06	2.21623e-06
4.60	2.11245e-06	2.01334e-06	1.91870e-06	1.82833e-06	1.74205e-06
4.65	1.65968e-06	1.58105e-06	1.50600e-06	1.43437e-06	1.36603e-06
4.70	1.30081e-06	1.23858e-06	1.17922e-06	1.12260e-06	1.06859e-06
4.75	1.01708e-06	9.67965e-07	9.21130e-07	8.76476e-07	8.33907e-07
4.80	7.93328e-07	7.54651e-07	7.17791e-07	6.82665e-07	6.49196e-07
4.85	6.17307e-07	5.86929e-07	5.57991e-07	5.30429e-07	5.04180e-07
4.90	4.79183e-07	4.55382e-07	4.32721e-07	4.11148e-07	3.90613e-07
4.95	3.71067e-07	3.52466e-07	3.34765e-07	3.17921e-07	3.01896e-07

NORMAL(0, 1) CUMULATIVE DISTRIBUTION FUNCTION

x add:	.00	.01	.02	.03	.04
5.00	2.86652e-07	2.72150e-07	2.58357e-07	2.45240e-07	2.32766e-07
5.05	2.20905e-07	2.09628e-07	1.98908e-07	1.88717e-07	1.79032e-07
5.10	1.69827e-07	1.61079e-07	1.52768e-07	1.44871e-07	1.37369e-07
5.15	1.30243e-07	1.23475e-07	1.17047e-07	1.10943e-07	1.05147e-07
5.20	9.96443e-08	9.44203e-08	8.94616e-08	8.47550e-08	8.02883e-08
5.25	7.60496e-08	7.20277e-08	6.82119e-08	6.45919e-08	6.11582e-08
5.30	5.79013e-08	5.48126e-08	5.18836e-08	4.91064e-08	4.64733e-08
5.35	4.39771e-08	4.16110e-08	3.93683e-08	3.72429e-08	3.52288e-08
5.40	3.33204e-08	3.15124e-08	2.97995e-08	2.81770e-08	2.66403e-08
5.45	2.51849e-08	2.38067e-08	2.25018e-08	2.12663e-08	2.00967e-08
5.50	1.89896e-08	1.79417e-08	1.69500e-08	1.60115e-08	1.51236e-08
5.55	1.42835e-08	1.34887e-08	1.27370e-08	1.20259e-08	1.13535e-08
5.60	1.07176e-08	1.01163e-08	9.54787e-09	9.01048e-09	8.50251e-09
5.65	8.02239e-09	7.56865e-09	7.13988e-09	6.73474e-09	6.35197e-09
5.70	5.99037e-09	5.64881e-09	5.32620e-09	5.02153e-09	4.73383e-09
5.75	4.46217e-09	4.20570e-09	3.96358e-09	3.73503e-09	3.51932e-09
5.80	3.31575e-09	3.12364e-09	2.94238e-09	2.77137e-09	2.61004e-09
5.85	2.45786e-09	2.31434e-09	2.17898e-09	2.05133e-09	1.93098e-09
5.90	1.81751e-09	1.71054e-09	1.60971e-09	1.51467e-09	1.42511e-09
5.95	1.34071e-09	1.26119e-09	1.18627e-09	1.11569e-09	1.04920e-09
6.00	9.86587e-10				

CRITICAL POINTS OF THE NORMAL(0, 1) DISTRIBUTION

Q =	2.5e-01	1.0e-01	5.0e-02	2.5e-02	1.0e-02
x =	0.674489750	1.281551566	1.644853627	1.959963985	2.326347874

Q =	5.0e-03	2.5e-03	1.0e-03	5.0e-04	2.5e-04
x =	2.575829304	2.807033768	3.090232306	3.290526731	3.480756404

Q =	1.0e-04	5.0e-05	2.5e-05	1.0e-05	5.0e-06
x =	3.719016485	3.890591886	4.055626981	4.264890794	4.417173413

Q =	2.5e-06	1.0e-06	5.0e-07	2.5e-07	1.0e-07
x =	4.564787730	4.753424309	4.891638475	5.026312836	5.199337581

CHI-SQUARED CUMULATIVE DISTRIBUTION FUNCTION

x	1 d.f.	2 d.f.	3 d.f.	4 d.f.	5 d.f.
1.0 E-4	0.99202129	0.99995000	0.99999973	1.00000000	1.00000000
1.5 E-4	0.99022819	0.99992500	0.99999951	1.00000000	1.00000000
2.0 E-4	0.98871658	0.99990000	0.99999925	1.00000000	1.00000000
2.5 E-4	0.98738486	0.99987501	0.99999895	0.99999999	1.00000000
3.0 E-4	0.98618092	0.99985001	0.99999862	0.99999999	1.00000000
3.5 E-4	0.98507382	0.99982502	0.99999826	0.99999998	1.00000000
4.0 E-4	0.98404337	0.99980002	0.99999787	0.99999998	1.00000000
4.5 E-4	0.98307558	0.99977503	0.99999746	0.99999997	1.00000000
5.0 E-4	0.98216025	0.99975003	0.99999703	0.99999997	1.00000000
5.5 E-4	0.98128966	0.99972504	0.99999657	0.99999996	1.00000000
6.0 E-4	0.98045785	0.99970004	0.99999609	0.99999996	1.00000000
6.5 E-4	0.97966006	0.99967505	0.99999559	0.99999995	1.00000000
7.0 E-4	0.97889242	0.99965006	0.99999508	0.99999994	1.00000000
7.5 E-4	0.97815176	0.99962507	0.99999454	0.99999993	1.00000000
8.0 E-4	0.97743543	0.99960008	0.99999398	0.99999992	1.00000000
8.5 E-4	0.97674116	0.99957509	0.99999341	0.99999991	1.00000000
9.0 E-4	0.97606705	0.99955010	0.99999282	0.99999990	1.00000000
9.5 E-4	0.97541144	0.99952511	0.99999221	0.99999989	1.00000000
1.0 E-3	0.97477288	0.99950012	0.99999159	0.99999988	1.00000000
1.5 E-3	0.96910579	0.99925028	0.99998456	0.99999972	1.00000000
2.0 E-3	0.96432941	0.99900050	0.99997623	0.99999950	0.99999999
2.5 E-3	0.96012239	0.99875078	0.99996678	0.99999922	0.99999998
3.0 E-3	0.95631990	0.99850112	0.99995634	0.99999888	0.99999997
3.5 E-3	0.95282403	0.99825153	0.99994499	0.99999847	0.99999996
4.0 E-3	0.94957097	0.99800200	0.99993280	0.99999800	0.99999995
4.5 E-3	0.94651639	0.99775253	0.99991982	0.99999747	0.99999993
5.0 E-3	0.94362802	0.99750312	0.99990611	0.99999688	0.99999991
5.5 E-3	0.94088149	0.99725378	0.99989170	0.99999623	0.99999988
6.0 E-3	0.93825788	0.99700450	0.99987661	0.99999551	0.99999985
6.5 E-3	0.93574211	0.99675528	0.99986090	0.99999473	0.99999982
7.0 E-3	0.93332199	0.99650612	0.99984456	0.99999389	0.99999978
7.5 E-3	0.93098745	0.99625702	0.99982764	0.99999299	0.99999974
8.0 E-3	0.92873007	0.99600799	0.99981015	0.99999202	0.99999970
8.5 E-3	0.92654276	0.99575902	0.99979211	0.99999099	0.99999965
9.0 E-3	0.92441941	0.99551011	0.99977353	0.99998991	0.99999959
9.5 E-3	0.92235479	0.99526126	0.99975443	0.99998875	0.99999953
1.0 E-2	0.92034433	0.99501248	0.99973483	0.99998754	0.99999947
1.5 E-2	0.90252325	0.99252805	0.99951359	0.99997202	0.99999854
2.0 E-2	0.88753708	0.99004983	0.99925224	0.99995033	0.99999701
2.5 E-2	0.87436706	0.98757780	0.99895654	0.99992252	0.99999479
3.0 E-2	0.86249023	0.98511194	0.99863039	0.99988862	0.99999180
3.5 E-2	0.85159566	0.98265224	0.99827668	0.99984865	0.99998796
4.0 E-2	0.84148058	0.98019867	0.99789766	0.99980265	0.99998322
4.5 E-2	0.83200403	0.97775124	0.99749515	0.99975064	0.99997751
5.0 E-2	0.82306327	0.97530991	0.99707067	0.99969266	0.99997079
5.5 E-2	0.81458070	0.97287468	0.99662551	0.99962874	0.99996300
6.0 E-2	0.80649594	0.97044553	0.99616079	0.99955890	0.99995409
6.5 E-2	0.79876096	0.96802245	0.99567749	0.99948318	0.99994402
7.0 E-2	0.79133678	0.96560542	0.99517648	0.99940161	0.99993274
7.5 E-2	0.78419123	0.96319442	0.99465854	0.99931421	0.99992022
8.0 E-2	0.77729741	0.96078944	0.99412437	0.99922102	0.99990642
8.5 E-2	0.77063255	0.95839047	0.99357461	0.99912206	0.99989130
9.0 E-2	0.76417716	0.95599748	0.99300984	0.99901737	0.99987483
9.5 E-2	0.75791440	0.95361047	0.99243062	0.99890697	0.99985696

CHI-SQUARED CUMULATIVE DISTRIBUTION FUNCTION

x	1 d.f.	2 d.f.	3 d.f.	4 d.f.	5 d.f.
1.0 E-1	0.75182963	0.95122942	0.99183742	0.99879090	0.99983768
1.5 E-1	0.69853536	0.92774349	0.98522606	0.99732425	0.99956059
2.0 E-1	0.65472085	0.90483742	0.97758930	0.99532116	0.99911386
2.5 E-1	0.61707508	0.88249690	0.96914040	0.99280902	0.99847918
3.0 E-1	0.58388242	0.86070798	0.96002848	0.98981417	0.99764309
3.5 E-1	0.55411313	0.83945702	0.95036612	0.98636200	0.99659563
4.0 E-1	0.52708926	0.81873075	0.94024249	0.98247690	0.99532959
4.5 E-1	0.50233495	0.79851622	0.92973057	0.97818237	0.99383991
5.0 E-1	0.47950012	0.77880078	0.91889141	0.97350098	0.99212329
5.5 E-1	0.45831769	0.75957212	0.90777705	0.96845446	0.99017793
6.0 E-1	0.43857803	0.74081822	0.89643237	0.96306369	0.98800324
6.5 E-1	0.42011268	0.72252735	0.88489653	0.95734874	0.98559970
7.0 E-1	0.40278369	0.70468809	0.87320395	0.95132892	0.98296868
7.5 E-1	0.38647623	0.68728928	0.86138508	0.94502276	0.98011229
8.0 E-1	0.37109337	0.67032005	0.84946703	0.93844806	0.97703334
8.5 E-1	0.35655234	0.65376979	0.83747403	0.93162194	0.97373518
9.0 E-1	0.34278171	0.63762815	0.82542781	0.92456082	0.97022164
9.5 E-1	0.32971930	0.62188506	0.81334791	0.91728046	0.96649697
1.0 E 0	0.31731051	0.60653066	0.80125196	0.90979599	0.96256577
1.5 E 0	0.22067136	0.47236655	0.68227033	0.82664147	0.91306981
2.0 E 0	0.15729921	0.36787944	0.57240670	0.73575888	0.84914504
2.5 E 0	0.11384630	0.28650480	0.47529108	0.64463579	0.77649507
3.0 E 0	0.08326452	0.22313016	0.39162518	0.55782540	0.69998584
3.5 E 0	0.06136883	0.17377394	0.32076212	0.47787834	0.62338763
4.0 E 0	0.04550026	0.13533528	0.26146413	0.40600585	0.54941595
4.5 E 0	0.03389485	0.10539922	0.21229029	0.34254748	0.47988344
5.0 E 0	0.02534732	0.08208500	0.17179714	0.28729750	0.41588019
5.5 E 0	0.01901647	0.06392786	0.13863862	0.23972948	0.35794588
6.0 E 0	0.01430588	0.04978707	0.11161023	0.19914827	0.30621892
6.5 E 0	0.01078745	0.03877421	0.08966250	0.16479038	0.26055846
7.0 E 0	0.00815097	0.03019738	0.07189777	0.13588823	0.22064031
7.5 E 0	0.00616990	0.02351775	0.05755845	0.11170929	0.18602983
8.0 E 0	0.00467773	0.01831564	0.04601171	0.09157819	0.15623563
8.5 E 0	0.00355146	0.01426423	0.03673311	0.07488723	0.13074779
9.0 E 0	0.00269980	0.01110900	0.02929089	0.06109948	0.10906416
9.5 E 0	0.00205472	0.00865170	0.02333136	0.04974725	0.09070739
1.0 E 1	0.00156540	0.00673795	0.01856614	0.04042768	0.07523525
1.5 E 1	0.00010751	0.00055308	0.00181665	0.00470122	0.01036234
2.0 E 1	7.7442e-06	4.5400e-05	0.00016974	0.00049940	0.00124973
2.5 E 1	5.7330e-07	3.7267e-06	1.5440e-05	5.0310e-05	0.00013933
3.0 E 1	4.3205e-08	3.0590e-07	1.3801e-06	4.8944e-06	1.4749e-05
3.5 E 1	3.2971e-09	2.5110e-08	1.2182e-07	4.6453e-07	1.5047e-06
4.0 E 1	2.5396e-10	2.0611e-09	1.0655e-08	4.3284e-08	1.4934e-07

x	6 d.f.	7 d.f.	8 d.f.	9 d.f.	10 d.f.
3.0 E-2	0.99999944	0.99999996	1.00000000	1.00000000	1.00000000
3.5 E-2	0.99999912	0.99999994	1.00000000	1.00000000	1.00000000
4.0 E-2	0.99999869	0.99999990	0.99999999	1.00000000	1.00000000
4.5 E-2	0.99999813	0.99999986	0.99999999	1.00000000	1.00000000
5.0 E-2	0.99999744	0.99999979	0.99999998	1.00000000	1.00000000
5.5 E-2	0.99999660	0.99999971	0.99999998	1.00000000	1.00000000
6.0 E-2	0.99999560	0.99999961	0.99999997	1.00000000	1.00000000
6.5 E-2	0.99999442	0.99999948	0.99999995	1.00000000	1.00000000

CHI-SQUARED CUMULATIVE DISTRIBUTION FUNCTION

x	6 d.f.	7 d.f.	8 d.f.	9 d.f.	10 d.f.
7.0 E-2	0.99999304	0.99999933	0.99999994	0.99999999	1.00000000
7.5 E-2	0.99999145	0.99999915	0.99999992	0.99999999	1.00000000
8.0 E-2	0.99998965	0.99999893	0.99999990	0.99999999	1.00000000
8.5 E-2	0.99998761	0.99999868	0.99999987	0.99999999	1.00000000
9.0 E-2	0.99998532	0.99999840	0.99999984	0.99999998	1.00000000
9.5 E-2	0.99998276	0.99999806	0.99999980	0.99999998	1.00000000
1.0 E-1	0.99997993	0.99999769	0.99999975	0.99999997	1.00000000
1.5 E-1	0.99993353	0.99999063	0.99999876	0.99999984	0.99999998
2.0 E-1	0.99984535	0.99997484	0.99999615	0.99999944	0.99999992
2.5 E-1	0.99970352	0.99994612	0.99999079	0.99999851	0.99999977
3.0 E-1	0.99949714	0.99989996	0.99998129	0.99999669	0.99999944
3.5 E-1	0.99921619	0.99983170	0.99996601	0.99999350	0.99999882
4.0 E-1	0.99885152	0.99973656	0.99994316	0.99998839	0.99999774
4.5 E-1	0.99839481	0.99960975	0.99991074	0.99998067	0.99999601
5.0 E-1	0.99783850	0.99944648	0.99986663	0.99996957	0.99999339
5.5 E-1	0.99717578	0.99924203	0.99980857	0.99995421	0.99998957
6.0 E-1	0.99640051	0.99899175	0.99973419	0.99993362	0.99998421
6.5 E-1	0.99550722	0.99869111	0.99964105	0.99990675	0.99997693
7.0 E-1	0.99449107	0.99833574	0.99952665	0.99987244	0.99996726
7.5 E-1	0.99334779	0.99792137	0.99938841	0.99982949	0.99995472
8.0 E-1	0.99207367	0.99744395	0.99922375	0.99977659	0.99993876
8.5 E-1	0.99066553	0.99689958	0.99903004	0.99971239	0.99991876
9.0 E-1	0.98912067	0.99628453	0.99880465	0.99963547	0.99989410
9.5 E-1	0.98743687	0.99559529	0.99854496	0.99954435	0.99986405
1.0 E 0	0.98561232	0.99482854	0.99824838	0.99943750	0.99982788
1.5 E 0	0.95949456	0.98230966	0.99270783	0.99714677	0.99893532
2.0 E 0	0.91969860	0.95984037	0.98101184	0.99146761	0.99634015
2.5 E 0	0.86846767	0.92709707	0.96173095	0.98088349	0.99087572
3.0 E 0	0.80884683	0.88500223	0.93435755	0.96429497	0.98142406
3.5 E 0	0.74396970	0.83522548	0.89918965	0.94114441	0.96709838
4.0 E 0	0.67667642	0.77977741	0.85712346	0.91141253	0.94734698
4.5 E 0	0.60933927	0.72071727	0.80943311	0.87553903	0.92198589
5.0 E 0	0.54381312	0.65996323	0.75757613	0.83430826	0.89117802
5.5 E 0	0.48145670	0.59918387	0.70303999	0.78872801	0.85537851
6.0 E 0	0.42319008	0.53974935	0.64723189	0.73991829	0.81526324
6.5 E 0	0.36956667	0.48272319	0.59140764	0.68901902	0.77165344
7.0 E 0	0.32084720	0.42887986	0.53663267	0.63711941	0.72544495
7.5 E 0	0.27706844	0.37873691	0.48376738	0.58520877	0.67754764
8.0 E 0	0.23810331	0.33259390	0.43347012	0.53414622	0.62883694
8.5 E 0	0.20371109	0.29057274	0.38621156	0.48464589	0.58011831
9.0 E 0	0.17357807	0.25265605	0.34229596	0.43727419	0.53210358
9.5 E 0	0.14734918	0.21872185	0.30188558	0.39245576	0.48539756
1.0 E 1	0.12465202	0.18857347	0.26502592	0.35048521	0.44049329
1.5 E 1	0.02025672	0.03599940	0.05914546	0.09093598	0.13206186
2.0 E 1	0.00276940	0.00556968	0.01033605	0.01791240	0.02925269
2.5 E 1	0.00034145	0.00075880	0.00155456	0.00297118	0.00534551
3.0 E 1	3.9308e-05	9.4960e-05	0.00021138	0.00043872	0.00085664
3.5 E 1	4.3095e-06	1.1184e-05	2.6738e-05	5.9583e-05	0.00012487
4.0 E 1	4.5551e-07	1.2588e-06	3.2037e-06	7.5985e-06	1.6945e-05
4.5 E 1	4.6802e-08	1.3676e-07	3.6800e-07	9.2266e-07	2.1747e-06
5.0 E 1	4.7011e-09	1.4445e-08	4.0868e-08	1.0772e-07	2.6691e-07

CHI-SQUARED CUMULATIVE DISTRIBUTION FUNCTION

x	11 d.f.	12 d.f.	13 d.f.	14 d.f.	15 d.f.
4.0 E-1	0.99999958	0.99999993	0.99999999	1.00000000	1.00000000
4.5 E-1	0.99999921	0.99999985	0.99999997	1.00000000	1.00000000
5.0 E-1	0.99999863	0.99999973	0.99999995	0.99999999	1.00000000
5.5 E-1	0.99999773	0.99999953	0.99999990	0.99999998	1.00000000
6.0 E-1	0.99999641	0.99999922	0.99999984	0.99999997	0.99999999
6.5 E-1	0.99999454	0.99999876	0.99999973	0.99999994	0.99999999
7.0 E-1	0.99999196	0.99999811	0.99999957	0.99999991	0.99999998
7.5 E-1	0.99998850	0.99999720	0.99999934	0.99999985	0.99999997
8.0 E-1	0.99998394	0.99999596	0.99999902	0.99999977	0.99999995
8.5 E-1	0.99997805	0.99999431	0.99999858	0.99999966	0.99999992
9.0 E-1	0.99997056	0.99999215	0.99999798	0.99999950	0.99999988
9.5 E-1	0.99996119	0.99998936	0.99999719	0.99999928	0.99999982
1.0 E 0	0.99994961	0.99998584	0.99999617	0.99999900	0.99999975
1.5 E 0	0.99961962	0.99986945	0.99995683	0.99998621	0.99999574
2.0 E 0	0.99849588	0.99940582	0.99977375	0.99991676	0.99997035
2.5 E 0	0.99582417	0.99816191	0.99921977	0.99967987	0.99987277
3.0 E 0	0.99072589	0.99554402	0.99793432	0.99907401	0.99959780
3.5 E 0	0.98233510	0.99086644	0.99544123	0.99779879	0.99896981
4.0 E 0	0.96991702	0.98343639	0.99119139	0.99546619	0.99773734
4.5 E 0	0.95294990	0.97263465	0.98461799	0.99162793	0.99558002
5.0 E 0	0.93116661	0.95797896	0.97519313	0.98581269	0.99212641
5.5 E 0	0.90456053	0.93916469	0.96247680	0.97756669	0.98697983
6.0 E 0	0.87336425	0.91608206	0.94615296	0.96649146	0.97974775
6.5 E 0	0.83801045	0.88881320	0.92605084	0.95227474	0.97007104
7.0 E 0	0.79908350	0.85761355	0.90215156	0.93471190	0.95764975
7.5 E 0	0.75726866	0.82288283	0.87458221	0.91371732	0.94226311
8.0 E 0	0.71330383	0.78513039	0.84360028	0.88932602	0.92378270
8.5 E 0	0.66793719	0.74493905	0.80957139	0.86168708	0.90217836
9.0 E 0	0.62189233	0.70293043	0.77294354	0.83105058	0.87751745
9.5 E 0	0.57584155	0.65973393	0.73422019	0.79775023	0.84995843
1.0 E 1	0.53038715	0.61596065	0.69393437	0.76218346	0.81973992
1.5 E 1	0.18249693	0.24143645	0.30735277	0.37815469	0.45141721
2.0 E 1	0.04534067	0.06708596	0.09521026	0.13014142	0.17193269
2.5 E 1	0.00911668	0.01482287	0.02308373	0.03456739	0.04994343
3.0 E 1	0.00158460	0.00279243	0.00470970	0.00763190	0.01192150
3.5 E 1	0.00024780	0.00046831	0.00084668	0.00147002	0.00245903
4.0 E 1	3.5775e-05	7.1909e-05	0.00013824	0.00025512	0.00045350
4.5 E 1	4.8522e-06	1.0305e-05	2.0927e-05	4.0794e-05	7.6573e-05
5.0 E 1	6.2594e-07	1.3971e-06	2.9815e-06	6.1063e-06	1.2041e-05
5.5 E 1	7.7499e-08	1.8099e-07	4.0410e-07	8.6579e-07	1.7859e-06
6.0 E 1	9.2722e-09	2.2573e-08	5.2535e-08	1.1732e-07	2.5221e-07

x	16 d.f.	17 d.f.	18 d.f.	19 d.f.	20 d.f.
1.0 E 0	0.99999994	0.99999999	1.00000000	1.00000000	1.00000000
1.5 E 0	0.99999872	0.99999963	0.99999989	0.99999997	0.99999999
2.0 E 0	0.99998975	0.99999656	0.99999887	0.99999964	0.99999989
2.5 E 0	0.99995094	0.99998161	0.99999329	0.99999761	0.99999917
3.0 E 0	0.99983043	0.99993050	0.99997226	0.99998921	0.99999590
3.5 E 0	0.99953187	0.99979314	0.99991099	0.99996265	0.99998470
4.0 E 0	0.99890328	0.99948293	0.99976255	0.99989366	0.99995350
4.5 E 0	0.99773291	0.99886863	0.99944994	0.99973914	0.99987920
5.0 E 0	0.99575330	0.99777084	0.99885975	0.99943096	0.99972265
5.5 E 0	0.99265319	0.99596428	0.99783917	0.99887101	0.99942378

CHI-SQUARED CUMULATIVE DISTRIBUTION FUNCTION

x	16 d.f.	17 d.f.	18 d.f.	19 d.f.	20 d.f.
6.0 E 0	0.98809550	0.99318566	0.99619701	0.99792846	0.99889751
6.5 E 0	0.98173903	0.98914646	0.99370889	0.99644000	0.99803135
7.0 E 0	0.97326108	0.98354890	0.99012634	0.99421326	0.99668506
7.5 E 0	0.96237866	0.97610356	0.98518866	0.99103317	0.99469283
8.0 E 0	0.94886638	0.96654666	0.97863657	0.98667088	0.99186776
8.5 E 0	0.93256980	0.95465564	0.97022625	0.98089428	0.98800847
9.0 E 0	0.91341353	0.94026180	0.95974269	0.97347940	0.98290727
9.5 E 0	0.89140415	0.92325931	0.94701116	0.96422157	0.97635931
1.0 E 1	0.86662833	0.90361029	0.93190637	0.95294580	0.96817194
1.5 E 1	0.52463853	0.59548165	0.66196712	0.72259733	0.77640761
2.0 E 1	0.22022065	0.27422927	0.33281968	0.39457818	0.45792971
2.5 E 1	0.06982546	0.09470961	0.12491620	0.16054222	0.20143110
3.0 E 1	0.01800219	0.02634508	0.03744649	0.05179846	0.06985366
3.5 E 1	0.00397430	0.00622120	0.00945240	0.01396683	0.02010428
4.0 E 1	0.00077859	0.00129420	0.00208726	0.00327232	0.00499541
4.5 E 1	0.00013879	0.00024351	0.00041441	0.00068540	0.00110347
5.0 E 1	2.2925e-05	4.2240e-05	7.5483e-05	0.00013106	0.00022148
5.5 E 1	3.5561e-06	6.8523e-06	1.2804e-05	2.3244e-05	4.1061e-05
6.0 E 1	5.2337e-07	1.0509e-06	2.0461e-06	3.8698e-06	7.1218e-06
6.5 E 1	7.3667e-08	1.5367e-07	3.1079e-07	6.1057e-07	1.1671e-06
7.0 E 1	9.9790e-09	2.1568e-08	4.5193e-08	9.1982e-08	1.8214e-07

x	21 d.f.	22 d.f.	23 d.f.	24 d.f.	25 d.f.
2.0 E 0	0.99999997	0.99999999	1.00000000	1.00000000	1.00000000
2.5 E 0	0.99999972	0.99999991	0.99999997	0.99999999	1.00000000
3.0 E 0	0.99999848	0.99999945	0.99999980	0.99999993	0.99999998
3.5 E 0	0.99999388	0.99999760	0.99999908	0.99999965	0.99999987
4.0 E 0	0.99998013	0.99999169	0.99999660	0.99999864	0.99999946
4.5 E 0	0.99994531	0.99997578	0.99998949	0.99999553	0.99999814
5.0 E 0	0.99986784	0.99993837	0.99997186	0.99998740	0.99999447
5.5 E 0	0.99971243	0.99985954	0.99993280	0.99996849	0.99998550
6.0 E 0	0.99942618	0.99970766	0.99985410	0.99992861	0.99996573
6.5 E 0	0.99893516	0.99943614	0.99970747	0.99985120	0.99992573
7.0 E 0	0.99814223	0.99898061	0.99945189	0.99971101	0.99985048
7.5 E 0	0.99692644	0.99825689	0.99903118	0.99947191	0.99971751
8.0 E 0	0.99514424	0.99716023	0.99837218	0.99908477	0.99949494
8.5 E 0	0.99263262	0.99556591	0.99738386	0.99848583	0.99913975
9.0 E 0	0.98921405	0.99333133	0.99595747	0.99759572	0.99859620
9.5 E 0	0.98470270	0.99029967	0.99396797	0.99631938	0.99779493
1.0 E 1	0.97891186	0.98630473	0.99127665	0.99454691	0.99665264
1.5 E 1	0.82295181	0.86223798	0.89463358	0.92075869	0.94138257
2.0 E 1	0.52126125	0.58303975	0.64191179	0.69677615	0.74682531
2.5 E 1	0.24716408	0.29707474	0.35028534	0.40576069	0.46237366
3.0 E 1	0.09198801	0.11846441	0.14940165	0.18475180	0.22428900
3.5 E 1	0.02823511	0.03874505	0.05201556	0.06840084	0.08820321
4.0 E 1	0.00743678	0.01081172	0.01536908	0.02138682	0.02916440
4.5 E 1	0.00173197	0.00265384	0.00397464	0.00582506	0.00836247
5.0 E 1	0.00036480	0.00058646	0.00092132	0.00141597	0.00213115
5.5 E 1	7.0693e-05	0.00011877	0.00019496	0.00031304	0.00049213
6.0 E 1	1.2772e-05	2.2349e-05	3.8206e-05	6.3877e-05	0.00010455
6.5 E 1	2.1737e-06	3.9500e-06	7.0118e-06	1.2172e-05	2.0685e-05
7.0 E 1	3.5140e-07	6.6144e-07	1.2161e-06	2.1865e-06	3.8479e-06
7.5 E 1	5.4324e-08	1.0568e-07	2.0080e-07	3.7307e-07	6.7843e-07

CHI-SQUARED CUMULATIVE DISTRIBUTION FUNCTION

x	21 d.f.	22 d.f.	23 d.f.	24 d.f.	25 d.f.
8.0 E 1	8.0748e-09	1.6202e-08	3.1752e-08	6.0842e-08	1.1411e-07

x	26 d.f.	27 d.f.	28 d.f.	29 d.f.	30 d.f.
4.0 E 0	0.99999979	0.99999992	0.99999997	0.99999999	1.00000000
4.5 E 0	0.99999924	0.99999969	0.99999988	0.99999995	0.99999998
5.0 E 0	0.99999762	0.99999899	0.99999958	0.99999983	0.99999993
5.5 E 0	0.99999345	0.99999710	0.99999873	0.99999946	0.99999977
6.0 E 0	0.99998385	0.99999252	0.99999660	0.99999848	0.99999933
6.5 E 0	0.99996361	0.99998248	0.99999171	0.99999614	0.99999823
7.0 E 0	0.99992404	0.99996209	0.99998140	0.99999102	0.99999574
7.5 E 0	0.99985160	0.99992340	0.99996113	0.99998060	0.99999047
8.0 E 0	0.99972628	0.99985423	0.99992367	0.99996068	0.99998007
8.5 E 0	0.99951996	0.99973675	0.99985805	0.99992470	0.99996068
9.0 E 0	0.99919486	0.99954614	0.99974841	0.99986279	0.99992634
9.5 E 0	0.99870218	0.99924918	0.99957282	0.99976086	0.99986821
1.0 E 1	0.99798115	0.99880304	0.99930201	0.99959948	0.99977375
1.5 E 1	0.95733413	0.96943196	0.97843535	0.98501495	0.98973957
2.0 E 1	0.79155648	0.83075612	0.86446442	0.89292709	0.91654153
2.5 E 1	0.51897522	0.57446199	0.62783534	0.67824747	0.72503188
3.0 E 1	0.26761103	0.31415383	0.36321784	0.41400364	0.46565371
3.5 E 1	0.11164885	0.13886592	0.16986733	0.20453980	0.24264044
4.0 E 1	0.03901199	0.05123690	0.06612764	0.08393690	0.10486428
4.5 E 1	0.01177110	0.01626055	0.02206231	0.02942403	0.03860176
5.0 E 1	0.00314412	0.00455081	0.00646748	0.00903166	0.01240206
5.5 E 1	0.00075824	0.00114591	0.00170001	0.00247766	0.00354993
6.0 E 1	0.00016770	0.00026379	0.00040728	0.00061766	0.00092068
6.5 E 1	3.4441e-05	5.6235e-05	9.0112e-05	0.00014182	0.00021935
7.0 E 1	6.6346e-06	1.1217e-05	1.8610e-05	3.0322e-05	4.8549e-05
7.5 E 1	1.2087e-06	2.1113e-06	3.6190e-06	6.0916e-06	1.0075e-05
8.0 E 1	2.0964e-07	3.7764e-07	6.6749e-07	1.1585e-06	1.9756e-06
8.5 E 1	3.4816e-08	6.4562e-08	1.1747e-07	2.0986e-07	3.6838e-07
9.0 E 1	5.5622e-09	1.0602e-08	1.9826e-08	3.6403e-08	6.5673e-08

x	31 d.f.	32 d.f.	33 d.f.	34 d.f.	35 d.f.
6.0 E 0	0.99999971	0.99999988	0.99999995	0.99999998	0.99999999
6.5 E 0	0.99999920	0.99999965	0.99999985	0.99999993	0.99999997
7.0 E 0	0.99999801	0.99999908	0.99999958	0.99999981	0.99999992
7.5 E 0	0.99999539	0.99999780	0.99999897	0.99999952	0.99999978
8.0 E 0	0.99999005	0.99999511	0.99999763	0.99999887	0.99999947
8.5 E 0	0.99997978	0.99998976	0.99999489	0.99999748	0.99999878
9.0 E 0	0.99996106	0.99997972	0.99998959	0.99999473	0.99999737
9.5 E 0	0.99992848	0.99996175	0.99997984	0.99998952	0.99999463
1.0 E 1	0.99987412	0.99993099	0.99996271	0.99998013	0.99998956
1.5 E 1	0.99307512	0.99539168	0.99697520	0.99804111	0.99874797
2.0 E 1	0.93580362	0.95125960	0.96346590	0.97295839	0.98023092
2.5 E 1	0.76771772	0.80602900	0.83987115	0.86930800	0.89453283
3.0 E 1	0.51729655	0.56808958	0.61725743	0.66412320	0.70813095
3.5 E 1	0.28380139	0.32754239	0.37329027	0.42040390	0.46820272
4.0 E 1	0.12904035	0.15651313	0.18723835	0.22107420	0.25778138
4.5 E 1	0.04985012	0.06341093	0.07950088	0.09829883	0.11993375
5.0 E 1	0.01675727	0.02229302	0.02921792	0.03774765	0.04809770
5.5 E 1	0.00500341	0.00694143	0.00948458	0.01277058	0.01695320

CHI-SQUARED CUMULATIVE DISTRIBUTION FUNCTION

x	31 d.f.	32 d.f.	33 d.f.	34 d.f.	35 d.f.
6.0 E 1	0.00134979	0.00194748	0.00276681	0.00387272	0.00534322
6.5 E 1	0.00033364	0.00049936	0.00073585	0.00106814	0.00152809
7.0 E 1	7.6437e-05	0.00011841	0.00018057	0.00027122	0.00040145
7.5 E 1	1.6385e-05	2.6216e-05	4.1290e-05	6.4046e-05	9.7890e-05
8.0 E 1	3.3125e-06	5.4640e-06	8.8714e-06	1.4185e-05	2.2347e-05
8.5 E 1	6.3574e-07	1.0793e-06	1.8035e-06	2.9677e-06	4.8113e-06
9.0 E 1	1.1648e-07	2.0322e-07	3.4895e-07	5.9005e-07	9.8297e-07
9.5 E 1	2.0469e-08	3.6650e-08	6.4585e-08	1.1207e-07	1.9159e-07
1.0 E 2	3.4644e-09	6.3580e-09	1.1484e-08	2.0424e-08	3.5785e-08

x	36 d.f.	37 d.f.	38 d.f.	39 d.f.	40 d.f.
8.0 E 0	0.99999975	0.99999989	0.99999995	0.99999998	0.99999999
8.5 E 0	0.99999941	0.99999972	0.99999987	0.99999994	0.99999997
9.0 E 0	0.99999870	0.99999937	0.99999970	0.99999986	0.99999993
9.5 E 0	0.99999728	0.99999864	0.99999933	0.99999967	0.99999984
1.0 E 1	0.99999458	0.99999723	0.99999860	0.99999930	0.99999965
1.5 E 1	0.99920998	0.99950772	0.99969700	0.99981573	0.99988925
2.0 E 1	0.98572239	0.98981093	0.99281350	0.99498931	0.99654566
2.5 E 1	0.91583667	0.93357689	0.94814825	0.95995801	0.96940587
3.0 E 1	0.74885875	0.78602254	0.81947171	0.84917789	0.87521878
3.5 E 1	0.51599664	0.56311518	0.60893402	0.65289723	0.69453423
4.0 E 1	0.29702840	0.33840199	0.38142195	0.42555940	0.47025727
4.5 E 1	0.14447400	0.17191886	0.20219295	0.23514400	0.27054435
5.0 E 1	0.06047504	0.07506882	0.09204086	0.11151627	0.13357483
5.5 E 1	0.02220009	0.02868959	0.03660628	0.04613558	0.05745735
6.0 E 1	0.00727022	0.00975993	0.01293270	0.01692215	0.02187347
6.5 E 1	0.00215551	0.00299938	0.00411882	0.00558408	0.00747712
7.0 E 1	0.00058584	0.00084322	0.00119759	0.00167900	0.00232451
7.5 E 1	0.00014750	0.00021918	0.00032135	0.00046503	0.00066447
8.0 E 1	3.4705e-05	5.3149e-05	8.0304e-05	0.00011975	0.00017630
8.5 E 1	7.6886e-06	1.2116e-05	1.8835e-05	2.8897e-05	4.3769e-05
9.0 E 1	1.6140e-06	2.6133e-06	4.1740e-06	6.5790e-06	1.0237e-05
9.5 E 1	3.2281e-07	5.3630e-07	8.7891e-07	1.4214e-06	2.2692e-06
1.0 E 2	6.1795e-08	1.0522e-07	1.7672e-07	2.9287e-07	4.7914e-07

CRITICAL POINTS OF THE CHI-SQUARED DISTRIBUTION

Q	1 d.f.	2 d.f.	3 d.f.	4 d.f.	5 d.f.
5.0e-02	3.84145882	5.99146455	7.81472790	9.48772904	11.07049769
2.5e-02	5.02388619	7.37775891	9.34840360	11.14328678	12.83250199
1.0e-02	6.63489660	9.21034037	11.34486673	13.27670414	15.08627247
5.0e-03	7.87943858	10.59663473	12.83815647	14.86025900	16.74960234
2.5e-03	9.14059346	11.98292909	14.32034710	16.42393612	18.38561256
1.0e-03	10.82756617	13.81551056	16.26623620	18.46682695	20.51500565
5.0e-04	12.11566515	15.20180492	17.72999623	19.99735500	22.10532678
2.5e-04	13.41214780	16.58809928	19.18788484	21.51727547	23.68098327
1.0e-04	15.13670523	18.42068074	21.10751347	23.51274244	25.74483196
5.0e-05	16.44811021	19.80697511	22.55474907	25.01334257	27.29370745
2.5e-05	17.76453626	21.19326947	23.99833330	26.50729515	28.83334372
1.0e-05	19.51142096	23.02585093	25.90174975	28.47325542	30.85618994

Q	6 d.f.	7 d.f.	8 d.f.	9 d.f.	10 d.f.
5.0e-02	12.59158724	14.06714045	15.50731306	16.91897760	18.30703805
2.5e-02	14.44937534	16.01276427	17.53454614	19.02276780	20.48317735
1.0e-02	16.81189383	18.47530691	20.09023503	21.66599433	23.20925116
5.0e-03	18.54758418	20.27773987	21.95495499	23.58935078	25.18817957
2.5e-03	20.24940205	22.04039059	23.77447432	25.46247870	27.11217103
1.0e-03	22.45774448	24.32188635	26.12448156	27.87716487	29.58829845
5.0e-04	24.10279899	26.01776771	27.86804640	29.66580810	31.41981251
2.5e-04	25.72959419	27.69213487	29.58710481	31.42716313	33.22142994
1.0e-04	27.85634124	29.87750391	31.82762800	33.71994844	35.56401394
5.0e-05	29.44972494	31.51244350	33.50172020	35.43118280	37.31067361
2.5e-05	31.03155887	33.13373235	35.16022442	37.12502426	39.03824512
1.0e-05	33.10705682	35.25853642	37.33159365	39.34065373	41.29615797

Q	11 d.f.	12 d.f.	13 d.f.	14 d.f.	15 d.f.
5.0e-02	19.67513757	21.02606982	22.36203249	23.68479130	24.99579014
2.5e-02	21.92004926	23.33666416	24.73560488	26.11894805	27.48839286
1.0e-02	24.72497031	26.21696731	27.68824961	29.14123774	30.57791417
5.0e-03	26.75684892	28.29951882	29.81947122	31.31934962	32.80132065
2.5e-03	28.72934952	30.31847913	31.88308547	33.42601051	34.94958514
1.0e-03	31.26413362	32.90949041	34.52817897	36.12327368	37.69729822
5.0e-04	33.13661500	34.82127464	36.47779372	38.10940393	39.71875979
2.5e-04	34.97673714	36.69836174	38.39048171	40.05647283	41.69910967
1.0e-04	37.36698644	39.13440388	40.87065501	42.57928895	44.26322494
5.0e-05	39.14758136	40.94762380	42.71533530	44.45438498	46.16779243
2.5e-05	40.90748557	42.73863015	44.53634852	46.30442153	48.04596235
1.0e-05	43.20595972	45.07614652	46.91155313	48.71609690	50.49300558

Q	16 d.f.	17 d.f.	18 d.f.	19 d.f.	20 d.f.
5.0e-02	26.29622760	27.58711164	28.86929943	30.14352721	31.41043284
2.5e-02	28.84535072	30.19100912	31.52637844	32.85232686	34.16960690
1.0e-02	31.99992691	33.40866361	34.80530573	36.19086913	37.56623479
5.0e-03	34.26718654	35.71846566	37.15645146	38.58225655	39.99684631
2.5e-03	36.45574943	37.94613878	39.42214703	40.88497373	42.33566008
1.0e-03	39.25235479	40.79021671	42.31239633	43.82019596	45.31474662
5.0e-04	41.30807372	42.87921296	44.43377074	45.97311956	47.49845189
2.5e-04	43.32070595	44.92321540	46.50830576	48.07741433	49.63179033
1.0e-04	45.92489905	47.56636956	49.18939447	50.79548967	52.38597327
5.0e-05	47.85807872	49.52737490	51.17750188	52.81003035	54.42632659
2.5e-05	49.76357138	51.45944770	53.13547110	54.79326363	56.43423673
1.0e-05	52.24497689	53.97429344	55.68290738	57.37250401	59.04455038

CRITICAL POINTS OF THE CHI-SQUARED DISTRIBUTION

Q	21 d.f.	22 d.f.	23 d.f.	24 d.f.	25 d.f.
5.0e-02	32.67057334	33.92443847	35.17246163	36.41502850	37.65248413
2.5e-02	35.47887591	36.78071208	38.07562725	39.36407703	40.64646912
1.0e-02	38.93217268	40.28936044	41.63839812	42.97982014	44.31410490
5.0e-03	41.40106477	42.79565500	44.18127525	45.55851194	46.92789016
2.5e-03	43.77511678	45.20414590	46.62345817	48.03368695	49.43539955
1.0e-03	46.79703804	48.26794229	49.72823247	51.17859778	52.61965578
5.0e-04	49.01081160	50.51111876	52.00018929	53.47875077	54.94745532
2.5e-04	51.17252772	52.70059100	54.21683575	55.72202514	57.21684336
1.0e-04	53.96200012	55.52458878	57.07464314	58.61296975	60.14029191
5.0e-05	56.02758799	57.61487098	59.18911330	60.75115190	62.30173755
2.5e-05	58.05962772	59.67052860	61.26790892	62.85263426	64.42548124
1.0e-05	60.70033312	62.34098809	63.96752419	65.58084237	67.18175123

Q	26 d.f.	27 d.f.	28 d.f.	29 d.f.	30 d.f.
5.0e-02	38.88513866	40.11327207	41.33713815	42.55696780	43.77297183
2.5e-02	41.92317010	43.19451097	44.46079184	45.72228580	46.97924224
1.0e-02	45.64168267	46.96294212	48.27823577	49.58788447	50.89218131
5.0e-03	48.28988233	49.64491530	50.99337627	52.33561779	53.67196193
2.5e-03	50.82910648	52.21526915	53.59430618	54.96659884	56.33249550
1.0e-03	54.05196239	55.47602021	56.89228539	58.30117349	59.70306430
5.0e-04	56.40689012	57.85758614	59.30002543	60.73464717	62.16185287
2.5e-04	58.70190660	60.17777222	61.64494631	63.10389007	64.55502519
1.0e-04	61.65726128	63.16446742	64.66244583	66.15168463	67.63263026
5.0e-05	63.84154680	65.37119186	66.89122891	68.40216501	69.90446403
2.5e-05	65.98714983	67.53827362	69.07942827	70.61113873	72.13388529
1.0e-05	68.77097979	70.34918806	71.91697592	73.47489049	75.02343244

Q	31 d.f.	32 d.f.	33 d.f.	34 d.f.	35 d.f.
5.0e-02	44.98534328	46.19425952	47.39988392	48.60236737	49.80184957
2.5e-02	48.23188959	49.48043774	50.72508007	51.96599520	53.20334854
1.0e-02	52.19139483	53.48577184	54.77553976	56.06090875	57.34207343
5.0e-03	55.00270388	56.32811496	57.64844526	58.96392588	60.27477090
2.5e-03	57.69231551	59.04635245	60.39487698	61.73813927	63.07637108
1.0e-03	61.09830608	62.48721906	63.87009852	65.24721746	66.61882884
5.0e-04	63.58201075	64.99545945	66.40251132	67.80345512	69.19855850
2.5e-04	65.99873850	67.43538579	68.86529528	70.28877048	71.70609273
1.0e-04	69.10569229	70.57124758	72.02964379	73.48120252	74.92622189
5.0e-05	71.39855157	72.88481931	74.36362867	75.83531397	77.30018521
2.5e-05	73.64810871	75.15421470	76.65257762	78.14354387	79.62743467
1.0e-05	76.56306136	78.09420036	79.61723994	81.13254149	82.64044019

Q	36 d.f.	37 d.f.	38 d.f.	39 d.f.	40 d.f.
5.0e-02	50.99846017	52.19231973	53.38354062	54.57222776	55.75847928
2.5e-02	54.43729363	55.66797326	56.89552054	58.12005973	59.34170714
1.0e-02	58.61921450	59.89250005	61.16208676	62.42812102	63.69073975
5.0e-03	61.58117911	62.88333545	64.18141236	65.47557090	66.76596183
2.5e-03	64.40978764	65.73858924	67.06296262	68.38308229	69.69911154
1.0e-03	67.98516763	69.34645250	70.70288741	72.05466295	73.40195752
5.0e-04	70.58807009	71.97222133	73.35122809	74.72529212	76.09460230
2.5e-04	73.11752340	74.52330585	75.92366708	77.31881927	78.70896109
1.0e-04	76.36497897	77.79773171	79.22472082	80.64617129	82.06229384
5.0e-05	78.75853051	80.21061817	81.65669856	83.09700574	84.53175893
2.5e-05	81.10454860	82.57516374	84.03953962	85.49791889	86.95052884
1.0e-05	84.14124764	85.63525412	87.12273053	88.60393026	90.07909064

CRITICAL POINTS OF THE CHI-SQUARED DISTRIBUTION

Q	41 d.f.	42 d.f.	43 d.f.	44 d.f.	45 d.f.
5.0e-02	56.94238715	58.12403768	59.30351203	60.48088658	61.65623338
2.5e-02	60.56057173	61.77675581	62.99035553	64.20146147	65.41015901
1.0e-02	64.95007134	66.20623628	67.45934792	68.70951297	69.95683207
5.0e-03	68.05272646	69.33599746	70.61589962	71.89255046	73.16606082
2.5e-03	71.01120348	72.31950189	73.62414200	74.92525119	76.22294959
1.0e-03	74.74493840	76.08376271	77.41857824	78.74952423	80.07673201
5.0e-04	77.45933575	78.81965886	80.17572814	81.52769106	82.87568675
2.5e-04	80.09427886	81.47494763	82.85113208	84.22298738	85.59065996
1.0e-04	83.47328610	84.87933377	86.28061161	87.67728426	89.06950713
5.0e-05	85.96116379	87.38541357	88.80469011	90.21916481	91.62899942
2.5e-05	88.39758272	89.83928092	91.27581204	92.70735386	94.13407414
1.0e-05	91.54843436	93.01217076	94.47049685	95.92359835	97.37165059

Q	46 d.f.	47 d.f.	48 d.f.	49 d.f.	50 d.f.
5.0e-02	62.82962041	64.00111197	65.17076890	66.33864886	67.50480655
2.5e-02	66.61652877	67.82064698	69.02258579	70.22241357	71.42019519
1.0e-02	71.20140025	72.44330738	73.68263852	74.91947431	76.15389125
5.0e-03	74.43653537	75.70407310	76.96876773	78.23070809	79.48997847
2.5e-03	77.51735068	78.80856178	80.09668449	81.38181516	82.66404518
1.0e-03	81.40032566	82.72042252	84.03713372	85.35056461	86.66081519
5.0e-04	84.21984665	85.56029507	86.89714977	88.23052238	89.56051887
2.5e-04	86.95428814	88.31400277	89.66992781	91.02218077	92.37087323
1.0e-04	90.45742704	91.84118289	93.22090628	94.59672198	95.96874848
5.0e-05	93.03434678	94.43535151	95.83215062	97.22487404	98.61364512
2.5e-05	95.55613146	96.97367583	98.38684943	99.79578706	101.20061674
1.0e-05	98.81481924	100.25326109	101.68712476	103.11655114	104.54167405

Q	51 d.f.	52 d.f.	53 d.f.	54 d.f.	55 d.f.
5.0e-02	68.66929391	69.83216034	70.99345283	72.15321617	73.31149303
2.5e-02	72.61599227	73.80986340	75.00186432	76.19204817	77.38046558
1.0e-02	77.38596202	78.61575571	79.84333812	81.06877191	82.29211683
5.0e-03	80.74665895	82.00082570	83.25255121	84.50190453	85.74895156
2.5e-03	83.94346142	85.22014647	86.49417897	87.76563384	89.03458254
1.0e-03	87.96798048	89.27215083	90.57341231	91.87184688	93.16753277
5.0e-04	90.88723993	92.21078135	93.53123433	94.84868578	96.16321862
2.5e-04	93.71611120	95.05799554	96.39662229	97.73208297	99.06446492
1.0e-04	97.33709836	98.70187873	100.06319162	101.42113424	102.77579937
5.0e-05	99.99858115	101.37979366	102.75738891	104.13146818	105.50212810
2.5e-05	102.60146017	103.99843310	105.39164584	106.78120350	108.16720638
1.0e-05	105.96262078	107.37951230	108.79246399	110.20158575	111.60698248

Q	56 d.f.	57 d.f.	58 d.f.	59 d.f.	60 d.f.
5.0e-02	74.46832416	75.62374847	76.77780316	77.93052381	79.08194449
2.5e-02	78.56716489	79.75219228	80.93559189	82.11740594	83.29767488
1.0e-02	83.51342993	84.73276571	85.95017625	87.16571140	88.37941890
5.0e-03	86.99375516	88.23637541	89.47686974	90.71529311	91.95169816
2.5e-03	90.30109330	91.56523128	92.82705882	94.08663555	95.34401859
1.0e-03	94.46054464	95.75095383	97.03882857	98.32423413	99.60723307
5.0e-04	97.47491199	98.78384155	100.09007962	101.39369545	102.69475534
2.5e-04	100.39385150	101.72032239	103.04395378	104.36481861	105.68298675
1.0e-04	104.12727559	105.47564755	106.82099623	108.16339913	109.50293050
5.0e-05	106.86946096	108.23355493	109.59449440	110.95236011	112.30722944
2.5e-05	109.54975031	110.92892680	112.30482349	113.67752418	115.04710924
1.0e-05	113.00875445	114.40699733	115.80180284	117.19325862	118.58144877

CRITICAL POINTS OF THE CHI-SQUARED DISTRIBUTION

Q	65 d.f.	70 d.f.	75 d.f.	80 d.f.	85 d.f.
5.0e-02	84.82064550	90.53122543	96.21667075	101.87947397	107.52174097
2.5e-02	89.17714500	95.02318419	100.83933840	106.62856773	112.39337362
1.0e-02	94.42207901	100.42518423	106.39292293	112.32879252	118.23574925
5.0e-03	98.10514381	104.21489878	110.28558336	116.32105651	122.32458070
2.5e-03	101.59986837	107.80822146	113.97409415	120.10167707	126.19451379
1.0e-03	105.98814309	112.31693185	118.59909476	124.83922402	131.04120375
5.0e-04	109.16385261	115.57758394	121.94177948	128.26131219	134.54030444
2.5e-04	112.23562837	118.72984327	125.17177624	131.56656429	137.91855400
1.0e-04	116.15990748	122.75465376	129.29370362	135.78252112	142.22573071
5.0e-05	119.03912407	125.70606968	132.31487775	138.87124531	145.37999479
2.5e-05	121.85088952	128.58712309	135.26288544	141.88409556	148.45576468
1.0e-05	125.47612445	132.29998398	139.06043392	145.76367171	152.41494556

Q	90 d.f.	95 d.f.	100 d.f.	105 d.f.	110 d.f.
5.0e-02	113.14527014	118.75161175	124.34211340	129.91795529	135.48017793
2.5e-02	118.13589256	123.85796660	129.56119719	135.24698697	140.91657279
1.0e-02	124.11631869	129.97267873	135.80672317	141.62011104	147.41430540
5.0e-03	128.29894360	134.24654953	140.16948944	146.06959531	151.94848294
2.5e-03	132.25563268	138.28764609	144.29282620	150.27316400	156.23041577
1.0e-03	137.20835413	143.34353978	149.44925278	155.52767718	161.58073983
5.0e-04	140.78228067	146.99028216	153.16695508	159.31461879	165.43531974
2.5e-04	144.23146210	150.50849629	156.75244790	162.96576382	169.15060328
1.0e-04	148.62728755	154.99060580	161.31865696	167.61404626	173.87907303
5.0e-05	151.84525139	158.27057652	164.65907025	171.01345087	177.33611773
2.5e-05	154.98218017	161.46704394	167.91357880	174.32461098	180.70263539
1.0e-05	159.01874650	165.57895295	172.09894203	178.58167564	185.02976893

Q	115 d.f.	120 d.f.	125 d.f.	130 d.f.	135 d.f.
5.0e-02	141.02970429	146.56735758	152.09387569	157.60992312	163.11610079
2.5e-02	146.57105181	152.21140273	157.83850288	163.45314242	169.05603592
1.0e-02	153.19060434	158.95016590	164.69402832	170.42312675	176.13830702
5.0e-03	157.80758605	163.64818381	169.47142319	175.27833734	181.06986079
2.5e-03	162.16614004	168.08172747	173.97842510	179.85735633	185.71953738
1.0e-03	167.61015088	173.61743646	179.60396526	185.57097039	191.51956746
5.0e-04	171.53087444	177.60290407	183.65286270	189.68206048	195.69168278
2.5e-04	175.30888309	181.44231409	187.55243093	193.64061644	199.70812188
1.0e-04	180.11577865	186.32598545	192.51132820	198.67328015	204.81317449
5.0e-05	183.62920125	189.89460338	196.13403050	202.34902049	208.54096511
2.5e-05	187.04986747	193.36828530	199.65966366	205.92560226	212.16754885
1.0e-05	191.44554430	197.83107564	204.18822383	210.51866645	216.82392175

Q	140 d.f.	145 d.f.	150 d.f.	155 d.f.	160 d.f.
5.0e-02	168.61295425	174.10098057	179.58063415	185.05233172	190.51645651
2.5e-02	174.64783219	180.22912239	185.80044700	191.36230169	196.91514233
1.0e-02	181.84033713	187.52991695	193.20768639	198.87423232	204.53009459
5.0e-03	186.84684216	192.61005485	198.36020600	204.09794423	209.82386618
2.5e-03	191.56589116	197.39725877	203.21440934	209.01804840	214.80882494
1.0e-03	197.45076969	203.36550060	209.26460477	215.14885695	221.01896990
5.0e-04	201.68280628	207.65641241	213.61339872	219.55458866	225.48073980
2.5e-04	205.75608384	211.78553840	217.79743321	223.79263766	229.77195169
1.0e-04	210.93222233	217.03152780	223.11210086	229.17486812	235.22068225
5.0e-05	214.71112882	220.86066440	226.99062641	233.10198242	239.19562283
2.5e-05	218.38681893	224.58461179	230.76202468	236.92006436	243.05965734
1.0e-05	223.10536955	229.36426770	235.60176730	241.81892448	248.01671129

'STUDENT'S t' PROBABILITY DENSITY FUNCTION

d.f.	x = 1 E-1	2 E-1	3 E-1	4 E-1	5 E-1
1	0.315158303	0.306067198	0.292027419	0.274405074	0.254647909
2	0.350918217	0.343205903	0.330963858	0.315006397	0.296296296
3	0.365114444	0.357943795	0.346453574	0.331274372	0.313180911
4	0.372666466	0.365786635	0.354709627	0.339975734	0.322261869
5	0.377338130	0.370639978	0.359824328	0.345378076	0.327918531
6	0.380508511	0.373934678	0.363299252	0.349053932	0.331776000
7	0.382799334	0.376315935	0.365812301	0.351715215	0.334573253
8	0.384531296	0.378116644	0.367713578	0.353730330	0.336693898
9	0.385886327	0.379525702	0.369201901	0.355308811	0.338356623
10	0.386975226	0.380658181	0.370398462	0.356578534	0.339695136
11	0.387869292	0.381588143	0.371381302	0.357621931	0.340795755
12	0.388616469	0.382365399	0.372202936	0.358494513	0.341716676
13	0.389250176	0.383024676	0.372899987	0.359235022	0.342498557
14	0.389794417	0.383590922	0.373498775	0.359871314	0.343170658
15	0.390266876	0.384082518	0.374018696	0.360423932	0.343754570
16	0.390680867	0.384513304	0.374474362	0.360908354	0.344266575
17	0.391046605	0.384893901	0.374876985	0.361336465	0.344719180
18	0.391372058	0.385232592	0.375235313	0.361717540	0.345122151
19	0.391663531	0.385535934	0.375556272	0.362058924	0.345483225
20	0.391926080	0.385809186	0.375845417	0.362366510	0.345808612
21	0.392163805	0.386056611	0.376107252	0.362645078	0.346103352
22	0.392380067	0.386281704	0.376345470	0.362898549	0.346371579
23	0.392577645	0.386487358	0.376563130	0.363130167	0.346616716
24	0.392758861	0.386675985	0.376762780	0.363342641	0.346841620
25	0.392925665	0.386849616	0.376946567	0.363538249	0.347048697
26	0.393079711	0.387009971	0.377116309	0.363718924	0.347239985
27	0.393222408	0.387158515	0.377273557	0.363886311	0.347417223
28	0.393354965	0.387296506	0.377419640	0.364041824	0.347581904
29	0.393478426	0.387425031	0.377555706	0.364186684	0.347735317
30	0.393593696	0.387545032	0.377682753	0.364321949	0.347878580
31	0.393701563	0.387657328	0.377801646	0.364448540	0.348012667
32	0.393802719	0.387762638	0.377913148	0.364567267	0.348138434
33	0.393897771	0.387861596	0.378017926	0.364678840	0.348256630
34	0.393987255	0.387954758	0.378116571	0.364783887	0.348367920
35	0.394071647	0.388042619	0.378209605	0.364882964	0.348472891
36	0.394151368	0.388125620	0.378297495	0.364976566	0.348572067
37	0.394226797	0.388204152	0.378380655	0.365065134	0.348665915
38	0.394298270	0.388278567	0.378459457	0.365149064	0.348754852
39	0.394366091	0.388349180	0.378534235	0.365228710	0.348839255
40	0.394430533	0.388416275	0.378605289	0.365304393	0.348919462
41	0.394491843	0.388480110	0.378672891	0.365376402	0.348995777
42	0.394550242	0.388540915	0.378737287	0.365444996	0.349068477
43	0.394605934	0.388598902	0.378798699	0.365510415	0.349137814
44	0.394659103	0.388654262	0.378857329	0.365572872	0.349204015
45	0.394709915	0.388707170	0.378913363	0.365632565	0.349267288
46	0.394758526	0.388757785	0.378966970	0.365689673	0.349327824
47	0.394805073	0.388806252	0.379018303	0.365744361	0.349385796
48	0.394849687	0.388852706	0.379067505	0.365796779	0.349441364
49	0.394892485	0.388897270	0.379114705	0.365847066	0.349494674
50	0.394933576	0.388940056	0.379160023	0.365895349	0.349545860

'STUDENT'S t' PROBABILITY DENSITY FUNCTION

d.f.	x = 1 E-1	2 E-1	3 E-1	4 E-1	5 E-1
55	0.395116673	0.389130712	0.379361967	0.366110517	0.349773988
60	0.395269323	0.389289668	0.379530344	0.366289935	0.349964235
65	0.395398537	0.389424223	0.379672880	0.366441828	0.350125312
70	0.395509328	0.389539596	0.379795101	0.366572081	0.350263451
75	0.395605373	0.389639615	0.379901060	0.366685009	0.350383226
80	0.395689433	0.389727154	0.379993800	0.366783853	0.350488070
85	0.395763619	0.389804411	0.380075650	0.366871094	0.350580612
90	0.395829574	0.389873098	0.380148422	0.366948662	0.350662896
95	0.395888597	0.389934565	0.380213546	0.367018080	0.350736539
100	0.395941724	0.389989894	0.380272168	0.367080569	0.350802833
105	0.395989799	0.390039961	0.380325215	0.367137117	0.350862827
110	0.396033508	0.390085482	0.380373447	0.367188533	0.350917378
115	0.396073421	0.390127050	0.380417490	0.367235485	0.350967195
120	0.396110012	0.390165158	0.380457868	0.367278531	0.351012867
125	0.396143678	0.390200220	0.380495020	0.367318138	0.351054892
130	0.396174757	0.390232588	0.380529317	0.367354702	0.351093690
135	0.396203537	0.390262561	0.380561077	0.367388562	0.351129619
140	0.396230262	0.390290396	0.380590571	0.367420006	0.351162985
145	0.396255146	0.390316312	0.380618032	0.367449284	0.351194054
150	0.396278373	0.390340503	0.380643665	0.367476613	0.351223054
155	0.396300102	0.390363134	0.380667646	0.367502181	0.351250186
160	0.396320475	0.390384352	0.380690130	0.367526152	0.351275625
165	0.396339613	0.390404285	0.380711252	0.367548673	0.351299523
170	0.396357627	0.390423046	0.380731133	0.367569870	0.351322018
175	0.396374612	0.390440737	0.380749879	0.367589857	0.351343229
180	0.396390654	0.390457445	0.380767584	0.367608735	0.351363263
185	0.396405830	0.390473251	0.380784333	0.367626593	0.351382216
190	0.396420208	0.390488225	0.380800201	0.367643512	0.351400172
195	0.396433848	0.390502432	0.380815256	0.367659565	0.351417208
200	0.396446807	0.390515929	0.380829559	0.367674815	0.351433394

d.f.	x = 6 E-1	7 E-1	8 E-1	9 E-1	10 E-1
1	0.234051387	0.213630796	0.194091394	0.175861816	0.159154943
2	0.275823964	0.254507731	0.233127824	0.212295369	0.192450090
3	0.293010680	0.271588359	0.249665905	0.227883066	0.206748336
4	0.302318708	0.280908832	0.258753537	0.236493144	0.214662526
5	0.308141010	0.286765458	0.264488357	0.241944344	0.219679797
6	0.312122533	0.290782716	0.268433522	0.245702817	0.223142291
7	0.315015547	0.293708135	0.271312505	0.248450062	0.225674920
8	0.317212115	0.295933008	0.273505568	0.250545396	0.227607580
9	0.318936410	0.297681800	0.275231524	0.252196066	0.229130733
10	0.320325811	0.299092418	0.276625132	0.253529951	0.230361989
11	0.321469166	0.300254233	0.277773896	0.254630209	0.231377879
12	0.322426471	0.301227696	0.278737095	0.255553249	0.232230341
13	0.323239694	0.302055151	0.279556312	0.256338680	0.232955863
14	0.323939069	0.302767140	0.280261572	0.257015130	0.233580824
15	0.324546930	0.303386246	0.280875100	0.257603803	0.234124773
16	0.325080127	0.303929525	0.281413695	0.258120739	0.234602498
17	0.325551616	0.304410100	0.281890289	0.258578295	0.235025398
18	0.325971520	0.304838230	0.282315005	0.258986144	0.235402396
19	0.326347864	0.305222055	0.282695874	0.259351968	0.235740580
20	0.326687089	0.305568112	0.283039350	0.259681943	0.236045649

'STUDENT'S t' PROBABILITY DENSITY FUNCTION

d.f.	x = 6 E-1	7 E-1	8 E-1	9 E-1	10 E-1
21	0.326994428	0.305881713	0.283350683	0.259981092	0.236322240
22	0.327274174	0.306167218	0.283634181	0.260253540	0.236574163
23	0.327529882	0.306428241	0.283893418	0.260502710	0.236804575
24	0.327764522	0.306667801	0.284131379	0.260731462	0.237016120
25	0.327980595	0.306888439	0.284350580	0.260942207	0.237211023
26	0.328180221	0.307092313	0.284553155	0.261136990	0.237391173
27	0.328365207	0.307281263	0.284740927	0.261317559	0.237558185
28	0.328537108	0.307456869	0.284915461	0.261485414	0.237713444
29	0.328697264	0.307620497	0.285078108	0.261641853	0.237858150
30	0.328846838	0.307773331	0.285230043	0.261788002	0.237993342
31	0.328986847	0.307916406	0.285372291	0.261924844	0.238119930
32	0.329118178	0.308050627	0.285505749	0.262053239	0.238238708
33	0.329241615	0.308176791	0.285631208	0.262173948	0.238350379
34	0.329357848	0.308295603	0.285749365	0.262287640	0.238455562
35	0.329467491	0.308407687	0.285860840	0.262394909	0.238554805
36	0.329571087	0.308513598	0.285966185	0.262496285	0.238648599
37	0.329669124	0.308613834	0.286065891	0.262592240	0.238737379
38	0.329762038	0.308708838	0.286160399	0.262683198	0.238821539
39	0.329850219	0.308799009	0.286250106	0.262769540	0.238901428
40	0.329934021	0.308884707	0.286335368	0.262851607	0.238977364
41	0.330013762	0.308966256	0.286416507	0.262929711	0.239049634
42	0.330089729	0.309043952	0.286493816	0.263004130	0.239118497
43	0.330162185	0.309118060	0.286567560	0.263075121	0.239184188
44	0.330231368	0.309188824	0.286637979	0.263142913	0.239246921
45	0.330297494	0.309256465	0.286705294	0.263207720	0.239306891
46	0.330360762	0.309321185	0.286769705	0.263269733	0.239364278
47	0.330421352	0.309383169	0.286831396	0.263329130	0.239419245
48	0.330479432	0.309442587	0.286890536	0.263386073	0.239471941
49	0.330535155	0.309499596	0.286947279	0.263440710	0.239522504
50	0.330588660	0.309554339	0.287001770	0.263493180	0.239571062
55	0.330827143	0.309798364	0.287244696	0.263727115	0.239787565
60	0.331026054	0.310001931	0.287447378	0.263922321	0.239968235
65	0.331194489	0.310174330	0.287619051	0.264087680	0.240121288
70	0.331338952	0.310322212	0.287766326	0.264229551	0.240252606
75	0.331464222	0.310450458	0.287894059	0.264352607	0.240366513
80	0.331573884	0.310562736	0.288005897	0.264460357	0.240466255
85	0.331670686	0.310661854	0.288104634	0.264555491	0.240554320
90	0.331756763	0.310749997	0.288192444	0.264640101	0.240632646
95	0.331833804	0.310828892	0.288271046	0.264715842	0.240702763
100	0.331903162	0.310899922	0.288341816	0.264784039	0.240765897
105	0.331965930	0.310964208	0.288405869	0.264845766	0.240823042
110	0.332023006	0.311022666	0.288464118	0.264901902	0.240875012
115	0.332075131	0.311076054	0.288517318	0.264953174	0.240922480
120	0.332122921	0.311125005	0.288566098	0.265000187	0.240966005
125	0.332166896	0.311170050	0.288610986	0.265043451	0.241006060
130	0.332207495	0.311211638	0.288652432	0.265083397	0.241043044
135	0.332245092	0.311250153	0.288690815	0.265120393	0.241077297
140	0.332280009	0.311285922	0.288726464	0.265154754	0.241109110
145	0.332312523	0.311319230	0.288759660	0.265186752	0.241138735
150	0.332342872	0.311350323	0.288790649	0.265216622	0.241166391

'STUDENT'S t' PROBABILITY DENSITY FUNCTION

d.f.	x = 6 E-1	7 E-1	8 E-1	9 E-1	10 E-1
155	0.332371267	0.311379413	0.288819643	0.265244570	0.241192268
160	0.332397890	0.311406688	0.288846829	0.265270776	0.241216531
165	0.332422903	0.311432314	0.288872371	0.265295397	0.241239328
170	0.332446446	0.311456435	0.288896413	0.265318574	0.241260787
175	0.332468646	0.311479180	0.288919085	0.265340429	0.241281023
180	0.332489614	0.311500664	0.288940500	0.265361073	0.241300137
185	0.332509450	0.311520989	0.288960759	0.265380603	0.241318220
190	0.332528244	0.311540245	0.288979954	0.265399107	0.241335354
195	0.332546076	0.311558516	0.288998166	0.265416665	0.241351610
200	0.332563017	0.311575874	0.289015470	0.265433346	0.241367056

d.f.	x = 1 E 0	2 E 0	3 E 0	4 E 0	5 E 0
1	0.159154943	0.063661977	0.031830989	0.018724111	0.012242688
2	0.192450090	0.068041382	0.027410122	0.013094570	0.007127781
3	0.206748336	0.067509661	0.022972037	0.009163361	0.004219354
4	0.214662526	0.066291261	0.019693498	0.006708204	0.002649636
5	0.219679797	0.065090310	0.017292579	0.005123727	0.001757438
6	0.223142291	0.064036123	0.015491933	0.004054578	0.001220841
7	0.225674920	0.063135337	0.014104683	0.003303177	0.000881543
8	0.227607580	0.062368085	0.013009418	0.002756306	0.000657605
9	0.229130733	0.061711568	0.012126091	0.002346299	0.000504300
10	0.230361989	0.061145766	0.011400549	0.002031034	0.000396001
11	0.231377879	0.060654323	0.010795166	0.001783310	0.000317391
12	0.232230341	0.060224175	0.010283130	0.001584979	0.000258964
13	0.232955863	0.059844941	0.009844896	0.001423587	0.000214627
14	0.233580824	0.059508341	0.009465921	0.001290364	0.000180359
15	0.234124773	0.059207732	0.009135184	0.001179000	0.000153436
16	0.234602498	0.058937743	0.008844197	0.001084860	0.000131973
17	0.235025398	0.058693998	0.008586323	0.001004481	0.000114637
18	0.235402396	0.058472898	0.008356306	0.000935234	0.000100466
19	0.235740580	0.058271466	0.008149932	0.000875092	8.87580e-05
20	0.236045649	0.058087215	0.007963787	0.000822474	7.89891e-05
21	0.236322240	0.057918058	0.007795075	0.000776131	7.07649e-05
22	0.236574163	0.057762226	0.007641488	0.000735066	6.37840e-05
23	0.236804575	0.057618218	0.007501105	0.000698473	5.78136e-05
24	0.237016120	0.057484744	0.007372313	0.000665699	5.26717e-05
25	0.237211023	0.057360697	0.007253748	0.000636207	4.82146e-05
26	0.237391173	0.057245117	0.007144252	0.000609551	4.43280e-05
27	0.237558185	0.057137170	0.007042834	0.000585361	4.09200e-05
28	0.237713444	0.057036127	0.006948638	0.000563327	3.79162e-05
29	0.237858150	0.056941350	0.006860929	0.000543187	3.52558e-05
30	0.237993342	0.056852275	0.006779063	0.000524716	3.28889e-05
31	0.238119930	0.056768405	0.006702480	0.000507726	3.07742e-05
32	0.238238708	0.056689297	0.006630690	0.000492053	2.88773e-05
33	0.238350379	0.056614559	0.006563258	0.000477556	2.71695e-05
34	0.238455562	0.056543839	0.006499804	0.000464112	2.56264e-05
35	0.238554805	0.056476823	0.006439987	0.000451617	2.42276e-05
36	0.238648599	0.056413227	0.006383506	0.000439976	2.29556e-05
37	0.238737379	0.056352797	0.006330091	0.000429109	2.17955e-05
38	0.238821539	0.056295304	0.006279500	0.000418943	2.07345e-05
39	0.238901428	0.056240537	0.006231518	0.000409416	1.97615e-05
40	0.238977364	0.056188309	0.006185947	0.000400471	1.88670e-05

'STUDENT'S t' PROBABILITY DENSITY FUNCTION

d.f.	x = 1 E 0	2 E 0	3 E 0	4 E 0	5 E 0
41	0.239049634	0.056138447	0.006142612	0.000392059	1.80427e-05
42	0.239118497	0.056090794	0.006101353	0.000384134	1.72813e-05
43	0.239184188	0.056045206	0.006062026	0.000376658	1.65766e-05
44	0.239246921	0.056001554	0.006024498	0.000369594	1.59229e-05
45	0.239306891	0.055959715	0.005988649	0.000362910	1.53155e-05
46	0.239364278	0.055919580	0.005954370	0.000356578	1.47499e-05
47	0.239419245	0.055881046	0.005921560	0.000350571	1.42223e-05
48	0.239471941	0.055844020	0.005890127	0.000344865	1.37293e-05
49	0.239522504	0.055808415	0.005859986	0.000339440	1.32679e-05
50	0.239571062	0.055774152	0.005831061	0.000334275	1.28355e-05
55	0.239787565	0.055620526	0.005702327	0.000311787	1.10299e-05
60	0.239968235	0.055491251	0.005595211	0.000293697	9.67022e-06
65	0.240121288	0.055380968	0.005504702	0.000278853	8.61743e-06
70	0.240252606	0.055285779	0.005427224	0.000266468	7.78308e-06
75	0.240366513	0.055202786	0.005360158	0.000255987	7.10870e-06
80	0.240466255	0.055129788	0.005301541	0.000247009	6.55437e-06
85	0.240554320	0.055065081	0.005249874	0.000239237	6.09202e-06
90	0.240632646	0.055007329	0.005203991	0.000232445	5.70146e-06
95	0.240702763	0.054955469	0.005162974	0.000226463	5.36785e-06
100	0.240765897	0.054908643	0.005126090	0.000221155	5.08006e-06
105	0.240823042	0.054866153	0.005092743	0.000216413	4.82960e-06
110	0.240875012	0.054827422	0.005062450	0.000212154	4.60992e-06
115	0.240922480	0.054791973	0.005034809	0.000208308	4.41585e-06
120	0.240966005	0.054759406	0.005009486	0.000204817	4.24333e-06
125	0.241006060	0.054729384	0.004986204	0.000201636	4.08906e-06
130	0.241043044	0.054701618	0.004964723	0.000198725	3.95037e-06
135	0.241077297	0.054675865	0.004944844	0.000196051	3.82510e-06
140	0.241109110	0.054651913	0.004926394	0.000193587	3.71144e-06
145	0.241138735	0.054629579	0.004909224	0.000191309	3.60790e-06
150	0.241166391	0.054608706	0.004893205	0.000189198	3.51321e-06
155	0.241192268	0.054589153	0.004878226	0.000187234	3.42631e-06
160	0.241216531	0.054570800	0.004864188	0.000185405	3.34631e-06
165	0.241239328	0.054553540	0.004851006	0.000183695	3.27243e-06
170	0.241260787	0.054537277	0.004838603	0.000182095	3.20402e-06
175	0.241281023	0.054521928	0.004826913	0.000180593	3.14050e-06
180	0.241300137	0.054507418	0.004815876	0.000179182	3.08137e-06
185	0.241318220	0.054493680	0.004805439	0.000177853	3.02621e-06
190	0.241335354	0.054480654	0.004795553	0.000176599	2.97464e-06
195	0.241351610	0.054468286	0.004786177	0.000175414	2.92632e-06
200	0.241367056	0.054456527	0.004777272	0.000174293	2.88097e-06

d.f.	x = 6 E 0	7 E 0	8 E 0	9 E 0	10 E 0
1	0.008602970	0.006366198	0.004897075	0.003881828	0.003151583
2	0.004268985	0.002745647	0.001865023	0.001322461	0.000970733
3	0.002174867	0.001223363	0.000736907	0.000468817	0.000311808
4	0.001185854	0.000586802	0.000314709	0.000180150	0.000108792
5	0.000688482	0.000301344	0.000144443	7.46017e-05	4.09898e-05
6	0.000421747	0.000164118	7.05638e-05	3.29691e-05	1.65141e-05
7	0.000270377	9.39921e-05	3.63756e-05	1.54139e-05	7.05193e-06
8	0.000180194	5.62136e-05	1.96463e-05	7.56873e-06	3.16863e-06
9	0.000124171	3.49095e-05	1.10527e-05	3.88035e-06	1.48919e-06
10	8.80851e-05	2.24070e-05	6.44609e-06	2.06701e-06	7.28469e-07

'STUDENT'S t' PROBABILITY DENSITY FUNCTION

d.f.	x = 6 E 0	7 E 0	8 E 0	9 E 0	10 E 0
11	6.40947e-05	1.48082e-05	3.88188e-06	1.13942e-06	3.69378e-07
12	4.76961e-05	1.00440e-05	2.40582e-06	6.47756e-07	1.93479e-07
13	3.62074e-05	6.97313e-06	1.53015e-06	3.78682e-07	1.04381e-07
14	2.79799e-05	4.94388e-06	9.96350e-07	2.27088e-07	5.78542e-08
15	2.19708e-05	3.57246e-06	6.62812e-07	1.39392e-07	3.28719e-08
16	1.75033e-05	2.62655e-06	4.49659e-07	8.74156e-08	1.91098e-08
17	1.41281e-05	1.96188e-06	3.10601e-07	5.59156e-08	1.13474e-08
18	1.15406e-05	1.48682e-06	2.18142e-07	3.64280e-08	6.87212e-09
19	9.53032e-06	1.14194e-06	1.55580e-07	2.41398e-08	4.23901e-09
20	7.94922e-06	8.87939e-07	1.12555e-07	1.62526e-08	2.66008e-09
21	6.69161e-06	6.98354e-07	8.25175e-08	1.11059e-08	1.69635e-09
22	5.68088e-06	5.55094e-07	6.12499e-08	7.69501e-09	1.09825e-09
23	4.86072e-06	4.45590e-07	4.59932e-08	5.40163e-09	7.21213e-10
24	4.18926e-06	3.60988e-07	3.49136e-08	3.83850e-09	4.80017e-10
25	3.63497e-06	2.94969e-07	2.67743e-08	2.75938e-09	3.23563e-10
26	3.17386e-06	2.42968e-07	2.07301e-08	2.00537e-09	2.20736e-10
27	2.78750e-06	2.01647e-07	1.61958e-08	1.47248e-09	1.52311e-10
28	2.46159e-06	1.68542e-07	1.27615e-08	1.09179e-09	1.06238e-10
29	2.18492e-06	1.41814e-07	1.01366e-08	8.17046e-10	7.48671e-11
30	1.94868e-06	1.20075e-07	8.11315e-09	6.16829e-10	5.32781e-11
31	1.74582e-06	1.02272e-07	6.54060e-09	4.69577e-10	3.82697e-11
32	1.57071e-06	8.75971e-08	5.30905e-09	3.60328e-10	2.77346e-11
33	1.41881e-06	7.54262e-08	4.33748e-09	2.78596e-10	2.02712e-11
34	1.28642e-06	6.52727e-08	3.56568e-09	2.16963e-10	1.49370e-11
35	1.17053e-06	5.67552e-08	2.94849e-09	1.70133e-10	1.10924e-11
36	1.06866e-06	4.95725e-08	2.45183e-09	1.34293e-10	8.29891e-12
37	9.78756e-07	4.34849e-08	2.04976e-09	1.06672e-10	6.25343e-12
38	8.99114e-07	3.83008e-08	1.72238e-09	8.52447e-11	4.74451e-12
39	8.28311e-07	3.38661e-08	1.45437e-09	6.85163e-11	3.62345e-12
40	7.65151e-07	3.00560e-08	1.23379e-09	5.53765e-11	2.78485e-12
41	7.08627e-07	2.67689e-08	1.05135e-09	4.49953e-11	2.15339e-12
42	6.57887e-07	2.39219e-08	8.99723e-10	3.67473e-11	1.67489e-12
43	6.12203e-07	2.14466e-08	7.73122e-10	3.01588e-11	1.31008e-12
44	5.70957e-07	1.92869e-08	6.66950e-10	2.48687e-11	1.03032e-12
45	5.33617e-07	1.73958e-08	5.77532e-10	2.05998e-11	8.14557e-13
46	4.99728e-07	1.57345e-08	5.01916e-10	1.71385e-11	6.47242e-13
47	4.68894e-07	1.42705e-08	4.37721e-10	1.43189e-11	5.16812e-13
48	4.40774e-07	1.29763e-08	3.83016e-10	1.20119e-11	4.14616e-13
49	4.15072e-07	1.18289e-08	3.36229e-10	1.01161e-11	3.34149e-13
50	3.91528e-07	1.08088e-08	2.96074e-10	8.55177e-12	2.70487e-13
55	2.99180e-07	7.11472e-09	1.63597e-10	3.89047e-12	9.99548e-14
60	2.36288e-07	4.91140e-09	9.62393e-11	1.91219e-12	4.04626e-14
65	1.91768e-07	3.52793e-09	5.96925e-11	1.00401e-12	1.77178e-14
70	1.59214e-07	2.62073e-09	3.87364e-11	5.58030e-13	8.30621e-15
75	1.34746e-07	2.00339e-09	2.61353e-11	3.25879e-13	4.13375e-15
80	1.15917e-07	1.56969e-09	1.82396e-11	1.98724e-13	2.16851e-15
85	1.01129e-07	1.25644e-09	1.31110e-11	1.25892e-13	1.19196e-15
90	8.93055e-08	1.02462e-09	9.67264e-12	8.24926e-14	6.83045e-16
95	7.97036e-08	8.49377e-10	7.30194e-12	5.57047e-14	4.06293e-16
100	7.17975e-08	7.14361e-10	5.62613e-12	3.86417e-14	2.49925e-16

'STUDENT'S t' PROBABILITY DENSITY FUNCTION

d.f.	x = 6 E 0	7 E 0	8 E 0	9 E 0	10 E 0
105	6.52073e-08	6.08570e-10	4.41483e-12	2.74614e-14	1.58473e-16
110	5.96532e-08	5.24417e-10	3.52155e-12	1.99464e-14	1.03287e-16
115	5.49260e-08	4.56560e-10	2.85079e-12	1.47771e-14	6.90256e-17
120	5.08666e-08	4.01167e-10	2.33880e-12	1.11457e-14	4.71953e-17
125	4.73524e-08	3.55443e-10	1.94213e-12	8.54550e-15	3.29514e-17
130	4.42879e-08	3.17315e-10	1.63061e-12	6.65064e-15	2.34526e-17
135	4.15974e-08	2.85227e-10	1.38291e-12	5.24743e-15	1.69898e-17
140	3.92209e-08	2.57991e-10	1.18368e-12	4.19282e-15	1.25103e-17
145	3.71099e-08	2.34692e-10	1.02175e-12	3.38930e-15	9.35193e-18
150	3.52248e-08	2.14619e-10	8.88866e-13	2.76931e-15	7.08934e-18
155	3.35335e-08	1.97210e-10	7.78827e-13	2.28531e-15	5.44437e-18
160	3.20093e-08	1.82018e-10	6.86949e-13	1.90335e-15	4.23190e-18
165	3.06300e-08	1.68685e-10	6.09641e-13	1.59884e-15	3.32669e-18
170	2.93770e-08	1.56921e-10	5.44125e-13	1.35379e-15	2.64273e-18
175	2.82346e-08	1.46491e-10	4.88228e-13	1.15484e-15	2.12012e-18
180	2.71897e-08	1.37201e-10	4.40240e-13	9.91979e-16	1.71659e-18
185	2.62309e-08	1.28890e-10	3.98800e-13	8.57633e-16	1.40192e-18
190	2.53485e-08	1.21425e-10	3.62817e-13	7.46002e-16	1.15425e-18
195	2.45342e-08	1.14694e-10	3.31412e-13	6.52612e-16	9.57608e-19
200	2.37807e-08	1.08604e-10	3.03870e-13	5.73980e-16	8.00195e-19

d.f.	x = 1 E 1	2 E 1	3 E 1	4 E 1	5 E 1
1	0.003151583	0.000793790	0.000353285	0.000198819	0.000127273
2	0.000970733	0.000124068	3.69139e-05	1.55957e-05	7.99041e-06
3	0.000311808	2.03682e-05	4.05683e-06	1.28735e-06	5.28008e-07
4	0.000108792	3.65787e-06	4.88383e-07	1.16458e-07	3.82468e-08
5	4.09898e-05	7.14297e-07	6.40174e-08	1.14767e-08	3.01871e-09
6	1.65141e-05	1.50170e-07	9.04641e-09	1.21988e-09	2.57034e-10
7	7.05193e-06	3.36873e-08	1.36588e-09	1.38605e-10	2.34005e-11
8	3.16863e-06	8.00400e-09	2.18722e-10	1.67106e-11	2.26102e-12
9	1.48919e-06	2.00201e-09	3.69202e-11	2.12473e-12	2.30452e-13
10	7.28469e-07	5.24517e-10	6.53646e-12	2.83484e-13	2.46527e-14
11	3.69378e-07	1.43337e-10	1.20865e-12	3.95217e-14	2.75632e-15
12	1.93479e-07	4.07117e-11	2.32586e-13	5.73691e-15	3.20939e-16
13	1.04381e-07	1.19817e-11	4.64381e-14	8.64429e-16	3.87992e-17
14	5.78542e-08	3.64434e-12	9.59456e-15	1.34848e-16	4.85716e-18
15	3.28719e-08	1.14294e-12	2.04663e-15	2.17284e-17	6.28208e-19
16	1.91098e-08	3.68855e-13	4.49818e-16	3.60908e-18	8.37738e-20
17	1.13474e-08	1.22276e-13	1.01682e-16	6.16847e-19	1.14979e-20
18	6.87212e-09	4.15715e-14	2.36029e-17	1.08311e-19	1.62159e-21
19	4.23901e-09	1.44742e-14	5.61797e-18	1.95104e-20	2.34667e-22
20	2.66008e-09	5.15443e-15	1.36939e-18	3.60073e-21	3.48011e-23
21	1.69635e-09	1.87521e-15	3.41427e-19	6.80050e-22	5.28266e-24
22	1.09825e-09	6.96216e-16	8.69825e-20	1.31296e-22	8.19915e-25
23	7.21213e-10	2.63536e-16	2.26206e-20	2.58883e-23	1.29992e-25
24	4.80017e-10	1.01614e-16	5.99968e-21	5.20838e-24	2.10335e-26
25	3.23563e-10	3.98779e-17	1.62161e-21	1.06830e-24	3.47049e-27
26	2.20736e-10	1.59164e-17	4.46300e-22	2.23228e-25	5.83479e-28
27	1.52311e-10	6.45643e-18	1.24987e-22	4.74850e-26	9.98870e-29
28	1.06238e-10	2.66005e-18	3.55942e-23	1.02763e-26	1.74003e-29
29	7.48671e-11	1.11244e-18	1.03015e-23	2.26110e-27	3.08251e-30
30	5.32781e-11	4.71965e-19	3.02821e-24	5.05549e-28	5.55013e-31

'STUDENT'S t' PROBABILITY DENSITY FUNCTION

d.f.	x = 1 E 1	2 E 1	3 E 1	4 E 1	5 E 1
31	3.82697e-11	2.03029e-19	9.03651e-25	1.14797e-28	1.01513e-31
32	2.77346e-11	8.85137e-20	2.73609e-25	2.64612e-29	1.88513e-32
33	2.02712e-11	3.90898e-20	8.40174e-26	6.18860e-30	3.55271e-33
34	1.49370e-11	1.74795e-20	2.61534e-26	1.46787e-30	6.79176e-34
35	1.10924e-11	7.91091e-21	8.24943e-27	3.52950e-31	1.31653e-34
36	8.29891e-12	3.62235e-21	2.63565e-27	8.60003e-32	2.58660e-35
37	6.25343e-12	1.67750e-21	8.52622e-28	2.12268e-32	5.14896e-36
38	4.74451e-12	7.85395e-22	2.79176e-28	5.30533e-33	1.03812e-36
39	3.62345e-12	3.71645e-22	9.24925e-29	1.34227e-33	2.11916e-37
40	2.78485e-12	1.77683e-22	3.09959e-29	3.43658e-34	4.37859e-38
41	2.15339e-12	8.58050e-23	1.05037e-29	8.90109e-35	9.15430e-39
42	1.67489e-12	4.18414e-23	3.59824e-30	2.33164e-35	1.93602e-39
43	1.31008e-12	2.05973e-23	1.24576e-30	6.17540e-36	4.14068e-40
44	1.03032e-12	1.02332e-23	4.35771e-31	1.65324e-36	8.95350e-41
45	8.14557e-13	5.12985e-24	1.53976e-31	4.47270e-37	1.95689e-41
46	6.47242e-13	2.59412e-24	5.49436e-32	1.22253e-37	4.32202e-42
47	5.16812e-13	1.32303e-24	1.97947e-32	3.37522e-38	9.64393e-43
48	4.14616e-13	6.80373e-25	7.19870e-33	9.41038e-39	2.17357e-43
49	3.34149e-13	3.52726e-25	2.64205e-33	2.64899e-39	4.94710e-44

d.f.	x = 6 E 1	7 E 1	8 E 1	9 E 1	10 E 1
1	8.83949e-05	6.49479e-05	4.97281e-05	3.92927e-05	3.18278e-05
2	4.62577e-06	2.91367e-06	1.95221e-06	1.37123e-06	9.99700e-07
3	2.54820e-07	1.37606e-07	8.06854e-08	5.03814e-08	3.30599e-08
4	1.53893e-08	7.12533e-09	3.65639e-09	2.02970e-09	1.19880e-09
5	1.01281e-09	4.02093e-10	1.80587e-10	8.91220e-11	4.73797e-11
6	7.19176e-11	2.44838e-11	9.62434e-12	4.22281e-12	2.02075e-12
7	5.46083e-12	1.59433e-12	5.48560e-13	2.13994e-13	9.21781e-14
8	4.40128e-13	1.10207e-13	3.31915e-14	1.15124e-14	4.46391e-15
9	3.74239e-14	8.03745e-15	2.11901e-15	6.53502e-16	2.28102e-16
10	3.34027e-15	6.15350e-16	1.42021e-16	3.89454e-17	1.22372e-17

d.f.	x = 1 E 2	2 E 2	3 E 2	4 E 2	5 E 2
1	3.18278e-05	7.95755e-06	3.53674e-06	1.98942e-06	1.27323e-06
2	9.99700e-07	1.24991e-07	3.70358e-08	1.56247e-08	7.99990e-09
3	3.30599e-08	2.06717e-09	4.08365e-10	1.29213e-10	5.29263e-11
4	1.19880e-09	3.74906e-11	4.93772e-12	1.17180e-12	3.83985e-13
5	4.73797e-11	7.41141e-13	6.50795e-14	1.15836e-14	3.03667e-15
6	2.02075e-12	1.58120e-14	9.25710e-16	1.23580e-16	2.59178e-17
7	9.21781e-14	3.60827e-16	1.40844e-17	1.41022e-18	2.36611e-19

d.f.	x = 6 E 2	7 E 2	8 E 2	9 E 2	10 E 2
1	8.84192e-07	6.49611e-07	4.97358e-07	3.92975e-07	3.18310e-07
2	4.62959e-09	2.91543e-09	1.95312e-09	1.37174e-09	9.99997e-10
3	2.55241e-11	1.37773e-11	8.07603e-12	5.04184e-12	3.30795e-12
4	1.54317e-13	7.13974e-14	3.66205e-14	2.03219e-14	1.19999e-14
5	1.01699e-15	4.03313e-16	1.81006e-16	8.92855e-17	4.74501e-17

'STUDENT'S t' PROBABILITY DENSITY FUNCTION

d.f.	x = 1 E 3	2 E 3	3 E 3	4 E 3	5 E 3
1	3.18310e-07	7.95775e-08	3.53678e-08	1.98944e-08	1.27324e-08
2	9.99997e-10	1.25000e-10	3.70370e-11	1.56250e-11	8.00000e-12
3	3.30795e-12	2.06748e-13	4.08392e-14	1.29218e-14	5.29276e-15
4	1.19999e-14	3.74999e-16	4.93827e-17	1.17187e-17	3.84000e-18

d.f.	x = 6 E 3	7 E 3	8 E 3	9 E 3	10 E 3
1	8.84194e-09	6.49612e-09	4.97359e-09	3.92975e-09	3.18310e-09
2	4.62963e-12	2.91545e-12	1.95312e-12	1.37174e-12	1.00000e-12
3	2.55245e-15	1.37775e-15	8.07611e-16	5.04187e-16	3.30797e-16

d.f.	x = 1 E 4	2 E 4	3 E 4	4 E 4	5 E 4
1	3.18310e-09	7.95775e-10	3.53678e-10	1.98944e-10	1.27324e-10
2	1.00000e-12	1.25000e-13	3.70370e-14	1.56250e-14	8.00000e-15
3	3.30797e-16	2.06748e-17	4.08392e-18	1.29218e-18	5.29276e-19

d.f.	x = 6 E 4	7 E 4	8 E 4	9 E 4	10 E 4
1	8.84194e-11	6.49612e-11	4.97359e-11	3.92975e-11	3.18310e-11
2	4.62963e-15	2.91545e-15	1.95312e-15	1.37174e-15	1.00000e-15

d.f.	x = 1 E 5	2 E 5	3 E 5	4 E 5	5 E 5
1	3.18310e-11	7.95775e-12	3.53678e-12	1.98944e-12	1.27324e-12
2	1.00000e-15	1.25000e-16	3.70370e-17	1.56250e-17	8.00000e-18

d.f.	x = 6 E 5	7 E 5	8 E 5	9 E 5	10 E 5
1	8.84194e-13	6.49612e-13	4.97359e-13	3.92975e-13	3.18310e-13
2	4.62963e-18	2.91545e-18	1.95312e-18	1.37174e-18	1.00000e-18

'STUDENT'S t' CUMULATIVE DISTRIBUTION FUNCTION

d.f.	x = 1 E-1	2 E-1	3 E-1	4 E-1	5 E-1
1	0.468274483	0.437167042	0.407226421	0.378881058	0.352416382
2	0.464732719	0.429985996	0.396242830	0.363917237	0.333333333
3	0.463326174	0.427135165	0.391881646	0.357967577	0.325723982
4	0.462577920	0.425618507	0.389560714	0.354798631	0.321664982
5	0.462115071	0.424680257	0.388124521	0.352836557	0.319149436
6	0.461800976	0.424043496	0.387149610	0.351504116	0.317440000
7	0.461574030	0.423583376	0.386445025	0.350540834	0.316203568
8	0.461402455	0.423235496	0.385912241	0.349812252	0.315268038
9	0.461268224	0.422963320	0.385495352	0.349242046	0.314535650
10	0.461160359	0.422744596	0.385160304	0.348783705	0.313946803
11	0.461071796	0.422565004	0.384885179	0.348407288	0.313463110
12	0.460997785	0.422414917	0.384655237	0.348092653	0.313058738
13	0.460935015	0.422287621	0.384460202	0.347825756	0.312715671
14	0.460881107	0.422178295	0.384292692	0.347596506	0.312420958
15	0.460834310	0.422083388	0.384147267	0.347397468	0.312165057
16	0.460793305	0.422000226	0.384019834	0.347223042	0.311940778
17	0.460757079	0.421926756	0.383907248	0.347068931	0.311742603
18	0.460724844	0.421861378	0.383807061	0.346931783	0.311566229
19	0.460695975	0.421802826	0.383717330	0.346808945	0.311408246
20	0.460669971	0.421750083	0.383636502	0.346698288	0.311265921
21	0.460646425	0.421702328	0.383563313	0.346598088	0.311137038
22	0.460625006	0.421658884	0.383496731	0.346506928	0.311019778
23	0.460605437	0.421619193	0.383435900	0.346423640	0.310912638
24	0.460587489	0.421582789	0.383380105	0.346347245	0.310814361
25	0.460570968	0.421549279	0.383328747	0.346276923	0.310723893
26	0.460555711	0.421518333	0.383281316	0.346211976	0.310640337
27	0.460541578	0.421489667	0.383237378	0.346151812	0.310562932
28	0.460528450	0.421463037	0.383196562	0.346095921	0.310491022
29	0.460516222	0.421438235	0.383158547	0.346043864	0.310424042
30	0.460504806	0.421415079	0.383123053	0.345995258	0.310361502
31	0.460494123	0.421393409	0.383089838	0.345949773	0.310302976
32	0.460484104	0.421373087	0.383058689	0.345907116	0.310248087
33	0.460474690	0.421353992	0.383029419	0.345867033	0.310196509
34	0.460465828	0.421336015	0.383001864	0.345829296	0.310147950
35	0.460457470	0.421319062	0.382975876	0.345793706	0.310102152
36	0.460449574	0.421303046	0.382951327	0.345760085	0.310058886
37	0.460442104	0.421287893	0.382928099	0.345728274	0.310017949
38	0.460435026	0.421273534	0.382906089	0.345698130	0.309979157
39	0.460428309	0.421259909	0.382885203	0.345669526	0.309942346
40	0.460421927	0.421246963	0.382865358	0.345642347	0.309907368
41	0.460415855	0.421234647	0.382846478	0.345616488	0.309874089
42	0.460410071	0.421222914	0.382828493	0.345591856	0.309842388
43	0.460404555	0.421211726	0.382811343	0.345568366	0.309812157
44	0.460399290	0.421201045	0.382794969	0.345545940	0.309783294
45	0.460394257	0.421190837	0.382779320	0.345524507	0.309755710
46	0.460389443	0.421181071	0.382764350	0.345504003	0.309729320
47	0.460384833	0.421171720	0.382750015	0.345484369	0.309704050
48	0.460380415	0.421162757	0.382736275	0.345465550	0.309679829
49	0.460376176	0.421154159	0.382723095	0.345447497	0.309656593
50	0.460372107	0.421145904	0.382710440	0.345430164	0.309634284

'STUDENT'S t' CUMULATIVE DISTRIBUTION FUNCTION

d.f.	x = 1 E-1	2 E-1	3 E-1	4 E-1	5 E-1
55	0.460353974	0.421109120	0.382654051	0.345352926	0.309534870
60	0.460338856	0.421078453	0.382607038	0.345288529	0.309451980
65	0.460326059	0.421052494	0.382567241	0.345234017	0.309381810
70	0.460315087	0.421030237	0.382533119	0.345187275	0.309321642
75	0.460305575	0.421010941	0.382503537	0.345146754	0.309269479
80	0.460297251	0.420994054	0.382477647	0.345111288	0.309223823
85	0.460289904	0.420979150	0.382454798	0.345079987	0.309183529
90	0.460283372	0.420965899	0.382434483	0.345052158	0.309147703
95	0.460277527	0.420954042	0.382416304	0.345027254	0.309115643
100	0.460272266	0.420943368	0.382399940	0.345004837	0.309086783
105	0.460267505	0.420933710	0.382385133	0.344984552	0.309060668
110	0.460263176	0.420924929	0.382371670	0.344966109	0.309036923
115	0.460259223	0.420916910	0.382359376	0.344949267	0.309015240
120	0.460255600	0.420909559	0.382348106	0.344933827	0.308995362
125	0.460252266	0.420902795	0.382337736	0.344919621	0.308977072
130	0.460249188	0.420896551	0.382328163	0.344906506	0.308960187
135	0.460246338	0.420890770	0.382319298	0.344894362	0.308944552
140	0.460243691	0.420885400	0.382311066	0.344883084	0.308930032
145	0.460241227	0.420880401	0.382303401	0.344872583	0.308916512
150	0.460238927	0.420875735	0.382296247	0.344862782	0.308903893
155	0.460236775	0.420871369	0.382289554	0.344853612	0.308892087
160	0.460234757	0.420867277	0.382283279	0.344845015	0.308881018
165	0.460232862	0.420863431	0.382277384	0.344836939	0.308870620
170	0.460231078	0.420859812	0.382271835	0.344829337	0.308860832
175	0.460229396	0.420856400	0.382266603	0.344822169	0.308851604
180	0.460227807	0.420853177	0.382261662	0.344815399	0.308842887
185	0.460226304	0.420850128	0.382256987	0.344808995	0.308834641
190	0.460224880	0.420847240	0.382252559	0.344802928	0.308826829
195	0.460223530	0.420844499	0.382248357	0.344797171	0.308819418
200	0.460222246	0.420841896	0.382244365	0.344791702	0.308812376

d.f.	x = 6 E-1	7 E-1	8 E-1	9 E-1	10 E-1
1	0.327979130	0.305599888	0.285223287	0.266737708	0.250000000
2	0.304716634	0.278196512	0.253817018	0.231552506	0.211324865
3	0.295400604	0.267163499	0.241099476	0.217225516	0.195501109
4	0.290420579	0.261250083	0.234263568	0.209502760	0.186950483
5	0.287330144	0.257574474	0.230007033	0.204685600	0.181608734
6	0.285228135	0.255071690	0.227105182	0.201397617	0.177958842
7	0.283706748	0.253258776	0.225001350	0.199011759	0.175308331
8	0.282555029	0.251885526	0.223406667	0.197202102	0.173296754
9	0.281653036	0.250809520	0.222156499	0.195782658	0.171718198
10	0.280927591	0.249943785	0.221150210	0.194639631	0.170446566
11	0.280331534	0.249232233	0.220322844	0.193699512	0.169400348
12	0.279833115	0.248637077	0.219630616	0.192912716	0.168524529
13	0.279410177	0.248131937	0.219042939	0.192244587	0.167780639
14	0.279046793	0.247697840	0.218537803	0.191670176	0.167140972
15	0.278731218	0.247320791	0.218098970	0.191171067	0.166585068
16	0.278454604	0.246990244	0.217714194	0.190733369	0.166097492
17	0.278210160	0.246698100	0.217374072	0.190346409	0.165666381
18	0.277992584	0.246438035	0.217071259	0.190001851	0.165282466
19	0.277797678	0.246205044	0.216799937	0.189693090	0.164938400
20	0.277622076	0.245995109	0.216555439	0.189414825	0.164628289

'STUDENT'S t' CUMULATIVE DISTRIBUTION FUNCTION

d.f.	x = 6 E-1	7 E-1	8 E-1	9 E-1	10 E-1
21	0.277463047	0.245804970	0.216333975	0.189162751	0.164347342
22	0.277318351	0.245631953	0.216132437	0.188933337	0.164091631
23	0.277186133	0.245473847	0.215948253	0.188723660	0.163857903
24	0.277064848	0.245328803	0.215779273	0.188531279	0.163643441
25	0.276953193	0.245195268	0.215623691	0.188354138	0.163445956
26	0.276850065	0.245071925	0.215479974	0.188190497	0.163263511
27	0.276754523	0.244957649	0.215346814	0.188038868	0.163094450
28	0.276665761	0.244851476	0.215223091	0.187897976	0.162937353
29	0.276583081	0.244752574	0.215107834	0.187766720	0.162790994
30	0.276505880	0.244660222	0.215000205	0.187644143	0.162654308
31	0.276433629	0.244573789	0.214899470	0.187529414	0.162526366
32	0.276365869	0.244492724	0.214804987	0.187421801	0.162406357
33	0.276302192	0.244416542	0.214716192	0.187320663	0.162293564
34	0.276242241	0.244344815	0.214632587	0.187225432	0.162187355
35	0.276185698	0.244277163	0.214553729	0.187135605	0.162087171
36	0.276132280	0.244213249	0.214479225	0.187050736	0.161992513
37	0.276081736	0.244152771	0.214408725	0.186970424	0.161902936
38	0.276033839	0.244095459	0.214341914	0.186894313	0.161818042
39	0.275988387	0.244041071	0.214278510	0.186822081	0.161737473
40	0.275945197	0.243989389	0.214218258	0.186753439	0.161660906
41	0.275904105	0.243940216	0.214160930	0.186688126	0.161588051
42	0.275864961	0.243893374	0.214106318	0.186625905	0.161518644
43	0.275827630	0.243848700	0.214054233	0.186566562	0.161452446
44	0.275791989	0.243806048	0.214004504	0.186509903	0.161389239
45	0.275757926	0.243765283	0.213956974	0.186455748	0.161328826
46	0.275725338	0.243726283	0.213911501	0.186403935	0.161271025
47	0.275694132	0.243688935	0.213867954	0.186354316	0.161215670
48	0.275664221	0.243653137	0.213826213	0.186306755	0.161162609
49	0.275635526	0.243618795	0.213786169	0.186261125	0.161111703
50	0.275607975	0.243585820	0.213747719	0.186217312	0.161062823
55	0.275485199	0.243438871	0.213576360	0.186022040	0.160844959
60	0.275382825	0.243316332	0.213433458	0.185859186	0.160663252
65	0.275296158	0.243212590	0.213312468	0.185721295	0.160509390
70	0.275221841	0.243123626	0.213208709	0.185603037	0.160377429
75	0.275157409	0.243046493	0.213118745	0.185500497	0.160263003
80	0.275101014	0.242978978	0.213039996	0.185410736	0.160162835
85	0.275051239	0.242919388	0.212970488	0.185331507	0.160074417
90	0.275006985	0.242866405	0.212908685	0.185261058	0.159995795
95	0.274967380	0.242818988	0.212853373	0.185198006	0.159925427
100	0.274931729	0.242776303	0.212803580	0.185141244	0.159862078
105	0.274899467	0.242737676	0.212758520	0.185089877	0.159804748
110	0.274870134	0.242702555	0.212717548	0.185043169	0.159752617
115	0.274843347	0.242670482	0.212680133	0.185000515	0.159705010
120	0.274818790	0.242641078	0.212645830	0.184961408	0.159661362
125	0.274796194	0.242614022	0.212614266	0.184925424	0.159621198
130	0.274775334	0.242589045	0.212585126	0.184892202	0.159584119
135	0.274756018	0.242565915	0.212558142	0.184861438	0.159549780
140	0.274738079	0.242544435	0.212533082	0.184832867	0.159517890
145	0.274721376	0.242524434	0.212509748	0.184806264	0.159488196
150	0.274705786	0.242505765	0.212487967	0.184781431	0.159460477

'STUDENT'S t' CUMULATIVE DISTRIBUTION FUNCTION

d.f.	x = 6 E-1	7 E-1	8 E-1	9 E-1	10 E-1
155	0.274691200	0.242488299	0.212467589	0.184758198	0.159434544
160	0.274677524	0.242471924	0.212448484	0.184736415	0.159410230
165	0.274664677	0.242456539	0.212430534	0.184715950	0.159387386
170	0.274652585	0.242442059	0.212413640	0.184696687	0.159365885
175	0.274641182	0.242428405	0.212397709	0.184678524	0.159345610
180	0.274630413	0.242415509	0.212382663	0.184661368	0.159326460
185	0.274620225	0.242403309	0.212368429	0.184645139	0.159308344
190	0.274610573	0.242391751	0.212354943	0.184629763	0.159291180
195	0.274601416	0.242380785	0.212342148	0.184615174	0.159274896
200	0.274592716	0.242370366	0.212329992	0.184601314	0.159259424

d.f.	x = 1 E 0	2 E 0	3 E 0	4 E 0	5 E 0
1	0.250000000	0.147583618	0.102416382	0.077979130	0.062832958
2	0.211324865	0.091751710	0.047732983	0.028595479	0.018874776
3	0.195501109	0.069662984	0.028834443	0.014004228	0.007696219
4	0.186950483	0.058058262	0.019970984	0.008065045	0.003745217
5	0.181608734	0.050969739	0.015049624	0.005161708	0.002052358
6	0.177958842	0.046213156	0.012004098	0.003559489	0.001226171
7	0.175308331	0.042809664	0.009971063	0.002594957	0.000782639
8	0.173296754	0.040258119	0.008535841	0.001974886	0.000526413
9	0.171718198	0.038276412	0.007478182	0.001555214	0.000369484
10	0.170446566	0.036694017	0.006671828	0.001259166	0.000268667
11	0.169400348	0.035401978	0.006039920	0.001043097	0.000201265
12	0.168524529	0.034327507	0.005533348	0.000880848	0.000154656
13	0.167780639	0.033420179	0.005119449	0.000756040	0.000121477
14	0.167140972	0.032643976	0.004775756	0.000658025	9.72577e-05
15	0.166585068	0.031972504	0.004486369	0.000579658	7.91848e-05
16	0.166097492	0.031385982	0.004239750	0.000516012	6.54338e-05
17	0.165666381	0.030869303	0.004027348	0.000463603	5.47891e-05
18	0.165282466	0.030410733	0.003842706	0.000419915	4.64207e-05
19	0.164938400	0.030001018	0.003680862	0.000383096	3.97499e-05
20	0.164628289	0.029632768	0.003537949	0.000351762	3.43651e-05
21	0.164347342	0.029300006	0.003410913	0.000324858	2.99688e-05
22	0.164091631	0.028997851	0.003297310	0.000301572	2.63421e-05
23	0.163857903	0.028722274	0.003195166	0.000281271	2.33217e-05
24	0.163643441	0.028469925	0.003102868	0.000263454	2.07843e-05
25	0.163445956	0.028237990	0.003019090	0.000247722	1.86355e-05
26	0.163263511	0.028024094	0.002942727	0.000233753	1.68024e-05
27	0.163094450	0.027826214	0.002872856	0.000221285	1.52278e-05
28	0.162937353	0.027642619	0.002808700	0.000210103	1.38667e-05
29	0.162790994	0.027471819	0.002749596	0.000200032	1.26832e-05
30	0.162654308	0.027312522	0.002694982	0.000190923	1.16483e-05
31	0.162526366	0.027163608	0.002644373	0.000182653	1.07389e-05
32	0.162406357	0.027024093	0.002597352	0.000175117	9.93584e-06
33	0.162293564	0.026893116	0.002553556	0.000168228	9.22349e-06
34	0.162187355	0.026769918	0.002512670	0.000161911	8.58892e-06
35	0.162087171	0.026653826	0.002474416	0.000156101	8.02141e-06
36	0.161992513	0.026544244	0.002438553	0.000150742	7.51197e-06
37	0.161902936	0.026440639	0.002404866	0.000145788	7.05303e-06
38	0.161818042	0.026342535	0.002373166	0.000141195	6.63821e-06
39	0.161737473	0.026249508	0.002343284	0.000136929	6.26209e-06
40	0.161660906	0.026161172	0.002315070	0.000132956	5.92003e-06

'STUDENT'S t' CUMULATIVE DISTRIBUTION FUNCTION

d.f.	x = 1 E 0	2 E 0	3 E 0	4 E 0	5 E 0
41	0.161588051	0.026077181	0.002288390	0.000129250	5.60808e-06
42	0.161518644	0.025997224	0.002263123	0.000125785	5.32281e-06
43	0.161452446	0.025921017	0.002239162	0.000122540	5.06129e-06
44	0.161389239	0.025848302	0.002216409	0.000119496	4.82094e-06
45	0.161328826	0.025778844	0.002194775	0.000116635	4.59955e-06
46	0.161271025	0.025712430	0.002174182	0.000113942	4.39517e-06
47	0.161215670	0.025648864	0.002154556	0.000111403	4.20610e-06
48	0.161162609	0.025587967	0.002135831	0.000109006	4.03083e-06
49	0.161111703	0.025529574	0.002117948	0.000106740	3.86806e-06
50	0.161062823	0.025473534	0.002100852	0.000104595	3.71661e-06
55	0.160844959	0.025224111	0.002025543	9.53944e-05	3.09668e-06
60	0.160663252	0.025016522	0.001963849	8.81612e-05	2.64401e-06
65	0.160509390	0.024841060	0.001912403	8.23411e-05	2.30268e-06
70	0.160377429	0.024690805	0.001868861	7.75668e-05	2.03834e-06
75	0.160263003	0.024560691	0.001831539	7.35859e-05	1.82896e-06
80	0.160162835	0.024446924	0.001799199	7.02200e-05	1.65990e-06
85	0.160074417	0.024346605	0.001770909	6.73396e-05	1.52114e-06
90	0.159995795	0.024257484	0.001745957	6.48487e-05	1.40559e-06
95	0.159925427	0.024177785	0.001723786	6.26747e-05	1.30816e-06
100	0.159862078	0.024106089	0.001703958	6.07618e-05	1.22509e-06
105	0.159804748	0.024041249	0.001686120	5.90663e-05	1.15356e-06
110	0.159752617	0.023982326	0.001669988	5.75538e-05	1.09144e-06
115	0.159705010	0.023928546	0.001655329	5.61964e-05	1.03705e-06
120	0.159661362	0.023879264	0.001641951	5.49719e-05	9.89095e-07
125	0.159621198	0.023833937	0.001629693	5.38619e-05	9.46543e-07
130	0.159584119	0.023792109	0.001618420	5.28512e-05	9.08562e-07
135	0.159549780	0.023753389	0.001608018	5.19273e-05	8.74481e-07
140	0.159517890	0.023717444	0.001598391	5.10795e-05	8.43748e-07
145	0.159488196	0.023683985	0.001589455	5.02989e-05	8.15910e-07
150	0.159460477	0.023652763	0.001581138	4.95779e-05	7.90591e-07
155	0.159434544	0.023623561	0.001573379	4.89100e-05	7.67473e-07
160	0.159410230	0.023596189	0.001566122	4.82896e-05	7.46291e-07
165	0.159387386	0.023570481	0.001559321	4.77118e-05	7.26818e-07
170	0.159365885	0.023546288	0.001552935	4.71725e-05	7.08862e-07
175	0.159345610	0.023523482	0.001546926	4.66679e-05	6.92256e-07
180	0.159326460	0.023501946	0.001541262	4.61948e-05	6.76858e-07
185	0.159308344	0.023481577	0.001535914	4.57504e-05	6.62545e-07
190	0.159291180	0.023462282	0.001530857	4.53321e-05	6.49208e-07
195	0.159274896	0.023443979	0.001526067	4.49378e-05	6.36754e-07
200	0.159259424	0.023426593	0.001521524	4.45655e-05	6.25099e-07

d.f.	x = 6 E 0	7 E 0	8 E 0	9 E 0	10 E 0
1	0.052568457	0.045167235	0.039583424	0.035223287	0.031725517
2	0.013335737	0.009901971	0.007634036	0.006060830	0.004926229
3	0.004636357	0.002993128	0.002038289	0.001447906	0.001064200
4	0.001941269	0.001096065	0.000661948	0.000421916	0.000281002
5	0.000923069	0.000458374	0.000246453	0.000141340	8.54738e-05
6	0.000482268	0.000211742	0.000101732	5.26357e-05	2.89599e-05
7	0.000271129	0.000105777	4.55746e-05	2.13285e-05	1.06971e-05
8	0.000161697	5.63193e-05	2.18341e-05	9.26559e-06	4.24409e-06
9	0.000101250	3.16237e-05	1.10674e-05	4.26903e-06	1.78912e-06
10	6.60544e-05	1.85780e-05	5.88747e-06	2.06902e-06	7.94777e-07

'STUDENT'S t' CUMULATIVE DISTRIBUTION FUNCTION

d.f.	x = 6 E 0	7 E 0	8 E 0	9 E 0	10 E 0
11	4.46306e-05	1.13480e-05	3.26611e-06	1.04808e-06	3.69650e-07
12	3.10837e-05	7.17176e-06	1.87995e-06	5.52045e-07	1.79066e-07
13	2.22300e-05	4.67058e-06	1.11808e-06	3.01074e-07	8.99622e-08
14	1.62739e-05	3.12398e-06	6.84728e-07	1.69419e-07	4.67070e-08
15	1.21633e-05	2.14006e-06	4.30547e-07	9.80736e-08	2.49845e-08
16	9.26107e-06	1.49794e-06	2.77272e-07	5.82562e-08	1.37343e-08
17	7.16982e-06	1.06915e-06	1.82497e-07	3.54309e-08	7.74141e-09
18	5.63499e-06	7.76768e-07	1.22536e-07	2.20214e-08	4.46547e-09
19	4.48962e-06	5.73578e-07	8.37977e-08	1.39638e-08	2.63151e-09
20	3.62185e-06	4.29893e-07	5.82831e-08	9.02021e-09	1.58189e-09
21	2.95525e-06	3.26649e-07	4.11764e-08	5.92808e-09	9.68712e-10
22	2.43664e-06	2.51362e-07	2.95163e-08	3.95902e-09	6.03581e-10
23	2.02843e-06	1.95709e-07	2.14460e-08	2.68397e-09	3.82231e-10
24	1.70365e-06	1.54046e-07	1.57800e-08	1.84532e-09	2.45777e-10
25	1.44266e-06	1.22488e-07	1.17488e-08	1.28557e-09	1.60320e-10
26	1.23099e-06	9.83197e-08	8.84459e-09	9.06795e-10	1.06004e-10
27	1.05783e-06	7.96209e-08	6.72779e-09	6.47145e-10	7.09922e-11
28	9.15031e-07	6.50143e-08	5.16784e-09	4.66971e-10	4.81244e-11
29	7.96395e-07	5.35014e-08	4.00629e-09	3.40495e-10	3.29995e-11
30	6.97138e-07	4.43498e-08	3.13291e-09	2.50742e-10	2.28766e-11
31	6.13548e-07	3.70172e-08	2.47013e-09	1.86386e-10	1.60242e-11
32	5.42716e-07	3.10979e-08	1.96276e-09	1.39788e-10	1.13359e-11
33	4.82347e-07	2.62854e-08	1.57114e-09	1.05730e-10	8.09494e-12
34	4.30613e-07	2.23466e-08	1.26648e-09	8.06180e-11	5.83308e-12
35	3.86050e-07	1.91023e-08	1.02770e-09	6.19437e-11	4.23902e-12
36	3.47476e-07	1.64140e-08	8.39228e-10	4.79457e-11	3.10651e-12
37	3.13933e-07	1.41737e-08	6.89454e-10	3.73711e-11	2.29415e-12
38	2.84637e-07	1.22965e-08	5.69669e-10	2.93249e-11	1.70757e-12
39	2.58944e-07	1.07155e-08	4.73276e-10	2.31581e-11	1.27947e-12
40	2.36323e-07	9.37733e-09	3.95255e-10	1.84013e-11	9.66103e-13
41	2.16331e-07	8.23947e-09	3.31747e-10	1.47071e-11	7.33826e-13
42	1.98600e-07	7.26756e-09	2.79778e-10	1.18215e-11	5.61432e-13
43	1.82821e-07	6.43387e-09	2.37031e-10	9.55304e-12	4.31704e-13
44	1.68734e-07	5.71582e-09	2.01697e-10	7.76034e-12	3.34147e-13
45	1.56119e-07	5.09494e-09	1.72353e-10	6.33556e-12	2.60068e-13
46	1.44788e-07	4.55609e-09	1.47874e-10	5.19752e-12	2.03780e-13
47	1.34583e-07	4.08675e-09	1.27363e-10	4.28331e-12	1.60188e-13
48	1.25367e-07	3.67655e-09	1.10105e-10	3.54573e-12	1.26632e-13
49	1.17022e-07	3.31687e-09	9.55273e-11	2.94790e-12	1.00719e-13
50	1.09447e-07	3.00047e-09	8.31651e-11	2.46121e-12	8.06749e-14
55	8.04431e-08	1.88604e-09	4.36059e-11	1.05658e-12	2.79025e-14
60	6.14342e-08	1.25078e-09	2.45018e-11	4.94013e-13	1.11769e-14
65	4.84182e-08	8.67347e-10	1.45931e-11	2.47610e-13	
70	3.91736e-08	6.24506e-10	9.13477e-12	1.32872e-13	
75	3.24015e-08	4.64277e-10	5.96629e-12	7.43574e-14	
80	2.73078e-08	3.54775e-10	4.04454e-12	4.44940e-14	
85	2.33883e-08	2.77622e-10	2.83080e-12	2.69604e-14	
90	2.03119e-08	2.21797e-10	2.03904e-12	1.76817e-14	
95	1.78549e-08	1.80447e-10	1.50526e-12	1.13490e-14	
100	1.58625e-08	1.49179e-10	1.13693e-12		

'STUDENT'S t' CUMULATIVE DISTRIBUTION FUNCTION

d.f.	x = 6 E 0	7 E 0	8 E 0	9 E 0	10 E 0
105	1.42246e-08	1.25094e-10	8.74930e-13		
110	1.28617e-08	1.06236e-10	6.86607e-13		
115	1.17153e-08	9.12483e-11	5.46296e-13		
120	1.07415e-08	7.91792e-11	4.42340e-13		
125	9.90698e-09	6.93398e-11	3.61902e-13		
130	9.18601e-09	6.12312e-11	3.00388e-13		
135	8.55856e-09	5.44802e-11	2.51204e-13		
140	8.00884e-09	4.88084e-11	2.12188e-13		
145	7.52423e-09	4.40039e-11	1.81704e-13		
150	7.09461e-09	3.99018e-11	1.57069e-13		
155	6.71173e-09	3.63725e-11	1.35240e-13		
160	6.36886e-09	3.33193e-11	1.19175e-13		
165	6.06044e-09	3.06587e-11	1.04522e-13		
170	5.78186e-09	2.83289e-11	9.29876e-14		
175	5.52924e-09	2.62760e-11	8.20739e-14		
180	5.29933e-09	2.44606e-11	7.41345e-14		
185	5.08939e-09	2.28452e-11	6.60081e-14		
190	4.89707e-09	2.14043e-11	6.06115e-14		
195	4.72037e-09	2.01102e-11	5.37291e-14		
200	4.55756e-09	1.89479e-11	5.01321e-14		

d.f.	x = 1 E 1	2 E 1	3 E 1	4 E 1	5 E 1
1	0.031725517	0.015902251	0.010606402	0.007956090	0.006365349
2	0.004926229	0.001245332	0.000554631	0.000312207	0.000199880
3	0.001064200	0.000136602	4.06764e-05	1.71903e-05	8.80858e-06
4	0.000281002	1.84416e-05	3.67643e-06	1.16701e-06	4.78723e-07
5	8.54738e-05	2.88776e-06	3.85932e-07	9.20598e-08	3.02388e-08
6	2.89599e-05	5.07128e-07	4.54957e-08	8.15918e-09	2.14645e-09
7	1.06971e-05	9.77441e-08	5.89423e-09	7.95109e-10	1.67563e-10
8	4.24409e-06	2.03696e-08	8.26763e-10	8.39289e-11	1.41723e-11
9	1.78912e-06	4.53976e-09	1.24185e-10	9.49159e-12	1.28457e-12
10	7.94777e-07	1.07303e-09	1.98093e-11	1.14074e-12	1.24030e-13
11	3.69650e-07	2.67216e-10	3.33352e-12	1.44679e-13	1.26313e-14
12	1.79066e-07	6.97396e-11	5.88866e-13	1.94628e-14	
13	8.99622e-08	1.89916e-11	1.08665e-13		
14	4.67070e-08	5.37700e-12	2.11619e-14		
15	2.49845e-08	1.57765e-12			
16	1.37343e-08	4.78744e-13			
17	7.74141e-09	1.49654e-13			
18	4.46547e-09	4.84348e-14			
19	2.63151e-09	1.60956e-14			
20	1.58189e-09				
21	9.68712e-10				
22	6.03581e-10				
23	3.82231e-10				
24	2.45777e-10				
25	1.60320e-10				
26	1.06004e-10				
27	7.09922e-11				
28	4.81244e-11				
29	3.29995e-11				
30	2.28766e-11				

'STUDENT'S t' CUMULATIVE DISTRIBUTION FUNCTION

d.f.	x = 1 E 1	2 E 1	3 E 1	4 E 1	5 E 1
31	1.60242e-11				
32	1.13359e-11				
33	8.09494e-12				
34	5.83308e-12				
35	4.23902e-12				
36	3.10651e-12				
37	2.29415e-12				
38	1.70757e-12				
39	1.27947e-12				
40	9.66103e-13				
41	7.33826e-13				
42	5.61432e-13				
43	4.31704e-13				
44	3.34147e-13				
45	2.60068e-13				
46	2.03780e-13				
47	1.60188e-13				
48	1.26632e-13				
49	1.00719e-13				

d.f.	x = 6 E 1	7 E 1	8 E 1	9 E 1	10 E 1
1	0.005304674	0.004546975	0.003978666	0.003536631	0.003182993
2	0.000138831	0.000102010	7.81067e-05	6.17170e-05	4.99925e-05
3	5.09980e-06	3.21239e-06	2.15242e-06	1.51189e-06	1.10226e-06
4	2.31053e-07	1.24778e-07	7.31660e-08	4.56871e-08	2.99800e-08
5	1.21682e-08	5.63423e-09	2.89133e-09	1.60504e-09	9.48001e-10
6	7.20225e-10	2.85950e-10	1.28430e-10	6.33835e-11	3.36971e-11
7	4.68882e-11	1.59638e-11	6.27554e-12	2.75366e-12	1.31784e-12
8	3.30783e-12	9.65997e-13	3.32561e-13	1.29903e-13	5.61122e-14
9	2.50153e-13	6.27117e-14	1.89535e-14		
10	2.04067e-14				

d.f.	x = 1 E 2	2 E 2	3 E 2	4 E 2	5 E 2
1	0.003182993	0.001591536	0.001061029	0.000795773	0.000636619
2	4.99925e-05	1.24995e-05	5.55546e-06	3.12497e-06	1.99999e-06
3	1.10226e-06	1.37820e-07	4.08375e-08	1.72286e-08	8.82114e-09
4	2.99800e-08	1.87469e-09	3.70343e-10	1.17183e-10	4.79988e-11
5	9.48001e-10	2.96489e-11	3.90499e-12	9.26744e-13	3.03704e-13
6	3.36971e-11	5.27379e-13	4.65309e-14		
7	1.31784e-12	1.05000e-14			

d.f.	x = 6 E 2	7 E 2	8 E 2	9 E 2	10 E 2
1	0.000530516	0.000454728	0.000397887	0.000353678	0.000318310
2	1.38888e-06	1.02041e-06	7.81248e-07	6.17283e-07	4.99999e-07
3	5.10485e-09	3.21472e-09	2.15362e-09	1.51256e-09	1.10265e-09
4	2.31478e-11	1.24947e-11	7.32424e-12	4.57253e-12	3.00007e-12
5	1.22072e-13	5.64959e-14	2.89928e-14	1.61031e-14	

'STUDENT'S t' CUMULATIVE DISTRIBUTION FUNCTION

d.f.	x = 1 E 3	2 E 3	3 E 3	4 E 3	5 E 3
1	0.000318310	0.000159155	0.000106103	7.95775e-05	6.36620e-05
2	4.99999e-07	1.25000e-07	5.55555e-08	3.12500e-08	2.00000e-08
3	1.10265e-09	1.37832e-10	4.08394e-11	1.72292e-11	8.82145e-12
4	3.00007e-12	1.87593e-13	3.71301e-14	1.18118e-14	

d.f.	x = 6 E 3	7 E 3	8 E 3	9 E 3	10 E 3
1	5.30516e-05	4.54728e-05	3.97887e-05	3.53678e-05	3.18310e-05
2	1.38889e-08	1.02041e-08	7.81250e-09	6.17284e-09	5.00000e-09
3	5.10509e-12	3.21493e-12	2.15382e-12	1.51275e-12	1.10285e-12

d.f.	x = 1 E 4	2 E 4	3 E 4	4 E 4	5 E 4
1	3.18310e-05	1.59155e-05	1.06103e-05	7.95775e-06	6.36620e-06
2	5.00000e-09	1.25000e-09	5.55556e-10	3.12500e-10	2.00000e-10
3	1.10285e-12	1.38021e-13	4.10282e-14	1.74181e-14	

d.f.	x = 6 E 4	7 E 4	8 E 4	9 E 4	10 E 4
1	5.30516e-06	4.54728e-06	3.97887e-06	3.53678e-06	3.18310e-06
2	1.38889e-10	1.02041e-10	7.81252e-11	6.17286e-11	5.00002e-11

d.f.	x = 1 E 5	2 E 5	3 E 5	4 E 5	5 E 5
1	3.18310e-06	1.59155e-06	1.06103e-06	7.95775e-07	6.36620e-07
2	5.00002e-11	1.25002e-11	5.55571e-12	3.12516e-12	2.00016e-12

d.f.	x = 6 E 5	7 E 5	8 E 5	9 E 5	10 E 5
1	5.30516e-07	4.54728e-07	3.97887e-07	3.53678e-07	3.18310e-07
2	1.38905e-12	1.02057e-12	7.81408e-13	6.17442e-13	5.00158e-13

CRITICAL POINTS OF 'STUDENT'S t' DISTRIBUTION

Q	1 d.f.	2 d.f.	3 d.f.	4 d.f.	5 d.f.
5.0e-02	6.31375151	2.91998558	2.35336343	2.13184679	2.01504837
2.5e-02	12.70620474	4.30265273	3.18244631	2.77644511	2.57058184
1.0e-02	31.82051595	6.96455673	4.54070286	3.74694739	3.36493000
5.0e-03	63.65674116	9.92484320	5.84090931	4.60409487	4.03214298
2.5e-03	127.32133647	14.08904728	7.45331851	5.59756837	4.77334060
1.0e-03	318.30883899	22.32712477	10.21453185	7.17318222	5.89342953
5.0e-04	636.61924877	31.59905458	12.92397864	8.61030158	6.86882663
2.5e-04	1.273239e+03	44.70458729	16.32633461	10.30625468	7.97565342
1.0e-04	3.183099e+03	70.70007107	22.20374227	13.03367172	9.67756630
5.0e-05	6.366198e+03	99.99249984	28.00013001	15.54410058	11.17771007
2.5e-05	1.273240e+04	141.41605288	35.29791945	18.52239843	12.89282532
1.0e-05	3.183099e+04	223.60344363	47.92772838	23.33218270	15.54685453

Q	6 d.f.	7 d.f.	8 d.f.	9 d.f.	10 d.f.
5.0e-02	1.94318028	1.89457861	1.85954804	1.83311293	1.81246112
2.5e-02	2.44691185	2.36462425	2.30600414	2.26215716	2.22813885
1.0e-02	3.14266840	2.99795157	2.89645945	2.82143793	2.76376946
5.0e-03	3.70742802	3.49948330	3.35538733	3.24983554	3.16927267
2.5e-03	4.31682710	4.02933718	3.83251869	3.68966239	3.58140620
1.0e-03	5.20762624	4.78528963	4.50079093	4.29680566	4.14370049
5.0e-04	5.95881618	5.40788252	5.04130543	4.78091259	4.58689386
2.5e-04	6.78833999	6.08175619	5.61741081	5.29065384	5.04897275
1.0e-04	8.02479277	7.06343283	6.44199982	6.01013213	5.69382010
5.0e-05	9.08234633	7.88458426	7.12000388	6.59368258	6.21105089
2.5e-05	10.26087355	8.78250922	7.85064754	7.21526925	6.75679644
1.0e-05	12.03165318	10.10268425	8.90702687	8.10205788	7.52695402

Q	11 d.f.	12 d.f.	13 d.f.	14 d.f.	15 d.f.
5.0e-02	1.79588482	1.78228756	1.77093340	1.76131014	1.75305036
2.5e-02	2.20098516	2.17881283	2.16036866	2.14478669	2.13144955
1.0e-02	2.71807918	2.68099799	2.65030884	2.62449407	2.60248030
5.0e-03	3.10580652	3.05453959	3.01227584	2.97684273	2.94671288
2.5e-03	3.49661417	3.42844424	3.37246794	3.32569582	3.28603857
1.0e-03	4.02470104	3.92963326	3.85198239	3.78739024	3.73283443
5.0e-04	4.43697934	4.31779128	4.22083173	4.14045411	4.07276520
2.5e-04	4.86333309	4.71645866	4.59746146	4.49915507	4.41661283
1.0e-04	5.45276209	5.26327301	5.11057890	4.98501316	4.87999829
5.0e-05	5.92119416	5.69446579	5.51251505	5.36341304	5.23908821
2.5e-05	6.41157201	6.14286773	5.92812203	5.75276304	5.60698535
1.0e-05	7.09735556	6.76516924	6.50114491	6.28654936	6.10886790

Q	16 d.f.	17 d.f.	18 d.f.	19 d.f.	20 d.f.
5.0e-02	1.74588368	1.73960673	1.73406361	1.72913281	1.72471824
2.5e-02	2.11990530	2.10981558	2.10092204	2.09302405	2.08596345
1.0e-02	2.58348719	2.56693398	2.55237963	2.53948319	2.52797700
5.0e-03	2.92078162	2.89823052	2.87844047	2.86093461	2.84533971
2.5e-03	3.25199287	3.22244991	3.19657422	3.17372453	3.15340053
1.0e-03	3.68615479	3.64576738	3.61048488	3.57940015	3.55180834
5.0e-04	4.01499633	3.96512627	3.92164583	3.88340585	3.84951627
2.5e-04	4.34634858	4.28582834	4.23316730	4.18693526	4.14602782
1.0e-04	4.79091010	4.71440652	4.64801416	4.58986460	4.53852085
5.0e-05	5.13389352	5.04376498	4.96570629	4.89746159	4.83730115
2.5e-05	5.48396178	5.37879976	5.28790556	5.20858253	5.13876911
1.0e-05	5.95944167	5.83209966	5.72233117	5.62676623	5.54283863

CRITICAL POINTS OF 'STUDENT'S t' DISTRIBUTION

Q	21 d.f.	22 d.f.	23 d.f.	24 d.f.	25 d.f.
5.0e-02	1.72074290	1.71714437	1.71387153	1.71088208	1.70814076
2.5e-02	2.07961384	2.07387307	2.06865761	2.06389856	2.05953855
1.0e-02	2.51764802	2.50832455	2.49986674	2.49215947	2.48510718
5.0e-03	2.83135956	2.81875606	2.80733568	2.79693950	2.78743581
2.5e-03	3.13520625	3.11882421	3.10399696	3.09051355	3.07819946
1.0e-03	3.52715367	3.50499203	3.48496437	3.46677730	3.45018873
5.0e-04	3.81927716	3.79213067	3.76762680	3.74539862	3.72514395
2.5e-04	4.10957893	4.07690006	4.04743706	4.02073902	3.99643531
1.0e-04	4.49286013	4.45199272	4.41520471	4.38191675	4.35165387
5.0e-05	4.78387712	4.73612406	4.69318900	4.65438115	4.61913523
2.5e-05	5.07686322	5.02160159	4.97197507	4.92716810	4.88651454
1.0e-05	5.46856114	5.40237179	5.34302654	5.28952260	5.24104300

Q	26 d.f.	27 d.f.	28 d.f.	29 d.f.	30 d.f.
5.0e-02	1.70561792	1.70328845	1.70113093	1.69912703	1.69726089
2.5e-02	2.05552944	2.05183052	2.04840714	2.04522964	2.04227246
1.0e-02	2.47862982	2.47265991	2.46714010	2.46202136	2.45726154
5.0e-03	2.77871453	2.77068296	2.76326246	2.75638590	2.74999565
2.5e-03	3.06690912	3.05652011	3.04692878	3.03804674	3.02979822
1.0e-03	3.43499718	3.42103362	3.40815518	3.39624029	3.38518487
5.0e-04	3.70661174	3.68959171	3.67390640	3.65940502	3.64595864
2.5e-04	3.97421855	3.95383169	3.93505814	3.91771412	3.90164268
1.0e-04	4.32402304	4.29869630	4.27539786	4.25389412	4.23398596
5.0e-05	4.58698435	4.55753948	4.53047400	4.50551163	4.48241718
2.5e-05	4.84946513	4.81556296	4.78442497	4.75572762	4.72919580
1.0e-05	5.19691558	5.15658243	5.11957667	5.08550470	5.05403242

Q	31 d.f.	32 d.f.	33 d.f.	34 d.f.	35 d.f.
5.0e-02	1.69551878	1.69388875	1.69236031	1.69092426	1.68957246
2.5e-02	2.03951345	2.03693334	2.03451530	2.03224451	2.03010793
1.0e-02	2.45282419	2.44867763	2.44479420	2.44114963	2.43772255
5.0e-03	2.74404192	2.73848148	2.73327664	2.72839437	2.72380559
2.5e-03	3.02211783	3.01494889	3.00824199	3.00195390	2.99604661
1.0e-03	3.37489928	3.36530593	3.35633728	3.34793431	3.34004520
5.0e-04	3.63345635	3.62180226	3.61091301	3.60071580	3.59114678
2.5e-04	3.88670900	3.87279661	3.85980441	3.84764416	3.83623858
1.0e-04	4.21550258	4.19829668	4.18224054	4.16722287	4.15314629
5.0e-05	4.46098914	4.44105394	4.42246122	4.40508013	4.38879625
2.5e-05	4.70459412	4.68172002	4.66039824	4.64047640	4.62182138
1.0e-05	5.02487441	4.99778541	4.97255353	4.94899476	4.92694857

Q	36 d.f.	37 d.f.	38 d.f.	39 d.f.	40 d.f.
5.0e-02	1.68829771	1.68709362	1.68595446	1.68487512	1.68385101
2.5e-02	2.02809400	2.02619246	2.02439416	2.02269092	2.02107539
1.0e-02	2.43449406	2.43144740	2.42856763	2.42584141	2.42325678
5.0e-03	2.71948463	2.71540872	2.71155760	2.70791318	2.70445927
2.5e-03	2.99048657	2.98524406	2.98029265	2.97560876	2.97117129
1.0e-03	3.33262426	3.32563105	3.31902966	3.31278808	3.30687771
5.0e-04	3.58214970	3.57367484	3.56567807	3.55812008	3.55096576
2.5e-04	3.82551962	3.81542718	3.80590797	3.79691457	3.78840462
1.0e-04	4.13992519	4.12748403	4.11575588	4.10468126	4.09420710
5.0e-05	4.37350908	4.35913003	4.34558067	4.33279134	4.32069996
2.5e-05	4.60431637	4.58785848	4.57235668	4.55773021	4.54390711
1.0e-05	4.90627426	4.88684800	4.86856040	4.85131447	4.83502390

CRITICAL POINTS OF 'STUDENT'S t' DISTRIBUTION

Q	41 d.f.	42 d.f.	43 d.f.	44 d.f.	45 d.f.
5.0e-02	1.68287800	1.68195236	1.68107070	1.68022998	1.67942739
2.5e-02	2.01954097	2.01808170	2.01669220	2.01536757	2.01410339
1.0e-02	2.42080299	2.41847036	2.41625013	2.41413437	2.41211588
5.0e-03	2.70118130	2.69806619	2.69510208	2.69227827	2.68958502
2.5e-03	2.96696132	2.96296179	2.95915731	2.95553397	2.95207913
1.0e-03	3.30127289	3.29595053	3.29088982	3.28607195	3.28147985
5.0e-04	3.54418364	3.53774545	3.53162568	3.52580131	3.52025146
2.5e-04	3.78034023	3.77268731	3.76541521	3.75849621	3.75190523
1.0e-04	4.08428594	4.07487517	4.06593644	4.05743516	4.04934001
5.0e-05	4.30925100	4.29839469	4.28808622	4.27828523	4.26895519
2.5e-05	4.53082310	4.51842055	4.50664769	4.49545781	4.48480875
1.0e-05	4.81961161	4.80500862	4.79115295	4.77798879	4.76546573

Q	46 d.f.	47 d.f.	48 d.f.	49 d.f.	50 d.f.
5.0e-02	1.67866041	1.67792672	1.67722420	1.67655089	1.67590503
2.5e-02	2.01289560	2.01174051	2.01063476	2.00957524	2.00855911
1.0e-02	2.41018810	2.40834505	2.40658127	2.40489176	2.40327192
5.0e-03	2.68701349	2.68455562	2.68220403	2.67995197	2.67779327
2.5e-03	2.94878131	2.94563005	2.94261580	2.93972982	2.93696409
1.0e-03	3.27709803	3.27291238	3.26891002	3.26507917	3.26140906
5.0e-04	3.51495721	3.50990128	3.50506797	3.50044289	3.49601288
2.5e-04	3.74561951	3.73961835	3.73388285	3.72839578	3.72314135
1.0e-04	4.04162261	4.03425716	4.02722013	4.02049006	4.01404732
5.0e-05	4.26006300	4.25157858	4.24347456	4.23572594	4.22830987
2.5e-05	4.47466227	4.46498369	4.45574140	4.44690660	4.43845296
1.0e-05	4.75353814	4.74216461	4.73130749	4.72093244	4.71100811

Q	55 d.f.	60 d.f.	65 d.f.	70 d.f.	75 d.f.
5.0e-02	1.67303397	1.67064886	1.66863598	1.66691448	1.66542537
2.5e-02	2.00404478	2.00029782	1.99713791	1.99443711	1.99210215
1.0e-02	2.39608105	2.39011947	2.38509682	2.38080748	2.37710181
5.0e-03	2.66821599	2.66028303	2.65360447	2.64790462	2.64298307
2.5e-03	2.92470104	2.91455258	2.90601529	2.89873377	2.89244999
1.0e-03	3.24514911	3.23170913	3.22041427	3.21078906	3.20248884
5.0e-04	3.47639836	3.46020047	3.44659835	3.43501452	3.42503092
2.5e-04	3.69989094	3.68070823	3.66461208	3.65091331	3.63911360
1.0e-04	3.98556213	3.96208936	3.94241373	3.92568324	3.91128296
5.0e-05	4.19554178	4.16856502	4.14596993	4.12676979	4.11025331
2.5e-05	4.40112380	4.37042095	4.34472522	4.32290492	4.30414530
1.0e-05	4.66722124	4.63125194	4.60118010	4.57566638	4.55374816

Q	80 d.f.	85 d.f.	90 d.f.	95 d.f.	100 d.f.
5.0e-02	1.66412458	1.66297850	1.66196108	1.66105182	1.66023433
2.5e-02	1.99006342	1.98826791	1.98667454	1.98525100	1.98397152
1.0e-02	2.37386827	2.37102204	2.36849748	2.36624296	2.36421737
5.0e-03	2.63869060	2.63491385	2.63156517	2.62857567	2.62589052
2.5e-03	2.88697205	2.88215430	2.87788418	2.87407336	2.87065152
1.0e-03	3.19525769	3.18890160	3.18327081	3.17824791	3.17373949
5.0e-04	3.41633746	3.40869929	3.40193531	3.39590357	3.39049131
2.5e-04	3.62884379	3.61982452	3.61184054	3.60472332	3.59833899
1.0e-04	3.89875796	3.88776443	3.87803777	3.86937096	3.86159979
5.0e-05	4.09589483	4.08329752	4.07215620	4.06223231	4.05333670
2.5e-05	4.28784493	4.27355020	4.26091255	4.24965973	4.23957602
1.0e-05	4.53471592	4.51803518	4.50329571	4.49017743	4.47842690

CRITICAL POINTS OF 'STUDENT'S t' DISTRIBUTION

Q	105 d.f.	110 d.f.	115 d.f.	120 d.f.	125 d.f.
5.0e-02	1.65949538	1.65882419	1.65821183	1.65765090	1.65713518
2.5e-02	1.98281527	1.98176528	1.98080754	1.97993041	1.97912411
1.0e-02	2.36238751	2.36072634	2.35921155	2.35782461	2.35654999
5.0e-03	2.62346550	2.62126454	2.61925798	2.61742115	2.61573338
2.5e-03	2.86756201	2.86475864	2.86220344	2.85986485	2.85771644
1.0e-03	3.16967039	3.16597937	3.16261608	3.15953874	3.15671237
5.0e-04	3.38560776	3.38117908	3.37714453	3.37345377	3.37006464
2.5e-04	3.59257993	3.58735858	3.58260299	3.57825353	3.57426030
1.0e-04	3.85459227	3.84824112	3.84245828	3.83717075	3.83231751
5.0e-05	4.04531747	4.03805122	4.03143668	4.02538995	4.01984094
2.5e-05	4.23048828	4.22225594	4.21476368	4.20791602	4.20163323
1.0e-05	4.46784091	4.45825456	4.44953270	4.44156346	4.43425348

Q	130 d.f.	135 d.f.	140 d.f.	145 d.f.	150 d.f.
5.0e-02	1.65665941	1.65621913	1.65581051	1.65543025	1.65507550
2.5e-02	1.97838041	1.97769228	1.97705372	1.97645956	1.97590533
1.0e-02	2.35537458	2.35428723	2.35327841	2.35233989	2.35146458
5.0e-03	2.61417724	2.61273791	2.61140271	2.61016074	2.60900257
2.5e-03	2.85573593	2.85390438	2.85220558	2.85062561	2.84915243
1.0e-03	3.15410747	3.15169900	3.14946554	3.14738869	3.14545253
5.0e-04	3.36694163	3.36405457	3.36137771	3.35888889	3.35656898
2.5e-04	3.57058126	3.56718075	3.56402828	3.56109768	3.55836632
1.0e-04	3.82784717	3.82371615	3.81988724	3.81632846	3.81301220
5.0e-05	4.01473064	4.01000900	4.00563333	4.00156693	3.99777816
2.5e-05	4.19584819	4.19050401	4.18555216	4.18095098	4.17666449
1.0e-05	4.42752422	4.42130914	4.41555150	4.41020259	4.40522039

Q	155 d.f.	160 d.f.	165 d.f.	170 d.f.	175 d.f.
5.0e-02	1.65474377	1.65443290	1.65414098	1.65386632	1.65360744
2.5e-02	1.97538713	1.97490156	1.97444563	1.97401671	1.97361246
1.0e-02	2.35064631	2.34987966	2.34915992	2.34848289	2.34784490
5.0e-03	2.60791998	2.60690582	2.60595379	2.60505836	2.60421462
2.5e-03	2.84777556	2.84648586	2.84527530	2.84413682	2.84306416
1.0e-03	3.14364325	3.14194875	3.14035847	3.13886306	3.13745429
5.0e-04	3.35440135	3.35237147	3.35046663	3.34867562	3.34698852
2.5e-04	3.55581456	3.55342524	3.55118333	3.54907561	3.54709037
1.0e-04	3.80991450	3.80701443	3.80429368	3.80173611	3.79932747
5.0e-05	3.99423952	3.99092702	3.98781968	3.98489900	3.98214866
2.5e-05	4.17266149	4.16891474	4.16540042	4.16209755	4.15898761
1.0e-05	4.40056844	4.39621497	4.39213215	4.38829550	4.38468342

Q	180 d.f.	185 d.f.	190 d.f.	195 d.f.	200 d.f.
5.0e-02	1.65336301	1.65313187	1.65291295	1.65270531	1.65250810
2.5e-02	1.97323082	1.97286995	1.97252818	1.97220405	1.97189622
1.0e-02	2.34724265	2.34667322	2.34613401	2.34562266	2.34513708
5.0e-03	2.60341823	2.60266530	2.60195238	2.60127636	2.60063444
2.5e-03	2.84205179	2.84109474	2.84018862	2.83932945	2.83851369
1.0e-03	3.13612484	3.13486819	3.13367853	3.13255063	3.13147981
5.0e-04	3.34539656	3.34389190	3.34246756	3.34111727	3.33983541
2.5e-04	3.54521724	3.54344698	3.54177135	3.54018296	3.53867517
1.0e-04	3.79705512	3.79490780	3.79287549	3.79094918	3.78912079
5.0e-05	3.97955418	3.97710267	3.97478264	3.97258378	3.97049685
2.5e-05	4.15605417	4.15328262	4.15065992	4.14817440	4.14581557
1.0e-05	4.38127675	4.37805845	4.37501333	4.37212777	4.36938955

F-DISTRIBUTION CRITICAL POINTS: a = 1

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	1.00000000	0.66666667	0.58506027	0.54863217	0.52807377
2.5e-01	5.82842712	2.57142857	2.02386269	1.80740479	1.69246840
1.0e-01	39.86345819	8.52631579	5.53831946	4.54477072	4.06041995
5.0e-02	1.61448e+02	18.51282051	10.12796449	7.70864742	6.60789097
2.5e-02	6.47789e+02	38.50632911	17.44344332	12.21786263	10.00698220
1.0e-02	4.05218e+03	98.50251256	34.11622156	21.19768958	16.25817704
5.0e-03	1.62107e+04	1.98501e+02	55.55195674	31.33277162	22.78478053
2.5e-03	6.48449e+04	3.98501e+02	89.58432564	45.67398135	31.40667071
1.0e-03	4.05284e+05	9.98500e+02	1.67029e+02	74.13729332	47.18077922
5.0e-04	1.62114e+06	1.99850e+03	2.66549e+02	1.06219e+02	63.61104746
2.5e-04	6.48456e+06	3.99850e+03	4.24528e+02	1.51591e+02	85.29436037
1.0e-04	4.05285e+07	9.99850e+03	7.84007e+02	2.41619e+02	1.24941e+02

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	0.51488977	0.50572263	0.49898205	0.49381842	0.48973692
2.5e-01	1.62141846	1.57321501	1.53838939	1.51206168	1.49146474
1.0e-01	3.77594960	3.58942809	3.45791890	3.36030302	3.28501532
5.0e-02	5.98737761	5.59144785	5.31765507	5.11735503	4.96460274
2.5e-02	8.81310063	8.07266888	7.57088210	7.20928325	6.93672817
1.0e-02	13.74502253	12.24638335	11.25862414	10.56143105	10.04428927
5.0e-03	18.63499624	16.23555809	14.68819947	13.61360857	12.82647038
2.5e-03	24.80730788	21.11070021	18.77965427	17.18756501	16.03626411
1.0e-03	35.50749025	29.24519336	25.41476047	22.85712515	21.03959527
5.0e-04	46.08155982	36.98775831	31.55530418	27.99101806	25.49212581
2.5e-04	59.40839262	46.43107951	38.86323948	33.98581945	30.61302820
1.0e-04	82.48901481	62.16666899	50.69445529	43.47665002	38.57715317

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	0.47190648	0.46615534	0.46331535	0.46162268	0.46049894
2.5e-01	1.40365966	1.37608119	1.36259407	1.35459666	1.34930417
1.0e-01	2.97465302	2.88069452	2.83535424	2.80865765	2.79106763
5.0e-02	4.35124350	4.17087679	4.08474573	4.03430971	4.00119138
2.5e-02	5.87149377	5.56753500	5.42393715	5.34032322	5.28561059
1.0e-02	8.09595806	7.56247609	7.31409993	7.17057680	7.07710579
5.0e-03	9.94393492	9.17967728	8.82785886	8.62575804	8.49461671
2.5e-03	11.94005188	10.88929806	10.41112914	10.13808722	9.96156676
1.0e-03	14.81877555	13.29301437	12.60935783	12.22210607	11.97298729
5.0e-04	17.18954670	15.22281558	14.35200960	13.86178151	13.54761307
2.5e-04	19.73805151	17.25179545	16.16417054	15.55568775	15.16722615
1.0e-04	23.39948244	20.09206373	18.66844813	17.87860433	17.37693431

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.45909954	0.45826271	0.45770600	0.45730892	0.45701142
2.5e-01	1.34273220	1.33881216	1.33620839	1.33435323	1.33296442
1.0e-01	2.76931061	2.75637802	2.74780650	2.74170845	2.73714823
5.0e-02	3.96035242	3.93614299	3.92012441	3.90874141	3.90023617
2.5e-02	5.21835369	5.17859390	5.15233148	5.13369107	5.11977512
1.0e-02	6.96268806	6.89530103	6.85089345	6.81942412	6.79595794
5.0e-03	8.33460762	8.24064017	8.17882695	8.13507666	8.10248175
2.5e-03	9.74689709	9.62119645	9.53865820	9.48031128	9.43688063
1.0e-03	11.67136163	11.49543133	11.38019033	11.29886011	11.23839448
5.0e-04	13.16850723	12.94804351	12.80389832	12.70229757	12.62683091
2.5e-04	14.70006181	14.42920490	14.25244103	14.12800730	14.03566617
1.0e-04	16.77635444	16.42953841	16.20376425	16.04509834	15.92749851

F-DISTRIBUTION CRITICAL POINTS: a = 2

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	1.50000000	1.00000000	0.88110158	0.82842712	0.79876978
2.5e-01	7.50000000	3.00000000	2.27976315	2.00000000	1.85275282
1.0e-01	49.50000000	9.00000000	5.46238325	4.32455532	3.77971608
5.0e-02	1.99500e+02	19.00000000	9.55209450	6.94427191	5.78613504
2.5e-02	7.99500e+02	39.00000000	16.04410643	10.64911064	8.43362074
1.0e-02	4.99950e+03	99.00000000	30.81652035	18.00000000	13.27393361
5.0e-03	1.99995e+04	1.99000e+02	49.79927840	26.28427125	18.31383019
2.5e-03	7.99995e+04	3.99000e+02	79.93252850	38.00000000	24.96401358
1.0e-03	5.00000e+05	9.99000e+02	1.48500e+02	61.24555320	37.12232981
5.0e-04	2.00000e+06	1.99900e+03	2.36610e+02	87.44271910	49.78197763
2.5e-04	8.00000e+06	3.99900e+03	3.76476e+02	1.24491e+02	66.48648307
1.0e-04	5.00000e+07	9.99900e+03	6.94738e+02	1.98000e+02	97.02679264

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	0.77976315	0.76654779	0.75682846	0.74938068	0.74349177
2.5e-01	1.76220316	1.70098001	1.65685425	1.62355500	1.59753955
1.0e-01	3.46330407	3.25744205	3.11311764	3.00645242	2.92446596
5.0e-02	5.14325285	4.73741413	4.45897011	4.25649473	4.10282102
2.5e-02	7.25985568	6.54152030	6.05946744	5.71470539	5.45639553
1.0e-02	10.92476650	9.54657802	8.64911064	8.02151731	7.55943216
5.0e-03	14.54410643	12.40395675	11.04241237	10.10671356	9.42699906
2.5e-03	19.10418899	15.88714043	13.88854382	12.53915554	11.57227009
1.0e-03	27.00000000	21.68899856	18.49365301	16.38714975	14.90535853
5.0e-04	34.79763150	27.20573317	22.74961220	19.86546674	17.86525260
2.5e-04	44.62203156	33.93070800	27.81082915	23.92302451	21.26527804
1.0e-04	61.63304070	45.13234230	36.00000000	30.34186572	26.54786722

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	0.71773463	0.70941184	0.70529848	0.70284567	0.70121676
2.5e-01	1.48698355	1.45237470	1.43546925	1.42545101	1.41882368
1.0e-01	2.58925412	2.48871602	2.44036909	2.41195490	2.39325487
5.0e-02	3.49282848	3.31582950	3.23172699	3.18260985	3.15041131
2.5e-02	4.46125550	4.18206059	4.05099208	3.97493086	3.92526544
1.0e-02	5.84893192	5.39034586	5.17850824	5.05661087	4.97743204
5.0e-03	6.98646465	6.35468938	6.06642641	5.90161721	5.79499075
2.5e-03	8.20564203	7.36464069	6.98565695	6.77037992	6.63165900
1.0e-03	9.95262315	8.77339789	8.25075089	7.95641846	7.76776235
5.0e-04	11.38469200	9.89773989	9.24701147	8.88294950	8.65054195
2.5e-04	12.91954539	11.07525665	10.27840510	9.83552887	9.55395555
1.0e-04	15.11886432	12.71774696	11.69786385	11.13599427	10.78069173

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.69918768	0.69797399	0.69716642	0.69659034	0.69615870
2.5e-01	1.41059695	1.40569133	1.40243352	1.40011266	1.39837537
1.0e-01	2.37014901	2.35642740	2.34733823	2.34087447	2.33604218
5.0e-02	3.11076617	3.08729589	3.07177940	3.06075954	3.05252908
2.5e-02	3.86432908	3.82836693	3.80463818	3.78780839	3.77525084
1.0e-02	4.88073817	4.82390981	4.78650974	4.76003028	4.74029801
5.0e-03	5.66523966	5.58922307	5.53929272	5.50398906	5.47770671
2.5e-03	6.46345399	6.36521970	6.30082262	6.25535120	6.22153233
1.0e-03	7.54008910	7.40768107	7.32110726	7.26008522	7.21475939
5.0e-04	8.37107522	8.20906244	8.10334084	8.02892279	7.97370081
2.5e-04	9.21658672	9.02163067	8.89466355	8.80541127	8.73924653
1.0e-04	10.35701647	10.11322173	9.95486407	9.84374467	9.76147634

F-DISTRIBUTION CRITICAL POINTS: a = 3

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	1.70922560	1.13494292	1.00000000	0.94053408	0.90714622
2.5e-01	8.19986189	3.15337453	2.35555130	2.04666749	1.88426785
1.0e-01	53.59324466	9.16179017	5.39077328	4.19086044	3.61947741
5.0e-02	2.15707e+02	19.16429213	9.27662815	6.59138212	5.40945132
2.5e-02	8.64163e+02	39.16549456	15.43918238	9.97919853	7.76358948
1.0e-02	5.40335e+03	99.16620137	29.45669513	16.69436924	12.05995369
5.0e-03	2.16147e+04	1.99166e+02	47.46722825	24.25911989	16.52977046
2.5e-03	8.64603e+04	3.99167e+02	76.05612747	34.95563637	22.42561643
1.0e-03	5.40379e+05	9.99167e+02	1.41108e+02	56.17718849	33.20246318
5.0e-04	2.16152e+06	1.99917e+03	2.24701e+02	80.09249621	44.42245127
2.5e-04	8.64607e+06	3.99917e+03	3.57396e+02	1.13913e+02	59.22657461
1.0e-04	5.40380e+07	9.99917e+03	6.59340e+02	1.81018e+02	86.29162845

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	0.88578325	0.87094425	0.86003781	0.85168395	0.84508058
2.5e-01	1.78443100	1.71692883	1.66827158	1.63154633	1.60284883
1.0e-01	3.28876156	3.07407199	2.92379629	2.81286300	2.72767314
5.0e-02	4.75706266	4.34683140	4.06618055	3.86254836	3.70826482
2.5e-02	6.59879852	5.88981917	5.41596234	5.07811865	4.82562149
1.0e-02	9.77953824	8.45128505	7.59099195	6.99191722	6.55231256
5.0e-03	12.91660132	10.88244749	9.59647499	8.71705528	8.08074665
2.5e-03	16.86661382	13.84339205	11.97855941	10.72649659	9.83337198
1.0e-03	23.70330865	18.77226982	15.82948958	13.90180319	12.55274539
5.0e-04	30.45347686	23.45718776	19.38654349	16.77002060	14.96550051
2.5e-04	38.95720180	29.16704262	23.61535997	20.11451738	17.73558305
1.0e-04	53.68032159	38.67645113	30.45613778	25.40363768	22.03762163

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	0.81621271	0.80688687	0.80227752	0.79952877	0.79770323
2.5e-01	1.48080855	1.44256173	1.42386895	1.41278794	1.40545599
1.0e-01	2.38008705	2.27607140	2.22609158	2.19672976	2.17741094
5.0e-02	3.09839121	2.92227719	2.83874540	2.79000841	2.75807830
2.5e-02	3.85869867	3.58935912	3.46325966	3.39018878	3.34251973
1.0e-02	4.93819338	4.50973956	4.31256921	4.19934345	4.12589193
5.0e-03	5.81770166	5.23879260	4.97584100	4.82586925	4.72899121
2.5e-03	6.75685670	5.99874960	5.65888664	5.46637136	5.34253655
1.0e-03	8.09837979	7.05445715	6.59453998	6.33637057	6.17123078
5.0e-04	9.19553822	7.89438915	7.32868940	7.01331530	6.81243723
2.5e-04	10.36966418	8.77218298	8.08683319	7.70737905	7.46670853
1.0e-04	12.04979982	9.99418991	9.12777806	8.65244176	8.35264736

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.79542910	0.79406874	0.79316354	0.79251781	0.79203397
2.5e-01	1.39635278	1.39092352	1.38731753	1.38474842	1.38282519
1.0e-01	2.15354589	2.13937624	2.12999144	2.12331802	2.11832928
5.0e-02	2.71878498	2.69553425	2.68016757	2.66925636	2.66110830
2.5e-02	3.28408126	3.24961885	3.22689026	3.21077499	3.19875340
1.0e-02	4.03629673	3.98369531	3.94909979	3.92461701	3.90637866
5.0e-03	4.61126656	4.54238189	4.49717148	4.46522233	4.44144666
2.5e-03	5.19262120	5.10519386	5.04793284	5.00752531	4.97748627
1.0e-03	5.97230473	5.85680693	5.78136832	5.72823308	5.68878605
5.0e-04	6.57137081	6.43187271	6.34094670	6.27699346	6.22956410
2.5e-04	7.17897451	7.01302633	6.90508368	6.82926894	6.77310049
1.0e-04	7.99600903	7.79122713	7.65838864	7.56526196	7.49636121

F-DISTRIBUTION CRITICAL POINTS: a = 4

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	1.82271484	1.20710678	1.06322570	1.00000000	0.96456230
2.5e-01	8.58094417	3.23205081	2.39010880	2.06417777	1.89268285
1.0e-01	55.83296112	9.24341649	5.34264448	4.10724954	3.52019625
5.0e-02	2.24583e+02	19.24679434	9.11718225	6.38823291	5.19216777
2.5e-02	8.99583e+02	39.24841766	15.10097893	9.60452988	7.38788575
1.0e-02	5.62458e+03	99.24937186	28.70989839	15.97702485	11.39192807
5.0e-03	2.24996e+04	1.99250e+02	46.19462233	23.15450144	15.55605980
2.5e-03	8.99996e+04	3.99250e+02	73.94848584	33.30274407	21.04780179
1.0e-03	5.62500e+05	9.99250e+02	1.37100e+02	53.43582912	31.08500557
5.0e-04	2.25000e+06	1.99925e+03	2.18251e+02	76.12415689	41.53441461
2.5e-04	9.00000e+06	3.99925e+03	3.47069e+02	1.08210e+02	55.32148842
1.0e-04	5.62500e+07	9.99925e+03	6.40191e+02	1.71871e+02	80.52679983

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	0.94191327	0.92619310	0.91464536	0.90580385	0.89881713
2.5e-01	1.78715455	1.71573804	1.66421544	1.62529780	1.59486649
1.0e-01	3.18076287	2.96053409	2.80642571	2.69268006	2.60533643
5.0e-02	4.53367695	4.12031173	3.83785335	3.63308851	3.47804969
2.5e-02	6.22716116	5.52259435	5.05263222	4.71807846	4.46834158
1.0e-02	9.14830103	7.84664506	7.00607662	6.42208546	5.99433866
5.0e-03	12.02753029	10.05049125	8.80512953	7.95588513	7.34280574
2.5e-03	15.65182375	12.73337945	10.94070768	9.74105900	8.88760116
1.0e-03	21.92354136	17.19799378	14.39158451	12.56031874	11.28275151
5.0e-04	28.11521342	21.44083788	17.57822751	15.10595725	13.40681009
2.5e-04	35.91493413	26.61139231	21.36609237	18.07366062	15.84474027
1.0e-04	49.41865819	35.22200612	27.49283567	22.76609372	19.63005182

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	0.86829269	0.85843675	0.85356585	0.85066120	0.84873213
2.5e-01	1.46518978	1.42443552	1.40449186	1.39266064	1.38482853
1.0e-01	2.24893440	2.14223486	2.09095000	2.06081565	2.04098590
5.0e-02	2.86608140	2.68962757	2.60597495	2.55717915	2.52521510
2.5e-02	3.51469516	3.24992538	3.12611417	3.05441497	3.00765937
1.0e-02	4.43069016	4.01787684	3.82829355	3.71954519	3.64904749
5.0e-03	5.17427991	4.62335729	4.37377546	4.23163204	4.13989373
2.5e-03	5.96652715	5.25263761	4.93360929	4.75320657	4.63728953
1.0e-03	7.09603407	6.12452095	5.69813414	5.45928316	5.30670156
5.0e-04	8.01847415	6.81679047	6.29655925	6.00720695	5.82317507
2.5e-04	9.00469087	7.53926013	6.91350477	6.56793070	6.34910409
1.0e-04	10.41472326	8.54372411	7.75922632	7.33005237	7.05986063

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.84632900	0.84489147	0.84393491	0.84325253	0.84274123
2.5e-01	1.37509999	1.36929540	1.36543912	1.36269120	1.36063382
1.0e-01	2.01648649	2.00193845	1.99230227	1.98544971	1.98032684
5.0e-02	2.48588494	2.46261493	2.44723651	2.43631746	2.42816381
2.5e-02	2.95036120	2.91658201	2.89430846	2.87851798	2.86673984
1.0e-02	3.56310963	3.51268406	3.47953139	3.45607544	3.43860516
5.0e-03	4.02850597	3.96337719	3.92065164	3.89046811	3.86801161
2.5e-03	4.49709824	4.41541461	4.36194547	4.32422836	4.29619731
1.0e-03	5.12312262	5.01665040	4.94715419	4.89822741	4.86191717
5.0e-04	5.60262071	5.47514664	5.39212129	5.33375571	5.29048694
2.5e-04	6.08787084	5.93740731	5.83961893	5.77097590	5.72014228
1.0e-04	6.73896435	6.55497858	6.43574167	6.35220428	6.29042743

F-DISTRIBUTION CRITICAL POINTS: a = 5

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	1.89367482	1.25192518	1.10235812	1.03673967	1.00000000
2.5e-01	8.81979304	3.27989441	2.40949587	2.07229963	1.89465977
1.0e-01	57.24007713	9.29262635	5.30915702	4.05057907	3.45298225
5.0e-02	2.30162e+02	19.29640965	9.01345517	6.25605650	5.05032906
2.5e-02	9.21848e+02	39.29822778	14.88482292	9.36447082	7.14638183
1.0e-02	5.76365e+03	99.29929648	28.23708084	15.52185754	10.96702065
5.0e-03	2.30558e+04	1.99300e+02	45.39164571	22.45642554	14.93960546
2.5e-03	9.22244e+04	3.99300e+02	72.62124836	32.26088089	20.17826615
1.0e-03	5.76405e+05	9.99300e+02	1.34580e+02	51.71156856	29.75239858
5.0e-04	2.30562e+06	1.99930e+03	2.14197e+02	73.63071456	39.71944679
2.5e-04	9.22248e+06	3.99930e+03	3.40581e+02	1.04628e+02	52.86992362
1.0e-04	5.76405e+07	9.99930e+03	6.28165e+02	1.66131e+02	76.91122960

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	0.97653640	0.96025973	0.94830808	0.93916021	0.93193316
2.5e-01	1.78521248	1.71105700	1.65750198	1.61701115	1.58532326
1.0e-01	3.10751167	2.88334450	2.72644692	2.61061255	2.52164069
5.0e-02	4.38737419	3.97152315	3.68749867	3.48165865	3.32583453
2.5e-02	5.98756513	5.28523685	4.81727556	4.48441131	4.23608567
1.0e-02	8.74589526	7.46043549	6.63182516	6.05694071	5.63632619
5.0e-03	11.46369566	9.52205882	8.30179885	7.47115811	6.87236676
2.5e-03	14.88421766	12.03115594	10.28342993	9.11637189	8.28754026
1.0e-03	20.80266396	16.20580032	13.48468945	11.71366731	10.48072247
5.0e-04	26.64521583	20.17264770	16.44033677	14.05834123	12.42508628
2.5e-04	34.00492000	25.00657806	19.95329797	16.79139457	14.65636896
1.0e-04	46.74657096	33.05626518	25.63499013	21.11234895	18.12031982

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	0.90037648	0.89019259	0.88516027	0.88215951	0.88016665
2.5e-01	1.44995229	1.40725869	1.38633195	1.37390574	1.36567462
1.0e-01	2.15822722	2.04924608	1.99681977	1.96599886	1.94571033
5.0e-02	2.71088984	2.53355455	2.44946643	2.40040913	2.36827024
2.5e-02	3.28905585	3.02646641	2.90372232	2.83265408	2.78631480
1.0e-02	4.10268463	3.69901881	3.51383983	3.40767951	3.33888442
5.0e-03	4.76157368	4.22757699	3.98604571	3.84860446	3.75994848
2.5e-03	5.46252153	4.77577584	4.46950063	4.29650034	4.18541724
1.0e-03	6.46056185	5.53391314	5.12826342	4.90134819	4.75652075
5.0e-04	7.27483822	6.13499834	5.64299818	5.36979062	5.19620585
2.5e-04	8.14484582	6.76168097	6.17301043	5.84850711	5.64326087
1.0e-04	9.38799089	7.63215150	6.89870471	6.49828696	6.24653069

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.87768409	0.87619905	0.87521088	0.87450595	0.87397774
2.5e-01	1.35544448	1.34933734	1.34527867	1.34238582	1.34021957
1.0e-01	1.92063596	1.90574201	1.89587480	1.88885698	1.88361004
5.0e-02	2.32872059	2.30531824	2.28985128	2.27886882	2.27066749
2.5e-02	2.72953189	2.69605896	2.67398832	2.65834216	2.64667192
1.0e-02	3.25504930	3.20587177	3.17354548	3.15067703	3.13364588
5.0e-03	3.65235549	3.58947293	3.54823223	3.51910315	3.49743416
2.5e-03	4.05115723	3.97297415	3.92181483	3.88573600	3.85892738
1.0e-03	4.58241281	4.48150773	4.41567581	4.36934360	4.33496692
5.0e-04	4.98836693	4.86834396	4.79021350	4.73530928	4.69461760
2.5e-04	5.39849817	5.25765603	5.16617603	5.10198800	5.05446807
1.0e-04	5.94788813	5.77684795	5.66607707	5.58850790	5.53116463

F-DISTRIBUTION CRITICAL POINTS: a = 6

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	1.94216329	1.28244070	1.12894436	1.06166888	1.02402737
2.5e-01	8.98325631	3.31205614	2.42178539	2.07656825	1.89447158
1.0e-01	58.20441643	9.32553045	5.28473156	4.00974931	3.40450658
5.0e-02	2.33986e+02	19.32953402	8.94064512	6.16313228	4.95028807
2.5e-02	9.37111e+02	39.33145796	14.73471841	9.19731108	6.97770186
1.0e-02	5.85899e+03	99.33258887	27.91065736	15.20686486	10.67225479
5.0e-03	2.34371e+04	1.99333e+02	44.83846833	21.97457925	14.51326301
2.5e-03	9.37496e+04	3.99333e+02	71.70803432	31.54293325	19.57813950
1.0e-03	5.85937e+05	9.99333e+02	1.32847e+02	50.52502195	28.83436098
5.0e-04	2.34375e+06	1.99933e+03	2.11412e+02	71.91599825	38.47029110
2.5e-04	9.37500e+06	3.99933e+03	3.36125e+02	1.02167e+02	51.18379538
1.0e-04	5.85937e+07	9.99933e+03	6.19905e+02	1.62187e+02	74.42612999

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.00000000	0.98333868	0.97110826	0.96174924	0.95435681
2.5e-01	1.78213518	1.70593189	1.65083882	1.60914497	1.57648745
1.0e-01	3.05455068	2.82739227	2.66833472	2.55085525	2.46058197
5.0e-02	4.28386571	3.86596885	3.58058032	3.37375365	3.21717455
2.5e-02	5.81975658	5.11859661	4.65169554	4.31972183	4.07213132
1.0e-02	8.46612534	7.19140479	6.37068073	5.80177031	5.38581104
5.0e-03	11.07303891	9.15533592	7.95199224	7.13385028	6.54463054
2.5e-03	14.35366220	11.54514000	9.82797076	8.68302680	7.87087430
1.0e-03	20.02965472	15.52084044	12.85802614	11.12812978	9.92561291
5.0e-04	25.63264742	19.29838451	15.65530173	13.33507576	11.74687111
2.5e-04	32.69044991	23.90146269	18.97982746	15.90735927	13.83663064
1.0e-04	44.90926558	31.56651615	24.35650380	19.97385057	17.08055583

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	0.92209307	0.91168594	0.90654402	0.90347807	0.90144198
2.5e-01	1.43660420	1.39231983	1.37057411	1.35764778	1.34907933
1.0e-01	2.09132249	1.98033315	1.92687856	1.89543103	1.87472025
5.0e-02	2.59897771	2.42052319	2.33585240	2.28643590	2.25405301
2.5e-02	3.12833996	2.86669615	2.74438158	2.67355497	2.62736959
1.0e-02	3.87142682	3.47347661	3.29101239	3.18643421	3.11867427
5.0e-03	4.47214659	3.94921391	3.71290608	3.57850226	3.49183152
2.5e-03	5.11052016	4.44187172	4.14405914	3.97595722	3.86806700
1.0e-03	6.01860847	5.12225568	4.73056833	4.51167581	4.37205461
5.0e-04	6.75896790	5.66110804	5.18823003	4.92594441	4.75942148
2.5e-04	7.54962353	6.22248494	5.65902893	5.34883843	5.15281145
1.0e-04	8.67889223	7.00168902	6.30305962	5.92224051	5.68304344

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.89890561	0.89738839	0.89637882	0.89565862	0.89511898
2.5e-01	1.33842281	1.33205721	1.32782510	1.32480780	1.32254788
1.0e-01	1.84911259	1.83389555	1.82381158	1.81663824	1.81127429
5.0e-02	2.21419280	2.19060094	2.17500625	2.16393185	2.15566118
2.5e-02	2.57077051	2.53740316	2.51540088	2.49980253	2.48816758
1.0e-02	3.03611087	2.98768450	2.95585400	2.93333727	2.91656857
5.0e-03	3.38667588	3.32523238	3.28494141	3.25648601	3.23531972
2.5e-03	3.73771799	3.66183950	3.61219932	3.57719723	3.55119162
1.0e-03	4.20429909	4.10712455	4.04374661	3.99915128	3.96606860
5.0e-04	4.56017134	4.44517798	4.37035025	4.31778086	4.27882717
2.5e-04	4.91922394	4.78490668	4.69770349	4.63653514	4.59126095
1.0e-04	5.39955540	5.23732804	5.13231929	5.05881163	5.00448520

F-DISTRIBUTION CRITICAL POINTS: a = 8

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	2.00408012	1.32130338	1.16273958	1.09331993	1.05450963
2.5e-01	9.19227857	3.35255771	2.43636625	2.08046390	1.89230439
1.0e-01	59.43898057	9.36677033	5.25167108	3.95493994	3.33927571
5.0e-02	2.38883e+02	19.37099290	8.84523846	6.04104448	4.81831954
2.5e-02	9.56656e+02	39.37302207	14.53988657	8.97958042	6.75717201
1.0e-02	5.98107e+03	99.37421482	27.48917703	14.79888879	10.28931105
5.0e-03	2.39254e+04	1.99375e+02	44.12557171	21.35198036	13.96096260
2.5e-03	9.57027e+04	3.99375e+02	70.53246151	30.61669240	18.80222981
1.0e-03	5.98144e+05	9.99375e+02	1.30619e+02	48.99618877	27.64947537
5.0e-04	2.39258e+06	1.99937e+03	2.07830e+02	69.70799604	36.85948243
2.5e-04	9.57031e+06	3.99937e+03	3.30395e+02	98.99810985	49.01092725
1.0e-04	5.98145e+07	9.99937e+03	6.09289e+02	1.57113e+02	71.22560842

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.02975131	1.01259144	1.00000000	0.99036782	0.98276167
2.5e-01	1.77596228	1.69687168	1.63957798	1.59614063	1.56206191
1.0e-01	2.98303561	2.75157958	2.58934906	2.46940565	2.37715002
5.0e-02	4.14680416	3.72572532	3.43810123	3.22958261	3.07165839
2.5e-02	5.59962301	4.89934065	4.43325989	4.10195570	3.85489088
1.0e-02	8.10165137	6.84004907	6.02887011	5.46712252	5.05669313
5.0e-03	10.56576351	8.67811474	7.49590591	6.69330016	6.11591875
2.5e-03	13.66631442	10.91432763	9.23582571	8.11877788	7.32760050
1.0e-03	19.03033312	14.63400663	12.04554124	10.36800037	9.20414986
5.0e-04	24.32513325	18.16800387	14.63906989	12.39776810	10.86705246
2.5e-04	30.99457530	22.47412492	17.72122257	14.76328871	12.77482756
1.0e-04	42.54086825	29.64445771	22.70563892	18.50259148	15.73590050

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	0.94958829	0.93889604	0.93361459	0.93046581	0.92837483
2.5e-01	1.41533425	1.36852491	1.34545147	1.33170427	1.32257756
1.0e-01	1.99853387	1.88412068	1.82886338	1.79629964	1.77482888
5.0e-02	2.44706375	2.26616327	2.18017045	2.12992276	2.09696831
2.5e-02	2.91279653	2.65125626	2.52886345	2.45794198	2.41167182
1.0e-02	3.56441205	3.17262396	2.99298087	2.89000772	2.82328022
5.0e-03	4.08997348	3.58005978	3.34978976	3.21885741	3.13443791
2.5e-03	4.64767036	4.00108804	3.71348230	3.55124821	3.44716499
1.0e-03	5.43999319	4.58142498	4.20703658	3.99804325	3.86482817
5.0e-04	6.08534985	5.04033981	4.59141906	4.34277627	4.18505663
2.5e-04	6.77410544	5.51792985	4.98629362	4.69412469	4.50968828
1.0e-04	7.75725282	6.18016566	5.52574668	5.16977318	4.94648043

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.92577020	0.92421220	0.92317551	0.92243597	0.92188183
2.5e-01	1.31121002	1.30441040	1.29988570	1.29665778	1.29423899
1.0e-01	1.74825210	1.73244278	1.72195924	1.71449812	1.70891699
5.0e-02	2.05637261	2.03232759	2.01642561	2.00512910	1.99669039
2.5e-02	2.35494113	2.32148052	2.29940990	2.28375964	2.27208403
1.0e-02	2.74196415	2.69426273	2.66290563	2.64072214	2.62420066
5.0e-03	3.03202536	2.97218984	2.93295508	2.90524639	2.88463589
2.5e-03	3.32145832	3.24830386	3.20045436	3.16671901	3.14165672
1.0e-03	3.70486836	3.61226059	3.55188188	3.50940693	3.47790266
5.0e-04	3.99649283	3.88774624	3.81701581	3.76734061	3.73054001
2.5e-04	4.29013024	4.16399355	4.08214785	4.02475994	3.98229605
1.0e-04	4.68216385	4.53107485	4.43334442	4.36496485	4.31444637

F-DISTRIBUTION CRITICAL POINTS: a = 10

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	2.04191262	1.34500479	1.18331912	1.11257336	1.07303833
2.5e-01	9.32014944	3.37701838	2.44466882	2.08196054	1.88985300
1.0e-01	60.19498034	9.39157278	5.23041127	3.91987560	3.29740167
5.0e-02	2.41882e+02	19.39589672	8.78552471	5.96437055	4.73506307
2.5e-02	9.68627e+02	39.39797460	14.41894204	8.84388097	6.61915433
1.0e-02	6.05585e+03	99.39919597	27.22873412	14.54590080	10.05101722
5.0e-03	2.42245e+04	1.99400e+02	43.68580112	20.96673027	13.61817961
2.5e-03	9.68990e+04	3.99400e+02	69.80799214	30.04435169	18.32152712
1.0e-03	6.05621e+05	9.99400e+02	1.29247e+02	48.05258912	26.91656759
5.0e-04	2.42248e+06	1.99940e+03	2.05625e+02	68.34597381	35.86394325
2.5e-04	9.68994e+06	3.99940e+03	3.26869e+02	97.04433586	47.66882750
1.0e-04	6.05621e+07	9.99940e+03	6.02755e+02	1.53985e+02	69.24988423

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.04782613	1.03035648	1.01754070	1.00773884	1.00000000
2.5e-01	1.77084828	1.68980277	1.63099449	1.58633970	1.55125573
1.0e-01	2.93693467	2.70251048	2.53803678	2.41631558	2.32260394
5.0e-02	4.05996279	3.63652312	3.34716312	3.13728011	2.97823702
2.5e-02	5.46132372	4.76111643	4.29512696	3.96386516	3.71679186
1.0e-02	7.87411853	6.62006267	5.81429386	5.25654199	4.84914680
5.0e-03	10.25003712	8.38032593	7.21063592	6.41715965	5.84667843
2.5e-03	13.23942867	10.52165948	8.86645540	7.76613562	6.98747494
1.0e-03	18.41092480	14.08325538	11.54005611	9.89430492	8.75386628
5.0e-04	23.51556871	17.46691232	14.00776672	11.81462932	10.31892882
2.5e-04	29.94541132	21.58974925	16.94027977	14.05246700	12.11430327
1.0e-04	41.07681527	28.45476856	21.68255525	17.58976366	14.90072411

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	0.96626389	0.95539658	0.95002982	0.94683051	0.94470610
2.5e-01	1.39948744	1.35071506	1.32658201	1.31216939	1.30258542
1.0e-01	1.93673830	1.81948544	1.76268577	1.72914957	1.70700876
5.0e-02	2.34787757	2.16457992	2.07724805	2.02614296	1.99259200
2.5e-02	2.77367138	2.51119130	2.38816109	2.31679416	2.27019826
1.0e-02	3.36818639	2.97909356	2.80054511	2.69813941	2.63175078
5.0e-03	3.84700175	3.34396482	3.11674828	2.98751918	2.90417997
2.5e-03	4.35463134	3.72047844	3.43848421	3.27942373	3.17737545
1.0e-03	5.07524621	4.23879176	3.87438608	3.67105214	3.54147524
5.0e-04	5.66183781	4.64824212	4.21341763	3.97274845	3.82014965
2.5e-04	6.28761962	5.07405129	4.56137406	4.27988874	4.10229917
1.0e-04	7.18053924	5.66408483	5.03629308	4.69522944	4.48145666

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.94205994	0.94047716	0.93942399	0.93867270	0.93810977
2.5e-01	1.29062971	1.28346777	1.27869738	1.27529187	1.27273871
1.0e-01	1.67956794	1.66322513	1.65237930	1.64465603	1.63887641
5.0e-02	1.95122032	1.92669249	1.91046106	1.89892540	1.89030517
2.5e-02	2.21302571	2.17928041	2.15701144	2.14121518	2.12942767
1.0e-02	2.55081190	2.50331113	2.47207675	2.44997541	2.43351248
5.0e-03	2.80305422	2.74395626	2.70519856	2.67782338	2.65745903
2.5e-03	3.05412127	2.98238950	2.93546807	2.90238559	2.87780749
1.0e-03	3.38591391	3.29586659	3.23716241	3.19586776	3.16524012
5.0e-04	3.63777320	3.53262738	3.46425175	3.41623640	3.38066866
2.5e-04	3.89100000	3.76966311	3.69095395	3.63577586	3.59495280
1.0e-04	4.22858731	4.08413337	3.99073189	3.92539904	3.87714110

F-DISTRIBUTION CRITICAL POINTS: a = 12

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	2.06741246	1.36096586	1.19716621	1.12551995	1.08549169
2.5e-01	9.40640006	3.39339207	2.45000766	2.08258018	1.88769064
1.0e-01	60.70521187	9.40813214	5.21561783	3.89552685	3.26823921
5.0e-02	2.43906e+02	19.41251115	8.74464066	5.91172911	4.67770379
2.5e-02	9.76708e+02	39.41461548	14.33655235	8.75115892	6.52454922
1.0e-02	6.10632e+03	99.41585240	27.05181926	14.37358701	9.88827549
5.0e-03	2.44264e+04	1.99416e+02	43.38738679	20.70468755	13.38447084
2.5e-03	9.77065e+04	3.99416e+02	69.31669341	29.65539625	17.99416442
1.0e-03	6.10668e+05	9.99417e+02	1.28316e+02	47.41180417	26.41796644
5.0e-04	2.44267e+06	1.99942e+03	2.04131e+02	67.42137437	35.18703535
2.5e-04	9.77069e+06	3.99942e+03	3.24479e+02	95.71835713	46.75663786
1.0e-04	6.10668e+07	9.99942e+03	5.98328e+02	1.51862e+02	67.90753125

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.05997008	1.04228905	1.02932008	1.01940231	1.01157283
2.5e-01	1.76678069	1.68431656	1.62439522	1.57883548	1.54299665
1.0e-01	2.90472051	2.66811142	2.50195779	2.37888486	2.28405130
5.0e-02	3.99993538	3.57467645	3.28393901	3.07294712	2.91297672
2.5e-02	5.36624395	4.66582972	4.19966746	3.86822032	3.62094548
1.0e-02	7.71833266	6.46909128	5.66671926	5.11143102	4.70586969
5.0e-03	10.03429221	8.17641253	7.01491723	6.22736742	5.66132597
2.5e-03	12.94813481	10.25321171	8.61349420	7.52424007	6.75381677
1.0e-03	17.98881077	13.70731634	11.19448648	9.57000509	8.44518506
5.0e-04	22.96425725	16.98876522	13.57661585	11.41585275	9.94364059
2.5e-04	29.23131868	20.98700934	16.40735853	13.56681797	11.66251381
1.0e-04	40.08086276	27.64449734	20.98497101	16.96669499	14.33008743

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	0.97745286	0.96646670	0.96104223	0.95780881	0.95566187
2.5e-01	1.38731466	1.33694673	1.31193225	1.29695822	1.28698474
1.0e-01	1.89236317	1.77270370	1.71456263	1.68016749	1.65742869
5.0e-02	2.27758057	2.09206319	2.00345940	1.95152768	1.91739590
2.5e-02	2.67583062	2.41203403	2.28815698	2.21620916	2.16919216
1.0e-02	3.23111983	2.84309520	2.66482736	2.56249676	2.49611595
5.0e-03	3.67790684	3.17873241	2.95310109	2.82470195	2.74186262
2.5e-03	4.15128771	3.52473568	3.24604603	3.08880286	2.98789516
1.0e-03	4.82291806	4.00061548	3.64246960	3.44263307	3.31528023
5.0e-04	5.36940785	4.37627238	3.95049782	3.71489598	3.56552721
2.5e-04	5.95224501	4.76673955	4.26641634	3.99184032	3.81865430
1.0e-04	6.78367218	5.30753728	4.69731651	4.36603879	4.15848974

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.95298775	0.95138830	0.95032406	0.94956488	0.94899604
2.5e-01	1.27452341	1.26704731	1.26206271	1.25850175	1.25583062
1.0e-01	1.62920909	1.61238102	1.60120371	1.59323959	1.58727703
5.0e-02	1.87526157	1.85025511	1.83369528	1.82192027	1.81311786
2.5e-02	2.11145183	2.07734219	2.05481988	2.03883734	2.02690713
1.0e-02	2.41513608	2.36758212	2.33629981	2.31415791	2.29766106
5.0e-03	2.64129879	2.58250308	2.54393186	2.51668235	2.49640806
2.5e-03	2.86598479	2.79501467	2.74858203	2.71583928	2.69151077
1.0e-03	3.16237783	3.07386111	3.01615026	2.97555185	2.94543910
5.0e-04	3.38702374	3.28411476	3.21719439	3.17020098	3.13539014
2.5e-04	3.61263707	3.49435290	3.41763112	3.36384943	3.32406108
1.0e-04	3.91307438	3.77292408	3.68232304	3.61895741	3.57215698

F-DISTRIBUTION CRITICAL POINTS: a = 16

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	2.09959413	1.38109746	1.21462015	1.14183020	1.10117402
2.5e-01	9.51532119	3.41393407	2.45645166	2.08290372	1.88434658
1.0e-01	61.34988225	9.42885877	5.19640464	3.86393631	3.23027956
5.0e-02	2.46464e+02	19.43329253	8.69228627	5.84411743	4.60376403
2.5e-02	9.86919e+02	39.43542317	14.23152005	8.63258076	6.40316106
1.0e-02	6.17010e+03	99.43667556	26.82685728	14.15385989	9.68016431
5.0e-03	2.46815e+04	1.99437e+02	43.00828756	20.37095625	13.08607221
2.5e-03	9.87269e+04	3.99437e+02	68.69290405	29.16043220	17.57663692
1.0e-03	6.17045e+05	9.99437e+02	1.27136e+02	46.59692605	25.78264593
5.0e-04	2.46818e+06	1.99944e+03	2.02235e+02	66.24595750	34.32494470
2.5e-04	9.87273e+06	3.99944e+03	3.21447e+02	94.03305949	45.59532412
1.0e-04	6.17046e+07	9.99944e+03	5.92712e+02	1.49165e+02	66.19915969

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.07525776	1.05730668	1.04414178	1.03407565	1.02613009
2.5e-01	1.76088455	1.67649170	1.61503629	1.56821374	1.53130954
1.0e-01	2.86263522	2.62301285	2.45450054	2.32949931	2.23304250
5.0e-02	3.92228336	3.49440809	3.20163427	2.98896556	2.82756643
2.5e-02	5.24386045	4.54281777	4.07609592	3.74409691	3.49627142
1.0e-02	7.51857375	6.27500976	5.47655111	4.92402234	4.52044822
5.0e-03	9.75815750	7.91481746	6.76328978	5.98286444	5.42209123
2.5e-03	12.57579409	9.90935409	8.28883179	7.21320246	6.45284803
1.0e-03	17.44991260	13.22647828	10.75171249	9.15379149	8.04839545
5.0e-04	22.26088542	16.37770177	13.02471932	10.90461187	9.46181316
2.5e-04	28.32073090	20.21721421	15.72571481	12.94475427	11.08303968
1.0e-04	38.81148854	26.61032433	20.09342009	16.16935140	13.59894517

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	0.99151848	0.98038066	0.97488281	0.97160614	0.96943067
2.5e-01	1.36988631	1.31704632	1.29062573	1.27473947	1.26412502
1.0e-01	1.83253414	1.70899831	1.64862761	1.61277664	1.58901067
5.0e-02	2.18398317	1.99462355	1.90374984	1.85031495	1.81511336
2.5e-02	2.54654003	2.27988920	2.15418253	2.08097558	2.03304226
1.0e-02	3.05119839	2.66318812	2.48442367	2.38160008	2.31479931
5.0e-03	3.45675593	2.96105443	2.73652824	2.60855771	2.52589677
2.5e-03	3.88612228	3.26772358	2.99228173	2.83669375	2.73675858
1.0e-03	4.49489069	3.68900394	3.33782390	3.14175292	3.01673209
5.0e-04	4.98997307	4.02124423	3.60596551	3.37611392	3.23034831
2.5e-04	5.51779779	4.36635238	3.88071806	3.61422918	3.44613495
1.0e-04	6.27050390	4.84402104	4.25512568	3.93559470	3.73545337

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.96672123	0.96510076	0.96402256	0.96325344	0.96267716
2.5e-01	1.25082121	1.24281552	1.23746700	1.23364048	1.23076701
1.0e-01	1.55943591	1.54175312	1.52998722	1.52159301	1.51530239
5.0e-02	1.77155674	1.74564723	1.72846299	1.71623044	1.70707833
2.5e-02	1.97406032	1.93914970	1.91606823	1.89967336	1.88742657
1.0e-02	2.23318182	2.18518032	2.15357073	2.13118025	2.11448860
5.0e-03	2.42542549	2.36661106	2.32799461	2.30069615	2.28037578
2.5e-03	2.61590787	2.54548606	2.49938087	2.46685280	2.44267455
1.0e-03	2.86653772	2.77953224	2.72278030	2.68284259	2.65321194
5.0e-04	3.05608634	2.95558042	2.89020220	2.84428071	2.81025748
2.5e-04	3.24614646	3.13130034	3.05679579	3.00456112	2.96591293
1.0e-04	3.49883095	3.36371112	3.27636263	3.21527105	3.17014936

F-DISTRIBUTION CRITICAL POINTS: a = 20

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	2.11906393	1.39327262	1.22517083	1.15168539	1.11064651
2.5e-01	9.58125473	3.42629923	2.46019186	2.08284563	1.88197592
1.0e-01	61.74029240	9.44130938	5.18448168	3.84433831	3.20665035
5.0e-02	2.48013e+02	19.44576849	8.66018980	5.80254189	4.55813150
2.5e-02	9.93103e+02	39.44791130	14.16738138	8.55994319	6.32855524
1.0e-02	6.20873e+03	99.44917085	26.68979051	14.01960868	9.55264616
5.0e-03	2.48360e+04	1.99450e+02	42.77750013	20.16727590	12.90348806
2.5e-03	9.93449e+04	3.99450e+02	68.31334093	28.85856839	17.32140862
1.0e-03	6.20908e+05	9.99450e+02	1.26418e+02	46.10025764	25.39462209
5.0e-04	2.48363e+06	1.99945e+03	2.01082e+02	65.52975122	33.79865879
2.5e-04	9.93453e+06	3.99945e+03	3.19604e+02	93.00638096	44.88660575
1.0e-04	6.20908e+07	9.99945e+03	5.89297e+02	1.47523e+02	65.15691217

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.08448922	1.06637291	1.05308797	1.04293082	1.03491397
2.5e-01	1.75687916	1.67123233	1.60876566	1.56109998	1.52347608
1.0e-01	2.83633965	2.59473154	2.42463734	2.29832236	2.20074392
5.0e-02	3.87418858	3.44452483	3.15032377	2.93645539	2.77401640
2.5e-02	5.16840094	4.46673962	3.99945297	3.66690550	3.41854352
1.0e-02	7.39583189	6.15543839	5.35909494	4.80799523	4.40539477
5.0e-03	9.58877073	7.75396067	6.60820487	5.83184098	5.27401675
2.5e-03	12.34766741	9.69821202	8.08905340	7.02142152	6.26692019
1.0e-03	17.12011020	12.93162574	10.47968282	8.89761271	7.80374705
5.0e-04	21.83068936	16.00327962	12.68595254	10.59026697	9.16507220
2.5e-04	27.76405791	19.74581303	15.30760753	12.56258680	10.72649585
1.0e-04	38.03583312	25.97740849	19.54696710	15.67992931	13.14953081

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	1.00000000	0.98876903	0.98322627	0.97992321	0.97773037
2.5e-01	1.35800933	1.30332015	1.27581155	1.25920379	1.24807450
1.0e-01	1.79384331	1.66730895	1.60515147	1.56810713	1.54348574
5.0e-02	2.12415521	1.93165348	1.83885935	1.78412482	1.74798413
2.5e-02	2.46448430	2.19516027	2.06771405	1.99329449	1.94446982
1.0e-02	2.93773528	2.54865918	2.36887612	2.26524280	2.19780591
5.0e-03	3.31778578	2.82304097	2.59841759	2.47016094	2.38720056
2.5e-03	3.71996743	3.10530817	2.83103059	2.67587333	2.57610024
1.0e-03	4.28996645	3.49278411	3.14498953	2.95060455	2.82655183
5.0e-04	4.75337842	3.79818583	3.38841951	3.16144666	3.01741163
2.5e-04	5.24733646	4.11528507	3.63770175	3.37550410	3.21003914
1.0e-04	5.95161262	4.55401187	3.97720089	3.66418322	3.46807649

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.97499948	0.97336626	0.97227961	0.97150448	0.97092370
2.5e-01	1.23408357	1.22563958	1.21998690	1.21593688	1.21289220
1.0e-01	1.51276462	1.49434796	1.48207166	1.47330182	1.46672313
5.0e-02	1.70316008	1.67643425	1.65868014	1.64602715	1.63655204
2.5e-02	1.88426718	1.84856071	1.82491961	1.80810983	1.79554318
1.0e-02	2.11527070	2.06664610	2.03458806	2.01186010	1.99490553
5.0e-03	2.28621828	2.22701711	2.18810657	2.16057927	2.14007660
2.5e-03	2.45529730	2.38481447	2.33862840	2.30602188	2.28177308
1.0e-03	2.67737934	2.59088001	2.53441832	2.49466390	2.46515719
5.0e-04	2.84508960	2.74562277	2.68088292	2.63538994	2.60167275
2.5e-04	3.01306829	2.89988470	2.82642490	2.77490463	2.73677454
1.0e-04	3.23614809	3.10365608	3.01797977	2.95804312	2.91376586

F-DISTRIBUTION CRITICAL POINTS: a = 24

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	2.13210973	1.40142948	1.23223763	1.15828486	1.11698845
2.5e-01	9.62544871	3.43455931	2.46263144	2.08269754	1.88023528
1.0e-01	62.00204594	9.44961589	5.17636482	3.83099448	3.19052294
5.0e-02	2.49052e+02	19.45408876	8.63850104	5.77438866	4.52715311
2.5e-02	9.97249e+02	39.45623819	14.12414582	8.51087345	6.27804014
1.0e-02	6.23463e+03	99.45750162	26.59752322	13.92906354	9.46647080
5.0e-03	2.49396e+04	1.99458e+02	42.62222531	20.03000071	12.78021005
2.5e-03	9.97593e+04	3.99458e+02	68.05804738	28.65521321	17.14918927
1.0e-03	6.23497e+05	9.99458e+02	1.25935e+02	45.76579762	25.13294244
5.0e-04	2.49399e+06	1.99946e+03	2.00307e+02	65.04754250	33.44383925
2.5e-04	9.97596e+06	3.99946e+03	3.18364e+02	92.31522585	44.40889292
1.0e-04	6.23498e+07	9.99946e+03	5.87001e+02	1.46417e+02	64.45452382

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.09066874	1.07244095	1.05907491	1.04885621	1.04079112
2.5e-01	1.75400048	1.66747155	1.60428651	1.55601638	1.51787212
1.0e-01	2.81834478	2.57532726	2.40409654	2.27682727	2.17842592
5.0e-02	3.84145690	3.41049438	3.11523980	2.90047376	2.73724765
2.5e-02	5.11719240	4.41499907	3.94722034	3.61419574	3.36536871
1.0e-02	7.31272081	6.07431925	5.27926439	4.72899757	4.32692916
5.0e-03	9.47419879	7.64496818	6.50294579	5.72917229	5.17319682
2.5e-03	12.19348377	9.55527857	7.95360165	6.89119649	6.14048728
1.0e-03	16.89736868	12.73220036	10.29543363	8.72386163	7.63759697
5.0e-04	21.54025883	15.75016244	12.45663696	10.37720983	8.96369706
2.5e-04	27.38835483	19.42726023	15.02471930	12.30370363	10.48468814
1.0e-04	37.51249228	25.54987978	19.17742238	15.34858365	12.84494145

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	1.00567205	0.99437785	0.98880465	0.98548374	0.98327918
2.5e-01	1.34938782	1.29325208	1.26486821	1.24766903	1.23611197
1.0e-01	1.76666715	1.63773739	1.57411084	1.53606637	1.51071818
5.0e-02	2.08245372	1.88735998	1.79293703	1.73707961	1.70011670
2.5e-02	2.40756156	2.13587873	2.00686832	1.93134276	1.88169626
1.0e-02	2.85936326	2.46892098	2.28799771	2.18348503	2.11536434
5.0e-03	3.22202662	2.72722133	2.50204003	2.37322904	2.28979076
2.5e-03	3.60569918	2.99280567	2.71878741	2.56353532	2.46357768
1.0e-03	4.14932842	3.35720403	3.01112992	2.81747058	2.69375743
5.0e-04	4.59121578	3.64430450	3.23767025	3.01221135	2.86901815
2.5e-04	5.06216840	3.94231956	3.46956151	3.20981715	3.04579029
1.0e-04	5.73356202	4.35453286	3.78524812	3.47616883	3.28243796

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.98053383	0.97889204	0.97779972	0.97702056	0.97643676
2.5e-01	1.22154226	1.21272414	1.20680948	1.20256568	1.19937183
1.0e-01	1.47900944	1.45995201	1.44722598	1.43812297	1.43128751
5.0e-02	1.65416788	1.62670811	1.60843710	1.59540027	1.58562884
2.5e-02	1.82035531	1.78389832	1.75972535	1.74251901	1.72964530
1.0e-02	2.03184705	1.98255615	1.95001810	1.92692838	1.90969161
5.0e-03	2.18807100	2.12834311	2.08904248	2.06121608	2.04047731
2.5e-03	2.34238905	2.27158253	2.22513831	2.19232523	2.16790881
1.0e-03	2.54482910	2.45837068	2.40188843	2.36209449	2.33254399
5.0e-04	2.69754289	2.59846551	2.53393217	2.48855935	2.45491675
2.5e-04	2.85037950	2.73799694	2.66501148	2.61379954	2.57588349
1.0e-04	3.05318646	2.92213752	2.83735242	2.77801658	2.73416989

F-DISTRIBUTION CRITICAL POINTS: a = 30

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	2.14520766	1.40961842	1.23933110	1.16490819	1.12335242
2.5e-01	9.66983109	3.44283271	2.46502719	2.08245947	1.87836139
1.0e-01	62.26496980	9.45792727	5.16811133	3.81742177	3.17408429
5.0e-02	2.50095e+02	19.46241141	8.61657587	5.74587698	4.49571226
2.5e-02	1.00141e+03	39.46456625	14.08052337	8.46127401	6.22687890
1.0e-02	6.26065e+03	99.46583286	26.50453370	13.83766034	9.37932921
5.0e-03	2.50436e+04	1.99466e+02	42.46580027	19.89150273	12.65564022
2.5e-03	1.00176e+05	3.99466e+02	67.80092564	28.45012228	16.97525293
1.0e-03	6.26099e+05	9.99467e+02	1.25449e+02	45.42858708	24.86877338
5.0e-04	2.50440e+06	1.99947e+03	1.99526e+02	64.56144114	33.08572863
2.5e-04	1.00176e+07	3.99947e+03	3.17116e+02	91.61856372	43.92683276
1.0e-04	6.26099e+07	9.99947e+03	5.84689e+02	1.45303e+02	63.74585885

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.09686895	1.07852863	1.06508065	1.05479970	1.04668577
2.5e-01	1.75094669	1.66349489	1.59955178	1.55063861	1.51193650
1.0e-01	2.79996008	2.55545696	2.38301567	2.25472010	2.15542594
5.0e-02	3.80816427	3.37580750	3.07940647	2.86365234	2.69955123
2.5e-02	5.06522684	4.36239305	3.89401592	3.56041018	3.31101667
1.0e-02	7.22853306	5.99201017	5.19812955	4.64858167	4.24693282
5.0e-03	9.35824456	7.53448952	6.39608957	5.62479231	5.07055061
2.5e-03	12.03753876	9.41050535	7.81621347	6.75892865	6.01190033
1.0e-03	16.67221633	12.53035505	10.10870971	8.54755577	7.46879771
5.0e-04	21.24677984	15.49407863	12.22435530	10.16114237	8.75924138
2.5e-04	27.00880233	19.10507763	14.73828476	12.04128367	10.23931010
1.0e-04	36.98391799	25.11762059	18.80339757	15.01287450	12.53602802

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	1.01135853	1.00000000	0.99439586	0.99105682	0.98884039
2.5e-01	1.34012048	1.28231362	1.25288950	1.23497346	1.22289034
1.0e-01	1.73822300	1.60647898	1.54107579	1.50179744	1.47553934
5.0e-02	2.03908590	1.84087169	1.74443196	1.68715693	1.64914101
2.5e-02	2.34860243	2.07394375	1.94291600	1.86594022	1.81520244
1.0e-02	2.77848490	2.38596735	2.20338205	2.09759344	2.02847852
5.0e-03	3.12340916	2.62778092	2.40147877	2.27168522	2.18743372
2.5e-03	3.48821737	2.87628645	2.60193164	2.44613068	2.34563269
1.0e-03	4.00499480	3.21709032	2.87210868	2.67869478	2.55494430
5.0e-04	4.42497903	3.48549895	3.08135264	2.85691325	2.71417308
2.5e-04	4.87253242	3.76403651	3.29545036	3.03765509	2.87466892
1.0e-04	5.51049470	4.14920845	3.58679472	3.28114342	3.08938437

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.98608042	0.98442997	0.98333194	0.98254871	0.98196188
2.5e-01	1.20759805	1.19830587	1.19205583	1.18756209	1.18417471
1.0e-01	1.44257530	1.42269116	1.40937888	1.39983827	1.39266359
5.0e-02	1.60173018	1.57330235	1.55434260	1.54079053	1.53061905
2.5e-02	1.75232953	1.71484888	1.68994363	1.67218749	1.65888581
1.0e-02	1.94352574	1.89325403	1.86000528	1.83637749	1.81871940
5.0e-03	2.08448664	2.02389230	1.98395244	1.95563653	1.93451146
2.5e-03	2.22353845	2.15204758	2.10508073	2.07185899	2.04711554
1.0e-03	2.40570884	2.31890870	2.26212523	2.22207733	2.19231388
5.0e-04	2.54297523	2.44389183	2.37927487	2.33380046	2.30005756
2.5e-04	2.68023522	2.56824836	2.49543971	2.44430865	2.40642722
1.0e-04	2.86221521	2.73219511	2.64799688	2.58902905	2.54542924

F-DISTRIBUTION CRITICAL POINTS: a = 40

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	2.15835716	1.41783944	1.24645148	1.17155577	1.12973891
2.5e-01	9.71439991	3.45111943	2.46737850	2.08212919	1.87634959
1.0e-01	62.52905173	9.46624353	5.15971942	3.80361474	3.15732383
5.0e-02	2.51143e+02	19.47073643	8.59441125	5.71699841	4.46379332
2.5e-02	1.00560e+03	39.47289548	14.03650907	8.41113239	6.17504970
1.0e-02	6.28678e+03	99.47416457	26.41081269	13.74537889	9.29118878
5.0e-03	2.51482e+04	1.99475e+02	42.30821029	19.75175318	12.52973500
2.5e-03	1.00594e+05	3.99475e+02	67.54195230	28.24325473	16.79954169
1.0e-03	6.28712e+05	9.99475e+02	1.24959e+02	45.08856129	24.60203096
5.0e-04	2.51485e+06	1.99947e+03	1.98740e+02	64.07135554	32.72421595
2.5e-04	1.00594e+07	3.99947e+03	3.15859e+02	90.91626499	43.44027869
1.0e-04	6.28712e+07	9.99947e+03	5.82362e+02	1.44179e+02	63.03070563

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.10309039	1.08463654	1.07110580	1.06076191	1.05259854
2.5e-01	1.74770998	1.65929078	1.59454555	1.54494590	1.50564321
1.0e-01	2.78116855	2.53509606	2.36136151	2.23195816	2.13169107
5.0e-02	3.77428628	3.34042965	3.04277782	2.82593265	2.66085521
2.5e-02	5.01247138	4.30887603	3.83978009	3.50547390	3.25539606
1.0e-02	7.14322190	5.90844856	5.11561040	4.56664872	4.16528690
5.0e-03	9.24084802	7.42244649	6.28753818	5.51858171	4.96593627
2.5e-03	11.87975594	9.26379556	7.67677000	6.62447554	5.88099192
1.0e-03	16.44454850	12.32596245	9.91935909	8.36851686	7.29714326
5.0e-04	20.95012006	15.23487207	11.98892540	9.94185450	8.55146554
2.5e-04	26.62523332	18.77907418	14.44808606	11.77508002	9.99008393
1.0e-04	36.44988310	24.68038192	18.42461746	14.67249719	12.22245393

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	1.01705989	1.00563572	1.00000000	0.99664249	0.99441395
2.5e-01	1.33011684	1.27034666	1.23965639	1.22084527	1.20809252
1.0e-01	1.70833404	1.57322807	1.50562495	1.46477897	1.43734220
5.0e-02	1.99381910	1.79179012	1.69279721	1.63368179	1.59427252
2.5e-02	2.28732204	2.00887239	1.87519738	1.79627499	1.74404643
1.0e-02	2.69474863	2.29921107	2.11423245	2.00659151	1.93601847
5.0e-03	3.02153176	2.52405683	2.29583947	2.16443676	2.07886767
2.5e-03	3.36706964	2.75501343	2.47947464	2.32245554	2.22087819
1.0e-03	3.85644435	3.07160874	2.72681593	2.53293187	2.40856709
5.0e-04	4.25409280	3.32086129	2.91826661	2.69410306	2.55121557
2.5e-04	4.67779678	3.57945381	3.11407870	2.85746643	2.69490137
1.0e-04	5.28170037	3.93695677	3.38043310	3.07742226	2.88699609

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.99163913	0.98997992	0.98887610	0.98808877	0.98749889
2.5e-01	1.19186135	1.18194035	1.17523889	1.17040504	1.16675207
1.0e-01	1.40271812	1.38171781	1.36760234	1.35745560	1.34980706
5.0e-02	1.54488737	1.51512527	1.49520239	1.48092200	1.47018028
2.5e-02	1.67904043	1.64010656	1.61414731	1.59559158	1.58166246
1.0e-02	1.84893238	1.79718143	1.76284878	1.73839325	1.72008259
5.0e-03	1.97393454	1.91193188	1.87094714	1.84182694	1.82006429
2.5e-03	2.09706735	2.02431292	1.97638990	1.94242311	1.91708413
1.0e-03	2.25815582	2.17039339	2.11284420	2.07218191	2.04191789
5.0e-04	2.37939044	2.27965443	2.21446979	2.16851820	2.13437524
2.5e-04	2.50051004	2.38825032	2.31511784	2.26367916	2.22552230
1.0e-04	2.66094326	2.53126014	2.44713035	2.38812814	2.34445402

F-DISTRIBUTION CRITICAL POINTS: a = 50

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	2.16627138	1.42278746	1.25073673	1.17555615	1.13358184
2.5e-01	9.74122985	3.45609785	2.46876769	2.08188582	1.87507430
1.0e-01	62.68805152	9.47123563	5.15461710	3.79521572	3.14710846
5.0e-02	2.51774e+02	19.47573259	8.58099627	5.69949150	4.44440562
2.5e-02	1.00812e+03	39.47789358	14.00991032	8.38078179	6.14362199
1.0e-02	6.30252e+03	99.47916381	26.35422509	13.68957976	9.23781079
5.0e-03	2.52111e+04	1.99480e+02	42.21309095	19.66729029	12.45353177
2.5e-03	1.00845e+05	3.99480e+02	67.38566970	28.11826383	16.69323758
1.0e-03	6.30285e+05	9.99480e+02	1.24664e+02	44.88316662	24.44071345
5.0e-04	2.52114e+06	1.99948e+03	1.98266e+02	63.77535227	32.50562664
2.5e-04	1.00846e+07	3.99948e+03	3.15101e+02	90.49212434	43.14612482
1.0e-04	6.30286e+07	9.99948e+03	5.80957e+02	1.43501e+02	62.59840642

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.10683372	1.08831130	1.07473052	1.06434855	1.05615524
2.5e-01	1.74567671	1.65665391	1.59140420	1.54136948	1.50168336
1.0e-01	2.76969078	2.52263291	2.34807862	2.21796691	2.11707253
5.0e-02	3.75366766	3.31885564	3.02039779	2.80284252	2.63712400
2.5e-02	4.98042430	4.27630794	3.80671620	3.47192500	3.22137191
1.0e-02	7.09147513	5.85768204	5.06539774	4.51671488	4.11545174
5.0e-03	9.16969105	7.35443440	6.22154873	5.45392128	4.90215635
2.5e-03	11.78417000	9.17479633	7.59206288	6.54268861	5.80125397
1.0e-03	16.30669370	12.20204658	9.80441825	8.25969892	7.19268336
5.0e-04	20.77053841	15.07777858	11.84607279	9.80863848	8.42509383
2.5e-04	26.39308947	18.58155169	14.27206021	11.61342708	9.83857068
1.0e-04	36.12673975	24.41553594	18.19494071	14.46588870	12.03191436

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	1.02048813	1.00902388	1.00336882	1.00000000	0.99776406
2.5e-01	1.32371476	1.26258666	1.23098991	1.21152129	1.19826669
1.0e-01	1.68961627	1.55215449	1.48295531	1.44094223	1.41261016
5.0e-02	1.96562794	1.76087918	1.66000315	1.59949547	1.55901109
2.5e-02	2.24929323	1.96806083	1.83238327	1.75195331	1.69854843
1.0e-02	2.64295447	2.24501192	2.05811339	1.94896422	1.87718703
5.0e-03	2.95863421	2.45940486	2.22950985	2.09670801	2.00998851
2.5e-03	3.29238900	2.67956602	2.40274963	2.24453383	2.14192219
1.0e-03	3.76502228	2.98128846	2.63599870	2.44133040	2.31617984
5.0e-04	4.14903380	3.21878441	2.81648357	2.59196007	2.44854722
2.5e-04	4.55818313	3.46514503	3.00103714	2.74458928	2.58182275
1.0e-04	5.14131006	3.80569095	3.25201945	2.95002601	2.75992642

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.99498016	0.99331561	0.99220828	0.99141847	0.99082672
2.5e-01	1.18131629	1.17090261	1.16384166	1.15873359	1.15486442
1.0e-01	1.37669713	1.35481044	1.34004665	1.32940458	1.32136505
5.0e-02	1.50806918	1.47723132	1.45651930	1.44163480	1.43041556
2.5e-02	1.63182403	1.59169408	1.56485365	1.54562145	1.53115653
1.0e-02	1.78830874	1.73529179	1.70001848	1.67483680	1.65594874
5.0e-03	1.90330402	1.84004290	1.79811410	1.76826067	1.74591253
2.5e-03	2.01647988	1.94252302	1.89368557	1.85900249	1.83308820
1.0e-03	2.16441384	2.07559441	2.01721800	1.97589647	1.94509686
5.0e-04	2.27566532	2.17503497	2.10912415	2.06258162	2.02795205
2.5e-04	2.38674909	2.27380246	2.20007440	2.14813408	2.10955531
1.0e-04	2.53380492	2.40377756	2.31926960	2.25991546	2.21592858

F-DISTRIBUTION CRITICAL POINTS: a = 60

Q	b = 1	b = 2	b = 3	b = 4	b = 5
5.0e-01	2.17155767	1.42609255	1.25359903	1.17822805	1.13614847
2.5e-01	9.75915313	3.45941947	2.46968475	2.08170443	1.87419501
1.0e-01	62.79427909	9.47456467	5.15118732	3.78956776	3.14023037
5.0e-02	2.52196e+02	19.47906383	8.57200411	5.68774413	4.43137970
2.5e-02	1.00980e+03	39.48122588	13.99209792	8.36043560	6.12252939
1.0e-02	6.31303e+03	99.48249674	26.31635086	13.65219819	9.20201493
5.0e-03	2.52531e+04	1.99483e+02	42.14944049	19.61072212	12.40244815
2.5e-03	1.01014e+05	3.99483e+02	67.28110372	28.03456810	16.62199408
1.0e-03	6.31337e+05	9.99483e+02	1.24466e+02	44.74565296	24.33262606
5.0e-04	2.52535e+06	1.99948e+03	1.97948e+02	63.57719044	32.35918338
2.5e-04	1.01014e+07	3.99948e+03	3.14594e+02	90.20819490	42.94907506
1.0e-04	6.31337e+07	9.99948e+03	5.80018e+02	1.43047e+02	62.30883943

Q	b = 6	b = 7	b = 8	b = 9	b = 10
5.0e-01	1.10933374	1.09076542	1.07715113	1.06674365	1.05853027
2.5e-01	1.74428199	1.65484654	1.58925011	1.53891482	1.49896250
1.0e-01	2.76195183	2.51421757	2.33909707	2.20849322	2.10716066
5.0e-02	3.73979661	3.30432288	3.00530258	2.78724856	2.62107716
2.5e-02	4.95889050	4.25439800	3.78444641	3.44930217	3.19840228
1.0e-02	7.05673682	5.82356564	5.03161771	4.48308696	4.08185527
5.0e-03	9.12194435	7.30875321	6.17718234	5.41040561	4.85919129
2.5e-03	11.72005267	9.11504287	7.53513871	6.48767608	5.74757030
1.0e-03	16.21425209	12.11888270	9.72721238	8.18654339	7.12239765
5.0e-04	20.65013650	14.97237128	11.75014418	9.71910873	8.34009506
2.5e-04	26.23746721	18.44904008	14.15388043	11.50481361	9.73669143
1.0e-04	35.91014207	24.23789034	18.04077529	14.32710736	11.90383375

Q	b = 20	b = 30	b = 40	b = 50	b = 60
5.0e-01	1.02277686	1.01128556	1.00561743	1.00224095	1.00000000
2.5e-01	1.31926210	1.25713641	1.22485624	1.20488187	1.19123501
1.0e-01	1.67677574	1.53756915	1.46715671	1.42423845	1.39520076
5.0e-02	1.94635792	1.73957362	1.63725182	1.57565390	1.53431418
2.5e-02	2.22335877	1.94000817	1.80277040	1.72114372	1.66679076
1.0e-02	2.60770829	2.20785394	2.01941122	1.90903161	1.83625936
5.0e-03	2.91588468	2.41514804	2.18384488	2.04986359	1.96216624
2.5e-03	3.24168199	2.62798479	2.35000444	2.19072479	2.08719612
1.0e-03	3.70301571	2.91962539	2.57366634	2.37818674	2.25226555
5.0e-04	4.07782709	3.14915760	2.74669792	2.52163119	2.37760845
2.5e-04	4.47716004	3.38723666	2.92360437	2.66694972	2.50377773
1.0e-04	5.04627801	3.71630742	3.16415232	2.86250583	2.67233953

Q	b = 80	b = 100	b = 120	b = 140	b = 160
5.0e-01	0.99720996	0.99554180	0.99443209	0.99364060	0.99304762
2.5e-01	1.17371223	1.16289894	1.15554227	1.15020619	1.14615567
1.0e-01	1.35825336	1.33564171	1.32034017	1.30928278	1.30091255
5.0e-02	1.48211138	1.45038565	1.42901318	1.41361762	1.40199091
2.5e-02	1.59865713	1.55752813	1.52994156	1.51013047	1.49520313
1.0e-02	1.74587702	1.69177963	1.65569320	1.62987752	1.61048127
5.0e-03	1.85397504	1.78961570	1.74685364	1.71634711	1.69347358
2.5e-03	1.96030034	1.88526226	1.83559532	1.80025758	1.77381416
1.0e-03	2.09919942	2.00937192	1.95020548	1.90825232	1.87693774
5.0e-04	2.20360424	2.10205829	2.03541225	1.98827352	1.95315333
2.5e-04	2.30781263	2.19407424	2.11968666	2.06720054	2.02816680
1.0e-04	2.44571379	2.31510618	2.23007026	2.17025933	2.12588137