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Curvature Relations in 3-D Symmetric Axes

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CURVATURE RELATIONS IN THREE-DIMENSIONAL SYMMETRIC AXES

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ABSTRACT

Aspects of symmetric axis geometry in three dimensions are discussed. A notion of radius curvature is defined and a relationship between symmetric axis curvature, radius curvature, and boundary curvature is derived.

1. Introduction

Shape is a concept of fundamental importance in many disciplines. Still, it remains difficult to define and even more difficult to measure. Blum[1, 2, 3, 4] has introduced a transformation, variously known as the symmetric axis transform (SAT), medial axis transform, or skeleton, that induces a unique, coordinate system-independent decomposition of a figure into simpler figures. Consequently, the divide-and-conquer strategy can be applied to describing figure shape: divide the figure into several smaller figures, describe each of them, then combine the results to yield a single description.

In [4], Blum and Nagel propose an elegant methodology for applying this strategy to describing figures bounded by piecewise smooth, simple closed curves in the continuous plane. Extending their methodology to three-dimensional figures appears very attractive. To do so, it is necessary to generalize to three dimensions the mathematical tools used by Blum and Nagel in two dimensions. Hence, in this paper, we develop the local differential geometry of the symmetric axis in three dimensions.

We begin by reviewing briefly several important properties of two-dimensional symmetric axes. Let an *outline* in R^2 be a smooth, simple closed curve. A *figure* is an outline together with its interior. The *symmetric axis* of a figure F is the locus of centers of all maximal discs of F, i.e., those discs contained in F but in no other disc in F. Equivalently, if C is the curve that bounds F, the symmetric axis, SA(C), is the set of points in F having at least two nearest neighbors on C.

The points of SA(C) can be classified into three types[4] depending on the *order* of the point, the number of disjoint connected subsets of C comprising its set of nearest neighbors. *End* points are of order one, *normal* points of order two, and *branch* points of order three or more, corresponding to maximal disc touchings in one, two, or more disjoint arcs respectively. Additionally, points are called *point contact* if each touching subset is a single point and *finite contact* otherwise. Under the assumptions made here, SA(C) is the union of simple arcs, each a sequence of normal points bounded at each end by a branch or end point, that intersect each other only at branch points[4]. See Figure 1.

Let τ be the mapping from C onto SA(C) that maps a point P_C in C to the center of the maximal disc tangent to C at P_C. With each contiguous interval of normal points, which Blum and Nagel call *simplified segments*, the inverse relation τ^{-1} associates two disjoint arcs of C. Consequently, F can be decomposed into the collection of two-sided parts associated with the simplified segments of SA(C) together with the collection of the (possibly degenerate) circular arcs associated with branch and end points. To describe the connection structure of

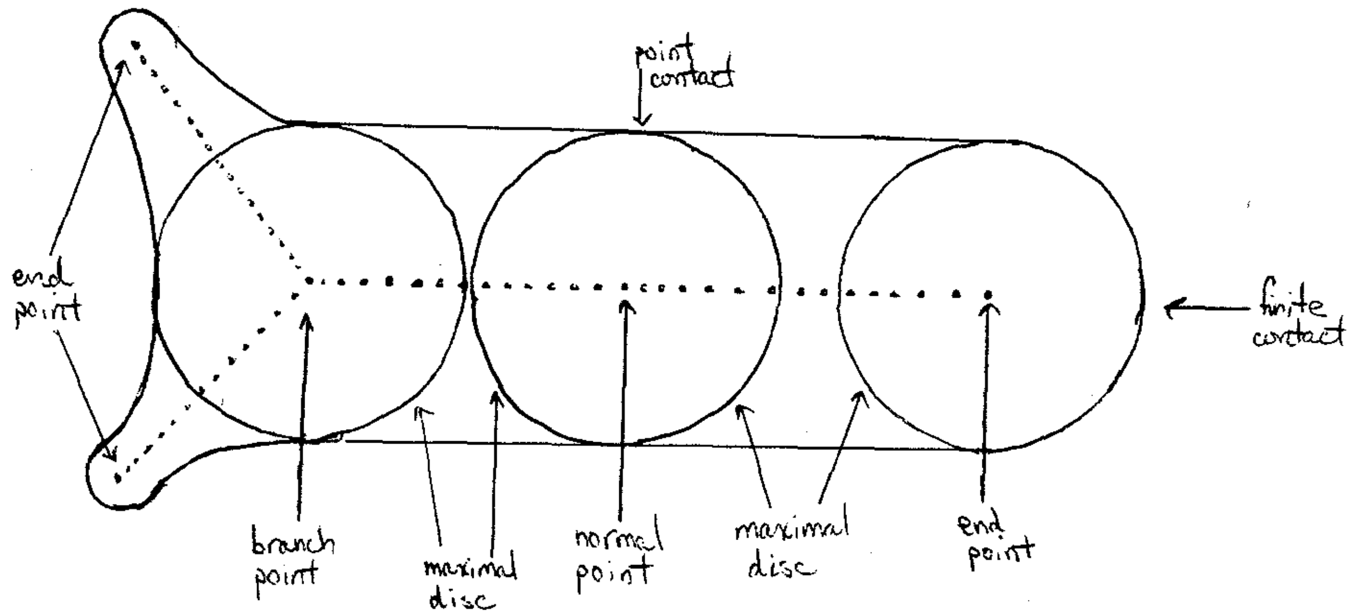

Figure 1: Symmetric Axis Point Types

the decomposition, Blum and Nagel[4] define a labeled graph with a node for each branch and end point. Other structures are possible; the choice depends on the goals of the analysis.

Choose a direction of traversing a simplified segment and call the two associated arcs of C the *left* and *right boundary* arcs. The angle between the tangent to C at a point P_C and the tangent to SA(C) at $\tau(P_C)$ is called the *object angle*, and is shown by Blum and Nagel to be the arcsin of the first derivative of the disc radius at $\tau(P_C)$ with respect to axis arc length. See Figure 2. The algebraic signs of the object angle and its derivative, the *object curvature*, partition the segment into *width shapes* juxtaposed one after the other.

As the axis becomes curved, the width shapes remain unchanged since they are a function of disc radius only. However, the associated boundary arcs change from convex, to straight, to concave in a manner depending on the axis curvature. Indeed, Blum and Nagel[4] give an explicit functional relationship among axis curvature, the boundary arc curvatures, object curvature, and object angle. It is that relationship that this paper generalizes to three

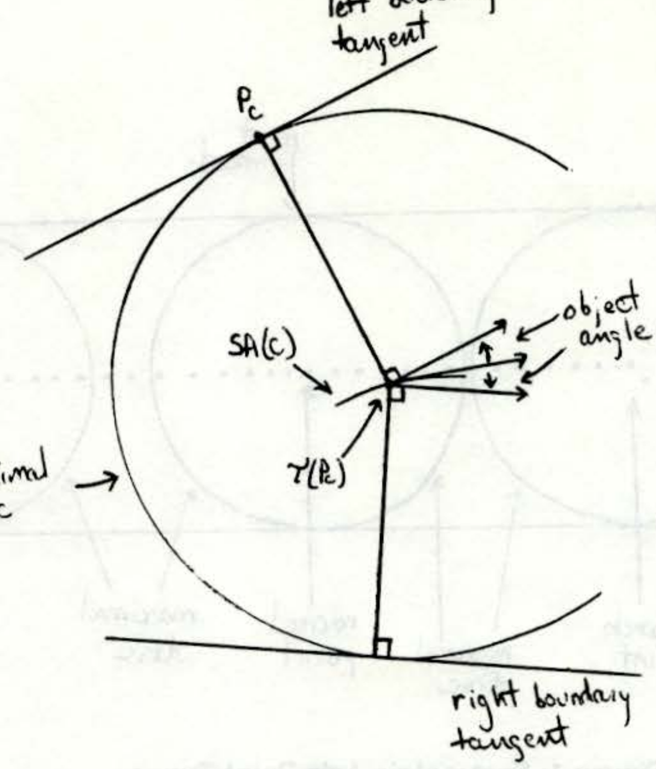

Figure 2: Normal Point Geometry (Point Contact)

dimensions.

The next section of this paper defines the SAT in three-dimensions and a notion of radius curvature. The following section examines the relationship between radius curvature, boundary curvature, and axis curvature. The paper concludes with a brief discussion of the intuitive meanings of radius and axis curvatures in the context of shape description and of other important problems requiring solution before Blum's transform can be applied to shape description in three dimensions.

2. The SAT in Three-dimensions

In R^3, an *outline* becomes a smooth, closed surface with no self-intersections, and maximal discs become maximal spheres. In general, the symmetric axis is a surface rather than a curve, though it sometimes degenerates into a space curve. Connected sets of normal points, again called *simplified segments*, are bounded by possibly degenerate space curves of branch and end points. As before, the figure can be decomposed into a collection of two-sided parts associated with simplified segments together with pieces of canal surfaces[1] associated with branch and end point curves[6]. Here, we consider the analysis of simplified segments and their associated boundary surfaces.

We first impose local curvilinear coordinate systems about normal points on simplified segments, thus bringing the techniques of calculus to bear. Let S be a simplified segment surface in $\mathbf{R}^3$. Except at finite contact normal points, which we ignore hereafter, S is a C^2 surface. Hence if we let U be an open subset of $\mathbf{R}^2$ with coordinates u^1 and u^2, we can let $\mathbf{s}: U \to S$ be a C^2 surface patch on S with linearly independent partial derivatives $\frac{\partial \mathbf{s}}{\partial u^i}$ denoted by $\mathbf{s}_i$. The tangent plane to S at $\mathbf{s}(u^1,u^2)$ is a two-dimensional subspace of $\mathbf{R}^3$ spanned by the *coordinate vectors* $\mathbf{s}_1$ and $\mathbf{s}_2$. Consequently, the unit normal at $\mathbf{s}(u^1,u^2)$, $\mathbf{n}_s(u^1,u^2)$, is $\frac{\mathbf{s}_1 \times \mathbf{s}_2}{|\mathbf{s}_1 \times \mathbf{s}_2|}$. Similarly, let B and C be the boundary surfaces associated with S as shown in Figure 3, let $\mathbf{b}(u^1,u^2)$ and $\mathbf{c}(u^1,u^2)$ be the points on B and C associated with $\mathbf{s}(u^1,u^2)$, and let $r: S \to \mathbf{R}^1$ map a point on S to the radius of the maximal sphere centered at that point. Finally, to distinguish between a vector, $\mathbf{X}$, and the n-tuple that represents it with respect to some basis, we denote the n-tuple by X.

In two dimensions, width shapes result from analyzing disc radius as a function of a single parameter, arc length along the symmetric axis. Unfortunately, in three dimensions no single parameter suffices. Instead, we examine the first and second derivatives of the radius function along curves in infinitely many directions through the point $P = \mathbf{s}(0,0)$, in much the same way a surface $z = f(x,y)$ is described by examining derivatives of f along lines in the (x,y) plane (directional derivatives).

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[1]A canal surface is the envelope of a family of spheres, possibly of varying radius, with centers lying on a space curve[5].

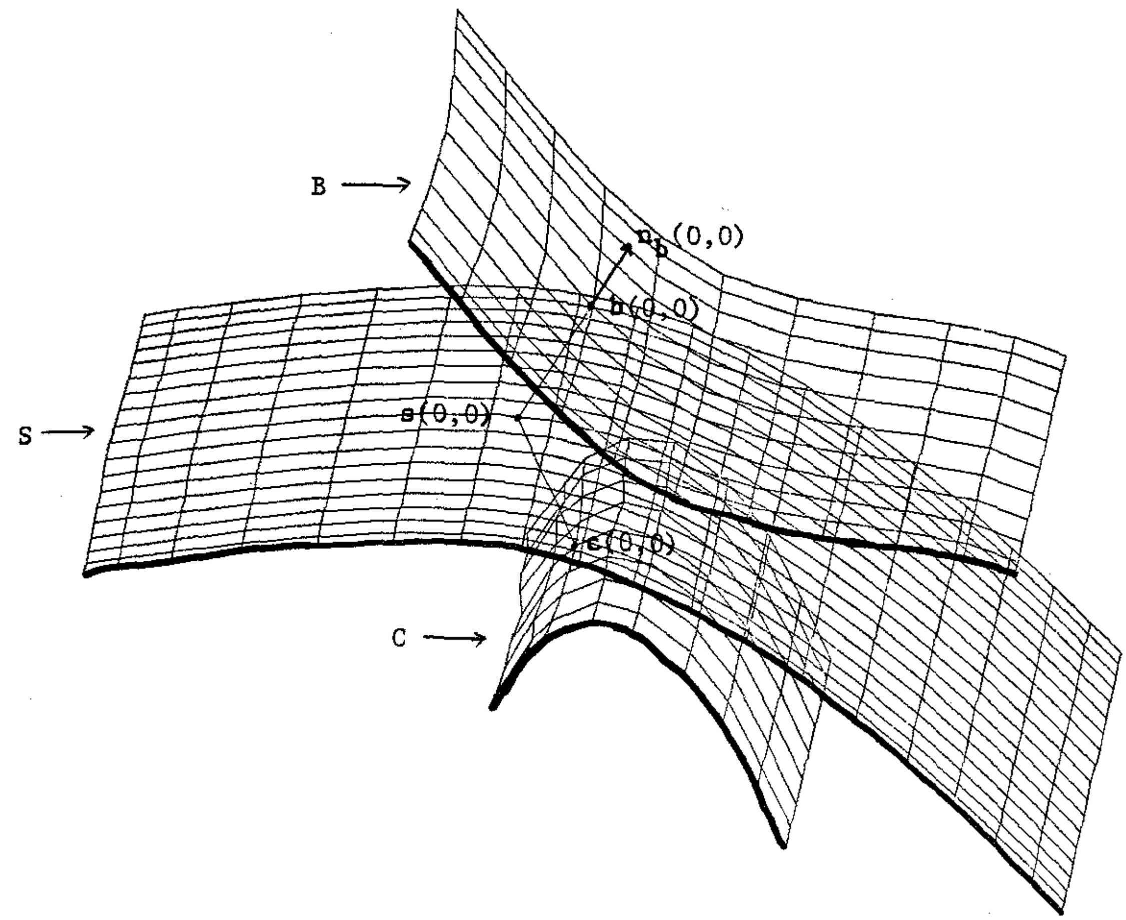

Figure 3: 3D SAT Geometry

Choose a curve on S passing through P, $\alpha(t): R^1 \to S$, where t is arc length along the curve and $\alpha(0) = P$. Let **X** be the tangent vector of α at P, $\frac{d\alpha}{dt}(0)$. Since α is parameterized by arc length and lies on S, **X** is a unit vector in the plane T_PS, the tangent plane of S at P. We define the *directional derivative of r in the* **X** *direction* to be $r_X = \frac{dr(\alpha)}{dt}(0)$. That r_X is well defined is shown by the following result[cf. 7, sec. 4-7]:

Lemma 1: r_X is independent of the choice of the curve α such that $X = \frac{d\alpha}{dt}(0)$.

Proof: For two scalar functions of t, α^1 and α^2, $\boldsymbol{\alpha}(t) = \mathbf{s}(\alpha^1(t), \alpha^2(t))$. Since T_PS is a vector space spanned by $\mathbf{s}_1$ and $\mathbf{s}_2$, there are scalars X^i such that $\mathbf{X} = \sum_{i=1}^{2} X^i \mathbf{s}_i$. Using the chain rule, $\frac{d\boldsymbol{\alpha}}{dt} = \sum_{i=1}^{2} \frac{d\alpha^i}{dt}\mathbf{s}_i$, so $\frac{d\alpha^i}{dt}(0) = X^i$. Applying the chain rule again,

$$\frac{dr(\boldsymbol{\alpha})}{dt}(t) = \sum_{i=1}^{2} \frac{\partial r(\mathbf{s})}{\partial u^i}\frac{d\alpha^i}{dt}. \tag{1}$$

Therefore $r_{\mathbf{X}} = \sum_{i=1}^{2}(\frac{\partial r(\mathbf{s})}{\partial u^i}X^i)(0,0)$, which is independent of the choice of $\boldsymbol{\alpha}$.

Similarly, the *second directional derivative of r in the* **X** *direction* is $r_{\mathbf{XX}} = \frac{d^2 r(\boldsymbol{\alpha})}{dt^2}(0)$. Differentiating (1) and substituting X^i for $\frac{d\alpha^i}{dt}$,

$$\frac{d^2 r(\boldsymbol{\alpha})}{dt^2}(t) = \sum_{i=1}^{2} \frac{\partial r(\mathbf{s})}{\partial u^i}\frac{d^2\alpha^i}{dt^2} + \sum_{i=1}^{2}\sum_{j=1}^{2} X^i X^j \frac{\partial^2 r(\mathbf{s})}{\partial u^i \partial u^j}. \tag{2}$$

Unlike $r_{\mathbf{X}}$, $r_{\mathbf{XX}}$ is not well defined without imposing an additional constraint on $\boldsymbol{\alpha}$. We would like $\boldsymbol{\alpha}$ to be straight in a small neighborhood of P, or more precisely, we require that in an infinitesimal neighborhood about P, the orthogonal projection of $\boldsymbol{\alpha}$ onto T_PS be a line in the **X** direction. There exists a unique curve with tangent vector **X** satisfying this condition, called a geodesic, such that $\boldsymbol{\alpha}(0) = P$ (cf. [7], sec. 4-5). The curve $\boldsymbol{\alpha}$ is characterized by the differential equations

$$\frac{d^2\alpha^k}{dt^2} = -\sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma_{ij}^k \frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}, \quad k = 1,2, \tag{3}$$

where the Γ_{ij}^k are the Christoffel symbols of the second kind of S [7,8], which measure the tangential components of the second partial derivatives $\mathbf{s}_{ij}$.

Combining (2) and (3), denoting $\frac{\partial r(\mathbf{s})}{\partial u^i}$ by r_i and $\frac{\partial^2 r(\mathbf{s})}{\partial u^i \partial u^j}$ by r_{ij}, and rearranging terms, we see that since $r_{ij} = r_{ji}$ and $\Gamma_{ij}^k = \Gamma_{ji}^k$, $r_{\mathbf{XX}}$ is a quadratic form in **X**:

$$r_{\mathbf{XX}} = Q(\mathbf{X}) = X^T Q X, \text{ with}$$

$$Q = [q_{ij}] = [r_{ij} - \sum_{k=1}^{2} r_k \Gamma_{ij}^k]. \tag{4}$$

For any unit vector **X** in $T_P S$, the second directional derivative of r in the direction defined by **X** is given by $Q(\mathbf{X})$.

Similarly, information about the curvature of S, i.e. the deviation of S from $T_P S$, is obtained by studying the curvature of curves on S through P. Consider the *normal sections at* P, those curves defined by the intersection of S with planes containing the normal to S at P. The *second fundamental form* of S, $\mathrm{II}(\mathbf{X})$, is a quadratic form over unit vectors in $T_P S$, with magnitude equal to the magnitude of the plane curvature of the normal section in the direction **X**, positive if the normal section lies on the side of $T_P S$ toward which the normal points and negative otherwise[7, 8]. Letting $L_s = [L_{s_{ij}}]$ be the matrix defining the second fundamental form with respect to the $\{\mathbf{s}_1, \mathbf{s}_2\}$ basis of $T_P S$, we have[7, 8]

$$\mathrm{II}(\mathbf{X}) = X^T L_s X. \tag{5}$$

Thus, two quadratic forms are defined at each point of S. One, the second fundamental form, gives the curvature of normal sections through the point in any direction, while the other gives the second derivative of the radius along the geodesic in the same direction. Since the normal to a geodesic is everywhere normal to the surface on which it lies, the geodesic and the normal section share a common normal vector. By construction, they have the same tangent vector and hence, the same curvature (cf. [8], sec IV-12). Therefore, one quadratic form measures the curvature of S along geodesics and the other measures the radius function second derivative along the same geodesics. Below, we see that the maximum and minimum values assumed by these forms yield much qualitative information about the behavior of both the symmetric surface and the radius function.

First, though, it is necessary to digress briefly to discuss inner products over vectors in $\mathbf{R}^3$. An inner product over $\mathbf{R}^3$ is defined by any symmetric, positive definite bilinear form. Choose a set of basis vectors for $\mathbf{R}^3$ and let $\mathbf{Y}$ and $\mathbf{Z}$ be two vectors represented in terms of that basis. Then, an inner product of $\mathbf{Y}$ and $\mathbf{Z}$, denoted $<\mathbf{Y},\mathbf{Z}>$, is given by $Y^T GZ$, where G is a 3 by 3 matrix such that $<\mathbf{Y},\mathbf{Z}> = <\mathbf{Z},\mathbf{Y}>$ and $<\mathbf{Y},\mathbf{Y}> > 0$ for all non-zero $\mathbf{Y}$. For the remainder of this paper, we will use the particular inner product defined by $G = I$ (the identity matrix) when the basis vectors are orthonormal. This is nothing more than the dot product, $Y^T Z$, often used in $\mathbf{R}^3$.

Consider the inner product of two vectors, $\mathbf{V}$ and $\mathbf{W}$, contained in the tangent plane T_PS to S at the point P. Since T_PS is spanned by $\mathbf{s}_1$ and $\mathbf{s}_2$, there are scalars V^i and W^i such that $\mathbf{V} = V^1\mathbf{s}_1+V^2\mathbf{s}_2$ and $\mathbf{W} = W^1\mathbf{s}_1+W^2\mathbf{s}_2$. By the bilinearity of inner products, $<\mathbf{V},\mathbf{W}> = \sum_{i=1}^{2}\sum_{j=1}^{2} V^i W^j <\mathbf{s}_i,\mathbf{s}_j>$, which can be written in matrix form as $V^T G_s W$, where $G_s = [g_{s_{ij}}] = [g_{s_{ji}}] = <\mathbf{s}_i,\mathbf{s}_j>$. Thus the bilinear form $V^T G_s W$, called the *first fundamental form of S*, is the inner product of $\mathbf{R}^3$ restricted to the two-dimensional subspace T_PS and expressed with respect to the basis $\{\mathbf{s}_1,\mathbf{s}_2\}$. Though the representation of the inner product depends on the basis vectors chosen, the inner product itself is basis independent. Hence we use $< >$ to denote the inner product of two vectors, regardless of the basis used to represent them.

Returning to the task of characterizing the behavior of the radius function, we seek the minimum and maximum values $Q(\mathbf{X})$ assumes over all unit vectors $\mathbf{X}$ in T_PS. Let Q' be the linear transformation such that $<Q'\mathbf{X},\mathbf{X}> = Q(\mathbf{X})$. Since both G_s and Q are symmetric, $Q' = G_s^{-1}Q$. Over all unit vectors $\mathbf{X}$ in T_PS, $Q(\mathbf{X})$ assumes its minimum value at the eigenvector of Q' corresponding to the smallest eigenvalue, γ_1 and its maximum value at the eigenvector corresponding to the largest eigenvalue, γ_2. Further, the values assumed are γ_1 and γ_2

respectively and the eigenvectors are orthogonal if the eigenvalues are distinct (cf. [9], sec. 2-17). By solving the characteristic equation of Q', it is easy to see that $\gamma_1\gamma_2 = \det(Q')$ and $\gamma_1+\gamma_2 = \mathrm{tr}(Q')$.

Similarly, over unit tangent vectors $\mathbf{X}$, $\mathrm{II}(\mathbf{X})$ assumes its maximum and minimum values, called *principal curvatures*, at the eigenvectors of $G_s^{-1}L_s$ with the extremal values being the associated eigenvalues λ_1 and λ_2. When the eigenvalues are distinct, the eigenvectors define two orthogonal directions called the *principal directions*. The product of the eigenvalues is K_S, the *Gaussian curvature* of the symmetric surface S, while their average is its *mean curvature*, H_S. Conversely, $\lambda_1 = H_S-\sqrt{H_S{}^2-K_S}$ and $\lambda_2 = H_S+\sqrt{H_S{}^2-K_S}$. As above, solving the characteristic equation of $G_s^{-1}L_s$ shows that $H_S = ½\mathrm{tr}(G_s^{-1}L_s)$ and $K_S = \det(G_s^{-1}L_s)$. Stretching the terminology, we define the *Gaussian* and *mean curvatures of the radius function* to be $K_R = \det(G_s^{-1}Q)$ and $H_R = ½\mathrm{tr}(G_s^{-1}Q)$ respectively.

The behavior of S at a point is characterized by the signs of the Gaussian and mean curvatures. For $K_S > 0$, all normal sections lie on one side of the tangent plane, the choice determined by the sign of the mean curvature. The surface is cup-shaped at the point. On the other hand, for $K_S < 0$ the normal sections about one principal direction lie above the tangent plane and those about the other lie below, giving S a saddle shape at the point. The remaining case, $K_S = 0$, is a transition between the two: in one principal direction the surface has flattened while in the other it remains curved. When both principal curvatures are zero, S is planar at the point and the principal directions cease to exist. For a fascinating discussion of this and other interpretations of both Gaussian and mean curvature see, Ch. IV of [5].

An analogous, though less geometric characterization of radius function behavior at a point on S is obtained by labeling the quadratic form Q as positive-definite, negative-definite, positive or negative semi-definite, or identically zero. Also note that eigenvalues and eigenvectors, and hence principal curvatures and directions, are invariant under change of basis. Consequently, the curvatures are independent of the choice of the parameterization $\mathbf{s}$ of S.

3. Boundary and Symmetric Surface Curvature Relations

3.1. Matrix Formulation

In the previous section we introduced notions of radius function curvature closely analogous to Gaussian and mean curvatures of the symmetric surface. In this section, we derive expressions for the curvatures of the associated boundary surfaces in terms of the symmetric surface curvatures and the radius function and its directional derivatives. We begin by deriving an equation relating the matrices that determine radius and symmetric surface curvature, Q and L_s, to the matrix defining the second fundamental form, and hence the curvature, of each boundary surface.

The maximal sphere centered at $\mathbf{s}(u^1,u^2)$ is tangent to the boundary surface B at $\mathbf{b}(u^1,u^2)$ with the boundary normal, $\mathbf{n}_b(u^1,u^2)$, lying along a radius of the sphere. Letting $r(u^1,u^2)$ denote $r(\mathbf{s}(u^1,u^2))$,

$$\mathbf{b}(u^1,u^2) = \mathbf{s}(u^1,u^2) \pm r(u^1,u^2)\mathbf{n}_b(u^1,u^2),$$

with the choice of sign determined by the direction of $\mathbf{n}_b$. Since $\mathbf{n}_b$ itself is determined only up to sign, choose $\mathbf{n}_b$ pointing away from S as shown in figure 3, giving

$$\mathbf{b}(u^1,u^2) = \mathbf{s}(u^1,u^2) + r(u^1,u^2)\mathbf{n}_b(u^1,u^2). \tag{6}$$

Similarly,

$$\mathbf{c}(u^1,u^2) = \mathbf{s}(u^1,u^2) + r(u^1,u^2)\mathbf{n}_c(u^1,u^2). \tag{7}$$

Dropping explicit mention of (u^1,u^2) and taking partial derivatives,

$$\mathbf{b}_i = \mathbf{s}_i + r_i \mathbf{n}_b + r\,\mathbf{n}_{b_i} \tag{8}$$

We can solve for r_i by taking the inner product of both sides of (8) with $\mathbf{n}_b$. Since $\mathbf{n}_b$ is a vector of constant magnitude, it is perpendicular to its derivative, $\mathbf{n}_{b_i}$. Thus, since $\mathbf{b}_i$ is perpendicular to $\mathbf{n}_b$ by definition,

$$r_i = -<\mathbf{s}_i, \mathbf{n}_b>. \tag{9}$$

Taking partial derivatives again,

$$r_{ij} = -<\mathbf{s}_{ij}, \mathbf{n}_b> - <\mathbf{s}_i, \mathbf{n}_{b_j}>.$$

Using Gauss's formulas[7, 8], $\mathbf{s}_{ij} = L_{s_{ij}} \mathbf{n}_s + \sum_{k=1}^{2} \Gamma_{ij}^k \mathbf{s}_k$, and the definition of the coefficients of the second fundamental form[7, 8], $L_{s_{ij}} = <\mathbf{s}_{ij}, \mathbf{n}_s>$, we obtain

$$r_{ij} = -L_{s_{ij}} <\mathbf{n}_s, \mathbf{n}_b> - \sum_{k=1}^{2} \Gamma_{ij}^k <\mathbf{s}_k, \mathbf{n}_b> - <\mathbf{s}_i, \mathbf{n}_{b_j}>. \tag{10}$$

Analogous results for boundary surface C follow from (7), though for brevity we defer further consideration of C until the end of this section.

Define the matrices $G_b = [G_{b_{ij}}]$ and $L_b = [L_{b_{ij}}]$ representing the first and second fundamental forms of B at $\mathbf{b}(u^1, u^2)$ with respect to the $\{\mathbf{b}_1(u^1, u^2), \mathbf{b}_2(u^1, u^2)\}$ basis of the tangent plane at $\mathbf{b}(u^1, u^2)$. Since $\mathbf{n}_b$ is a vector of constant magnitude, the $\mathbf{n}_{b_j}$ are perpendicular to it. Hence, they lie in the tangent plane and are expressed as a linear combination of the $\mathbf{b}_i$ by *Weingarten's equations*

$$\mathbf{n}_{b_j} = -\sum_{i=1}^{2} W_{b\,j}^{\,i} \mathbf{b}_i, \tag{11}$$

where $W_b = [W_{b\,j}^{\,i}] = G_b^{-1} L_b$, and is called the *Weingarten map* of B [7,8]. Letting $A = [<\mathbf{s}_i, \mathbf{b}_j>]$ and combining Weingarten's equations with (4), (9), and (10),

$$AW_b = [r_{ij}]+<\mathbf{n}_s,\mathbf{n}_b>L_s-\sum_{k=1}^{2} r_k\Gamma^k$$

$$= Q+<\mathbf{n}_s,\mathbf{n}_b>L_s. \tag{12}$$

Equation (12) relates boundary curvature, as expressed by W_b, to radius curvature, as expressed by Q, and to symmetric surface curvature, as expressed by L_s. We seek a better understanding of this relationship. Our approach is to solve for the two invariants of the matrix equation (12), the determinant and trace. We then solve the resulting two equations simultaneously for the boundary curvatures in terms of properties of the radius and symmetric surface.

3.2. Determinant Equations

Substitute Weingarten's equations into (8) and solve for the $\mathbf{s}_i$, giving

$$\mathbf{s}_1 = (1+rW_{b\,1}^{\,1})\mathbf{b}_1+rW_{b\,2}^{\,1}\mathbf{b}_2-r_1\mathbf{n}_b, \text{ and} \tag{13}$$

$$\mathbf{s}_2 = rW_{b\,2}^{\,1}\mathbf{b}_1+(1+rW_{b\,2}^{\,2})\mathbf{b}_2-r_1\mathbf{n}_b. \tag{14}$$

Recalling that $A = [<\mathbf{s}_i,\mathbf{b}_j>]$, we use (13) and (14) to obtain $A = TG_b$ and consequently, since $W_b = G_b^{-1}L_b$, that $AW_b = TL_b$, where $T = \begin{bmatrix} 1+rW_{b\,1}^{\,1} & rW_{b\,2}^{\,1} \\ rW_{b\,2}^{\,1} & 1+rW_{b\,2}^{\,2} \end{bmatrix}$. Left multiplying (12) by G_s^{-1}, then gives

$$G_s^{-1}TL_b = G_s^{-1}Q+<\mathbf{n}_s,\mathbf{n}_b>G_s^{-1}L_s. \tag{15}$$

To evaluate the determinant of the left side of (15), we use two intermediate results:

Lemma 2: $<\mathbf{n}_s,\mathbf{n}_b>^2 = 1-[r_1\ r_2]G_s^{-1}[r_1\ r_2]^T$.

Proof: Let x^1, x^2, and x^3 be scalars such that $\mathbf{n}_b = x^1\mathbf{s}_1+x^2\mathbf{s}_2+x^3\mathbf{n}_s$. Recalling that $G_s = [<\mathbf{s}_i,\mathbf{s}_j>]$ and taking the inner product of $\mathbf{n}_b$ with itself,

$$1 = \langle \mathbf{n}_b, \mathbf{n}_b \rangle$$

$$= [x^1\, x^2] G_s [x^1\, x^2]^T + (x^3)^2.$$

Taking inner products of $\mathbf{n}_b$ with $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{n}_s$ produces $[\langle \mathbf{n}_b, \mathbf{s}_1 \rangle\ \langle \mathbf{n}_b, \mathbf{s}_2 \rangle]^T = G_s [x^1\, x^2]^T$, and $\langle \mathbf{n}_s, \mathbf{n}_b \rangle = x^3$. Using (9), $G_s [x^1\, x^2]^T = -[r_1\, r_2]^T$. Hence, $\langle \mathbf{n}_s, \mathbf{n}_b \rangle^2 = 1 - [r_1\, r_2] G_s^{-1} [r_1\, r_2]^T$.

Lemma 3: Letting $g_s = \det(G_s)$ and $g_b = \det(G_b)$, $g_b \det^2(T) = g_s \langle \mathbf{n}_s, \mathbf{n}_b \rangle^2$.

Proof: By substituting (13) and (14) into $G_s = [\langle \mathbf{s}_i, \mathbf{s}_j \rangle]$, it is not difficult to show that

$$g_b \det^2(T) T^{-1} G_b^{-1} T^{-1} = g_s G_s^{-1} - R, \tag{16}$$

where $R = \begin{bmatrix} r_2^2 & -r_1 r_2 \\ -r_1 r_2 & r_1^2 \end{bmatrix}$. Taking the determinant of both sides and applying lemma 2,

$$g_b \det^2(T) = g_s - g_s [r_1\, r_2] G_s^{-1} [r_1\, r_2]^T$$

$$= g_s \langle \mathbf{n}_s, \mathbf{n}_b \rangle^2.$$

Thus the determinant of the left side of (15) is

$$\det(G_s^{-1} T L_b) = \frac{1}{g_s} \det(T) \det(L_b)$$

$$= \frac{\langle \mathbf{n}_s, \mathbf{n}_b \rangle^2 \det(G_b^{-1} L_b)}{\det(T)}$$

$$= \frac{\langle \mathbf{n}_s, \mathbf{n}_b \rangle^2 K_B}{\det(T)}, \tag{17}$$

where K_B is the Gaussian curvature of B.

We now evaluate the determinant of the right side of (15). Recalling that the determinant is invariant under change of basis, we change from the $\{\mathbf{s}_1, \mathbf{s}_2\}$ basis of TpS to that defined by the eigenvectors of $G_s^{-1} L_s$. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be eigenvectors of $G_s^{-1} L_s$ corresponding to the eigenvalues λ_1 and λ_2 respectively. Since eigenvectors are determined only up to a non-zero multiplicative constant and since $\mathbf{e}_1$ and $\mathbf{e}_2$ lie in the tangent plane T_pS and are orthogonal to each other, we can, without loss of generality, choose the $\mathbf{e}_i$ to be unit vectors so that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{n}_s$. Similarly, let $\mathbf{f}_1$ and $\mathbf{f}_2$ be unit eigenvectors of $G_s^{-1}Q$ corresponding to the eigenvalues γ_1 and γ_2 so that $\mathbf{f}_1 \times \mathbf{f}_2 = \mathbf{n}_s$. In terms of their respective eigenvector bases, the transformations represented by $G_s^{-1}L_s$ and $G_s^{-1}Q$ in terms of the $\{\mathbf{s}_1, \mathbf{s}_2\}$ basis, are represented by $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$, i.e. $G_s^{-1}L_s \approx \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $G_s^{-1}Q \approx \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$, where $\approx$ denotes matrix similarity.

Representing both transformations in terms of the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis requires examining the relationship between the $\mathbf{e}_i$ and the $\mathbf{f}_i$. Let ϑ be the counter-clockwise angle from $\mathbf{e}_1$ to $\mathbf{f}_1$. Then, with respect to the $\{\mathbf{f}_1, \mathbf{f}_2\}$ basis, $\mathbf{e}_i = \Theta \mathbf{f}_i$, where $\Theta = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}$. As shown in figure 4, ϑ is determined only up to a multiple of π; thus, Θ is determined only up to sign. Changing from the $\{\mathbf{f}_1, \mathbf{f}_2\}$ basis to the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis, $\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \approx \pm\Theta^{-1} \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} (\pm\Theta) = \Theta^T \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \Theta$. Therefore, $G_s^{-1}Q + \langle \mathbf{n}_s, \mathbf{n}_b \rangle G_s^{-1}L_s$ is similar to $\Theta^T \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \Theta + \langle \mathbf{n}_s, \mathbf{n}_b \rangle \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, which is easily seen to have a determinant of

$$\langle \mathbf{n}_s, \mathbf{n}_b \rangle^2 \lambda_1\lambda_2 + \gamma_1\gamma_2 + \langle \mathbf{n}_s, \mathbf{n}_b \rangle (\lambda_1\gamma_1 + \lambda_2\gamma_2 - (\gamma_1 - \gamma_2)(\lambda_1 - \lambda_2)\cos^2\vartheta) \tag{18}$$

Note that (18) is independent of ϑ if either $\gamma_1 = \gamma_2$ or $\lambda_1 = \lambda_2$. Consequently, when either pair of eigenvalues fail to be distinct and the principal directions are not well-defined, arbitrary directions can be chosen.

Combining (17) and (18) and rearranging terms,

$$\frac{K_B}{\det(T)} = \lambda_1\lambda_2 + \frac{\gamma_1\gamma_2}{\langle \mathbf{n}_s, \mathbf{n}_b \rangle^2} + \frac{\lambda_1(\gamma_1 \sin^2\vartheta + \gamma_2 \cos^2\vartheta) + \lambda_2(\gamma_1 \cos^2\vartheta + \gamma_2 \sin^2\vartheta)}{\langle \mathbf{n}_s, \mathbf{n}_b \rangle}. \tag{19}$$

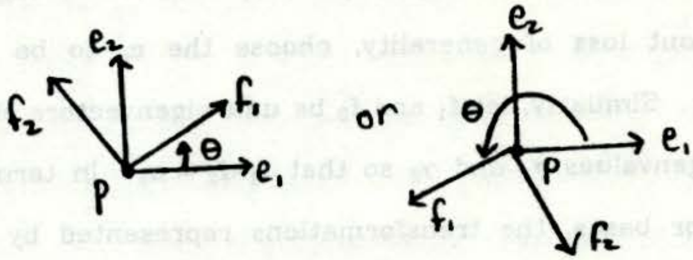

Figure 4: Relation between principal directions

Equation (19) can be simplified by using a theorem due to Euler[7], given here in the context of the radius function.

Lemma 4: Let $\mathbf{X}$ be a unit vector in T_PS. Then $Q(\mathbf{X}) = r_{\mathbf{XX}} = \gamma_1\cos^2\varphi+\gamma_2\sin^2\varphi$, where φ is the angle between $\mathbf{X}$ and $\mathbf{f}_1$.

Proof: With respect to the $\{\mathbf{f}_1,\mathbf{f}_2\}$ basis, $\mathbf{X}$ is represented by $[\cos\varphi \ \sin\varphi]^T$ and Q by $\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}$. Hence

$$Q(\mathbf{X}) = [\cos\varphi \ \sin\varphi]\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}[\cos\varphi \ \sin\varphi]^T = \gamma_1\cos^2\varphi+\gamma_2\sin^2\varphi.$$

Recalling that $\mathbf{e}_1 = \mathbf{f}_1\cos\vartheta+\mathbf{f}_2\sin\vartheta$ and $\mathbf{e}_2 = -\mathbf{f}_1\sin\vartheta+\mathbf{f}_2\cos\vartheta$ and applying lemma 4 twice, (19) becomes

$$\frac{K_B}{\det(T)} = \lambda_1\lambda_2+\frac{\gamma_1\gamma_2+\lambda_1 r_{\mathbf{e}_2\mathbf{e}_2}+\lambda_2 r_{\mathbf{e}_1\mathbf{e}_1}}{<\mathbf{n}_s,\mathbf{n}_b>^2}. \tag{20}$$

3.3. Trace Equations

The second equation relating boundary curvature to radius and symmetric surface curvature results from taking the trace of (12). Recalling that $A = TG_b$, it follows from (12) and (16) that

$$g_b \det^2(T) T^{-1} W_b = g_s G_s^{-1}(Q + \langle \mathbf{n}_s, \mathbf{n}_b \rangle L_s) - R(Q + \langle \mathbf{n}_s, \mathbf{n}_b \rangle L_s),$$

and hence, after taking the trace of both sides, that

$$2g_b(rK_B + H_B)\det(T) = 2g_s(H_R + \langle \mathbf{n}_s, \mathbf{n}_b \rangle H_S) - \mathrm{tr}(RQ) - \langle \mathbf{n}_s, \mathbf{n}_b \rangle \mathrm{tr}(RL_s). \quad (21)$$

Two observations enable us to evaluate $\mathrm{tr}(RL_s)$ and, by analogous reasoning, $\mathrm{tr}(RQ)$. First, simple algebra reveals that $\mathrm{tr}(RL_s)$ is nothing more than the second fundamental form of S, evaluated at $[r_2\ -r_1]$, $[r_2\ -r_1]L_s[r_2\ -r_1]^T$. Second, with respect to the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis, the second fundamental form is represented by the diagonal matrix $\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$. Hence, letting $[a^1\ a^2]$ represent, with respect to $\{\mathbf{e}_1, \mathbf{e}_2\}$, the vector represented by $[r_2\ -r_1]$ in the $\{\mathbf{s}_1, \mathbf{s}_2\}$ basis, $\mathrm{tr}(RL_s) = [a^1\ a^2]\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}[a^1\ a^2]^T$.

Let V be the matrix of transition from the $\{\mathbf{s}_1, \mathbf{s}_2\}$ basis to the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis[9], i.e. the matrix such that $[r_2\ -r_1]^T = V[a^1\ a^2]^T$. Since the columns of V are the coordinates of the $\mathbf{e}_i$ in the $\{\mathbf{s}_1, \mathbf{s}_2\}$ basis, and since the $\mathbf{e}_i$ are orthonormal, $V^T G_s V = I$, where I is the two-by-two identity matrix. Thus, $\det(V) = \frac{\pm 1}{\sqrt{g_s}}$ which is non-zero. Therefore, we can solve for $[a^1\, a^2]$ obtaining $\pm\sqrt{g_s}[r_1 V_{12} + r_2 V_{22}\ -r_1 V_{11} - r_2 V_{21}]$. Since by the definition of V, $\mathbf{e}_i = \sum_{j=1}^{2} V_{ji}\mathbf{s}_i$, by using (9) we see that $[a^1\, a^2] = \pm\sqrt{g_s}[-\langle \mathbf{n}_b, \mathbf{e}_2 \rangle\ \langle \mathbf{n}_b, \mathbf{e}_1 \rangle]$ and hence, that $\mathrm{tr}(RL_s) = g_s(\lambda_1 \langle \mathbf{n}_b, \mathbf{e}_2 \rangle^2 + \lambda_2 \langle \mathbf{n}_b, \mathbf{e}_1 \rangle^2)$. Analogously, $\mathrm{tr}(RQ) = g_s(\gamma_1 \langle \mathbf{n}_b, \mathbf{f}_2 \rangle^2 + \gamma_2 \langle \mathbf{n}_b, \mathbf{f}_1 \rangle^2)$. Finally, combining these results with (21), lemma 3, and the definition of mean curvature as the average of principal curvatures, we obtain

$$2\langle \mathbf{n}_s, \mathbf{n}_b \rangle^2 \frac{rK_B + H_B}{\det(T)} = \gamma_1(1 - \langle \mathbf{n}_b, \mathbf{f}_2 \rangle^2) + \gamma_2(1 - \langle \mathbf{n}_b, \mathbf{f}_1 \rangle^2) + \langle \mathbf{n}_s, \mathbf{n}_b \rangle(\lambda_1(1 - \langle \mathbf{n}_b, \mathbf{e}_2 \rangle^2) + \lambda_2(1 - \langle \mathbf{n}_b, \mathbf{e}_1 \rangle^2)) \quad (22)$$

To simplify (22) we show that inner products between the boundary normal, n_b, and unit vectors in the symmetric surface tangent plane are simply directional derivatives of the radius function.

Lemma 5: Let $\mathbf{X}$ be a unit vector in T_pS. Then the directional derivative of r in the $\mathbf{X}$ direction, $r_{\mathbf{X}}$, is $-<\mathbf{n}_b, \mathbf{X}>$.

Proof: Let X^1 and X^2 be the components of X in the $\{s_1, s_2\}$ basis, i.e. $\mathbf{X} = \sum_{i=1}^{2} X^i \mathbf{s}_i$. So, $<\mathbf{n}_b, \mathbf{X}> = \sum_{i=1}^{2} X^i <\mathbf{n}_b, \mathbf{s}_i>$ which, by (9), is $-\sum_{i=1}^{2} X^i r_i$. Thus, by the proof of lemma 1, $r_{\mathbf{X}} = -<\mathbf{n}_b, \mathbf{X}>$.

Therefore (22) can be rewritten as

$$\frac{rK_B + H_B}{\det(T)} = \frac{\gamma_1(1-r_{\mathbf{f}_2}^2)+\gamma_2(1-r_{\mathbf{f}_1}^2)}{2<\mathbf{n}_s, \mathbf{n}_b>^2} + \frac{\lambda_1(1-r_{\mathbf{e}_2}^2)+\lambda_2(1-r_{\mathbf{e}_1}^2)}{2<\mathbf{n}_s, \mathbf{n}_b>}. \tag{23}$$

3.4. Boundary Curvature Equations

Call the right sides of equations (20) and (23) k and h respectively:

$$k = \lambda_1\lambda_2 + \frac{\gamma_1\gamma_2 + \lambda_1 r_{\mathbf{e}_2\mathbf{e}_2} + \lambda_2 r_{\mathbf{e}_1\mathbf{e}_1}}{<\mathbf{n}_s, \mathbf{n}_b>^2}, \text{ and} \tag{24}$$

$$h = \frac{\gamma_1(1-r_{\mathbf{f}_2}^2)+\gamma_2(1-r_{\mathbf{f}_1}^2)}{2<\mathbf{n}_s, \mathbf{n}_b>^2} + \frac{\lambda_1(1-r_{\mathbf{e}_2}^2)+\lambda_2(1-r_{\mathbf{e}_1}^2)}{2<\mathbf{n}_s, \mathbf{n}_b>}. \tag{25}$$

Recall that $K_B = \det(W_b)$ and $H_B = \operatorname{tr}(W_b)$. Then, by straightfoward algebra, $\det(T) = 1 + r^2 K_B + 2rH_B$. Substituting into (20) and (23), we obtain a linear system of two equations in the two unknowns H_B and K_B with solutions

$$H_B = \frac{h - rk}{1 - 2rh + r^2 k} \tag{26}$$

and

$$K_B = \frac{k}{1 - 2rh + r^2 k}. \tag{27}$$

These equations give the Gaussian and mean curvatures of the boundary surface, B, in terms of properties of the radius and symmetric surface, together with the angle between the boundary normal, $\mathbf{n}_b$, and the symmetric surface normal, $\mathbf{n}_s$. Analogous equations for boundary surface C are obtained when the qualifying subscripts b and B are replaced by c and C respectively.

At first glance, it appears that knowledge of the boundary normal is prerequisite to evaluating h and k, and hence the boundary curvatures. This is not the case. Since $\mathbf{n}_s$, $\mathbf{e}_1$, and $\mathbf{e}_2$ are orthonormal, $<\mathbf{n}_s,\mathbf{n}_b>^2+<\mathbf{n}_b,\mathbf{e}_1>^2+<\mathbf{n}_b,\mathbf{e}_2>^2 = 1$. Hence, up to sign, $<\mathbf{n}_s,\mathbf{n}_b>$ is determined by $r_{\mathbf{e}_1}$ and $r_{\mathbf{e}_2}$.

Choosing the sign of $<\mathbf{n}_s,\mathbf{n}_b>$ chooses either boundary surface B or C. As symmetry suggests, and application of lemma 5 proves, $\mathbf{n}_b$ and $\mathbf{n}_c$ are reflections of each other through the symmetric surface tangent plane. Thus, by symmetry about the tangent plane, $<\mathbf{n}_s,\mathbf{n}_b> = <\mathbf{n}_c,-\mathbf{n}_s>$ and hence $<\mathbf{n}_s,\mathbf{n}_b> = -<\mathbf{n}_c,\mathbf{n}_s>$. Consequently, if we replace $<\mathbf{n}_s,\mathbf{n}_b>$ by $\pm<\mathbf{n}_s,\mathbf{n}_b>$ in (24) and (25), the curvature relations hold for either boundary, the choice determined by the sign.

4. Discussion

To understand the geometric significance of h and k, consider the surface B' defined by

$$\mathbf{b}'(u^1,u^2) = \mathbf{s}(u^1,u^2)+r'(u^1,u^2)\mathbf{n}_b(u^1,u^2),$$

where $r'(u^1,u^2) = r(u^1,u^2)-r(0,0)$. It is not difficult to see that B' passes through the point $P = \mathbf{s}(0,0)$ and at each (u^1,u^2) has the same unit normal vector as does B. B' and B are called *parallel surfaces*. See Figure 5. Since the derivatives of r' and r are identical, we can evaluate (26) and (27) substituting r' for r, obtaining $k = K_{B'}$, and $h = H_{B'}$. Thus, the terms h and k in (26) and (27) are the mean and Gaussian curvatures, respectively, of the surface parallel to B passing through P. Therefore, (26) and (27) express the change in boundary curvature due to change in distance from the symmetric surface. Blum and Nagel[4] use a similar relationship in the two-dimensional case to derive boundary curvature from parallel curve curvature. Analogous results hold for the surface parallel to C through P when the sign of $<\mathbf{n}_s, \mathbf{n}_b>$ is changed.

Though the symmetric surface and radius function together contain no information not contained in the boundary surfaces, examining each alone reveals different aspects of the shape of the boundary surface. Intuitively, sym-

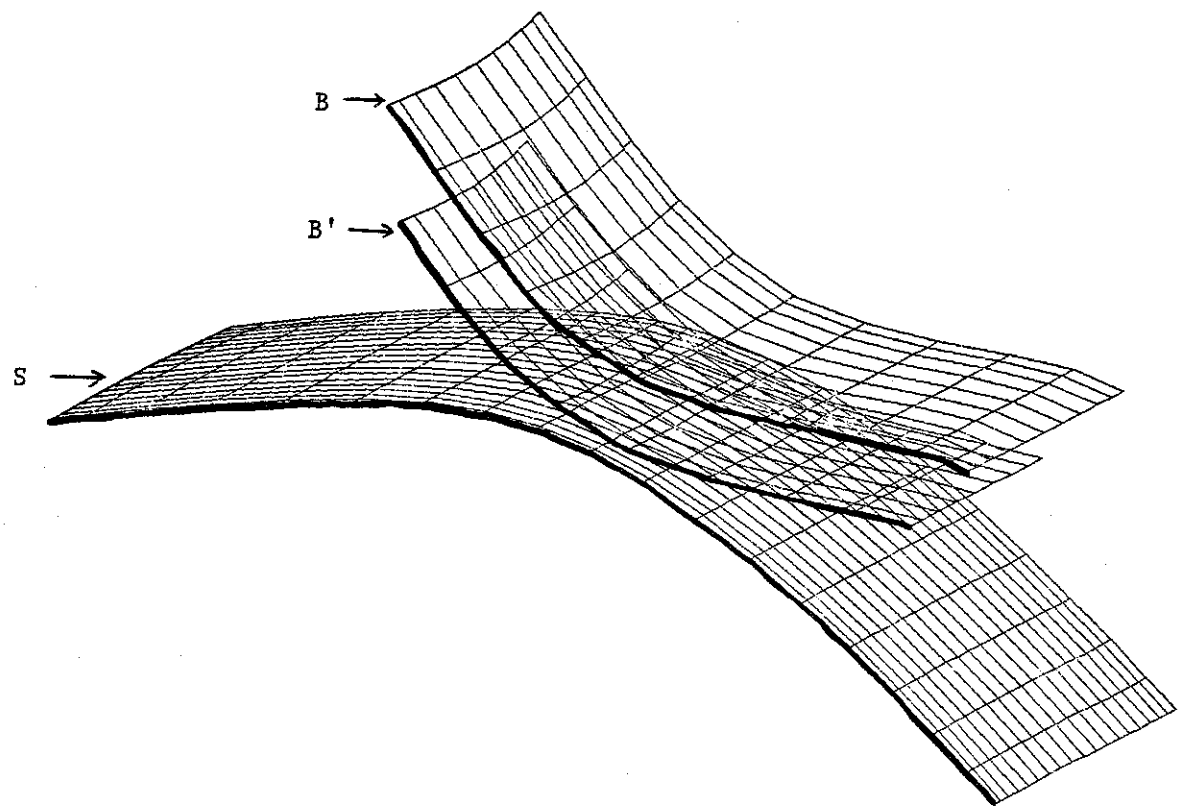

Figure 5: Surface Parallel to Boundary Surface

metric surface curvature reflects the overall "curvature trend" of the two-sided piece, i.e. the degree to which both boundary surfaces curve in the same direction. Radius curvature, on the other hand, reflects the symmetry of the boundary surfaces about the symmetric surface, i.e. the degree to which both boundary surfaces curve in opposite directions.

To see this, observe in (25) that symmetric surface curvatures λ_1 and λ_2 contribute with equal magnitude but opposite sign to the mean curvature of the two boundary surface parallels, while radius curvatures γ_1 and γ_2 contribute equally to each. Since the boundary surface normals each point away from the symmetric surface, boundary surface mean curvatures of opposite sign imply curvature in the same direction. Further, it can be shown that the signs of the Gaussian and mean curvatures of each boundary surface are the same as the signs of the curvatures of the corresponding parallel surface. Hence, our intuitive notions of the meanings of symmetric surface curvature and radius curvature are confirmed.

5. Summary and Conclusions

Blum's symmetric axis transform defines a unique decompostion of a figure into disjoint, two-sided pieces, each with its own surface (axis) of symmetry and associated boundary surfaces. In previous sections of this paper, we have defined measures of the radius function and have shown how these measures and the symmetric surface curvatures are related to the boundary surface curvatures. In particular, we have shown that the Gaussian and mean curvatures of the boundary surfaces are determined by nine measures, each with a geometric interpretation:

(1) the symmetric surface curvature as determined by two principal curvatures and a principal direction;

(2) the radius curvature as determined by two principal curvatures and a principal direction;

(3) directional derivatives of the radius function as determined by the angles between either boundary normal and the two symmetric surface principal directions, called *width angles* after Blum[2]; and

(4) the radius function itself.

Other, equivalent sets of measures are easily found. It can also be shown that these measures, and the curvature relationship derived from them, subsume the two-dimensional measures and curvature relationship given by Blum and Nagel[4].

It appears possible to use the measures defined here to further partition a simplified segment into a set of canonical two-sided pieces, yielding a symbolic description, much as Blum and Nagel have done in two-dimensions. We refrain from doing so until a suitable algorithm for computing symmetric surfaces of three-dimensional figures is developed. At that time it will be possible to evaluate which of several possible schemes is suitable for specific three-dimensional shape description problems. Indeed, the purpose of this paper is not to propose specific features for three-dimensional shape description, but rather to provide mathematical tools for further study of Blum's transform in three dimensions.

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