Dissipation Bounds: Recovering from Overload

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Abstract

The MC^2 mixed-criticality framework has been previously proposed for mixing safety-critical hard real-time (HRT) and mission-critical soft real-time (SRT) software on the same multicore computer. This paper focuses on the execution of SRT software within this framework. When determining SRT guarantees, jobs are provisioned based on a provisioned worst-case execution time (PWCET) that is not very pessimistic and could be overrun. In this paper, we propose a mechanism to recover from the overload created by such overruns. We propose a modification to the previously-proposed G-EDF-like (GEL) class of schedulers that uses virtual time to increase task periods. We then show how to compute *dissipation bounds* that indicate how long it takes to return to normal behavior after a transient overload.

1 Introduction

Future cyber-physical systems will require mixing tasks of varying importance. For example, future unmanned aerial vehicles (UAVs) will require more stringent timing requirements for adjusting flight surfaces than for long-term decision-making (Herman et al., 2012). The *mixed criticality* (*MC*) framework MC^2 has been previously proposed in order to allow these workloads to be simultaneously supported on a single multicore machine (Herman et al., 2012; Mollison et al., 2010). Using a single machine allows reductions in size, weight, and power.

In any mixed-criticality system, there are a number of *criticality levels*. For example, MC^2 has four criticality levels, denoted A (highest) through D (lowest). Each MC task is assigned a

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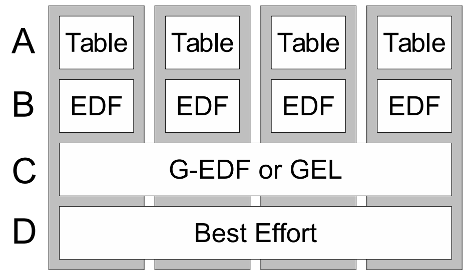


Figure 1: MC^2 architecture.

distinct criticality level. When analyzing a MC system, each task is assigned a separate *provisioned worst-case execution time* (*PWCET*) for each criticality level at or below its own criticality level. For example, under MC^2 a level-B task has PWCETs at levels B, C, and D. Guarantees are provided for level-ℓ tasks by assuming that no task with criticality at or above level ℓ exceeds its level-ℓ PWCET. For example, when analyzing level C, level-A, -B, and -C tasks are considered using level-C PWCETs.

As noted by Burns and Davis (2013), most proposed mixed-criticality frameworks do not provide any guarantees for level ℓ if any job exceeds its level-ℓ PWCET. This assumption could be highly problematic in practice. For example, suppose that a level-A flight-control job on a UAV exceeds its level-C PWCET. Then, no guarantees are provided for level-C mission control tasks from that point forward. The primary purpose of this paper is to provide guarantees in such situations.

Specifically, we consider response-time bounds for tasks at level C in MC^2. The architecture of MC^2 as described by Herman et al. (2012) is depicted in Figure 1: levels A and B are scheduled on a per-processor basis using table-driven and EDF scheduling, respectively. Level C was proposed by Mollison et al. (2010) to be scheduled using the global earliest-deadline-first (G-EDF) scheduler, but here we consider the more general class of *G-EDF-like* (*GEL*) schedulers that can yield better response-time bounds (Erickson et al., 2014; Leontyev et al., 2011). In prior work (Herman et al., 2012; Mollison et al., 2010), level C was analyzed using *restricted supply* analysis from Leontyev and Anderson (2010). “Restricted supply” indicates that some processors are not fully available to tasks at level C. MC^2 statically prioritizes tasks at levels A and B above those at level C, so execution at levels A and B can be considered as restricted supply when analyzing level C. We continue to use this general strategy, but reduce pessimism.

Contributions. We provide response-time bounds for MC^2 at level C, using arbitrary GEL schedulers. Our analysis is sufficiently general to account for level-A, -B, and -C jobs that overrun their level-C PWCETs.[1] When any job at or above level C overruns its level-C PWCET, the system at level C may be *overloaded*. As noted above, this can compromise level-C guarantees. Using the normal MC^2 framework, a task may have its per-job response times permanently increased as a result of even a single overload event, and multiple overload events could cause such increases to build up over time. For example, if a system is fully utilized, then there is no "slack" with which to recover from overload. Therefore, we must alter scheduling decisions to attempt to recover from transient overload conditions. We do so by scaling task inter-release times and modifying scheduling priorities. We also provide *dissipation bounds* on the time required for response-time bounds to settle back to normal.

Comparison to Related Work. Other techniques for managing overload have been provided in other settings, although most previously proposed techniques either focus exclusively on uniprocessors (Baruah et al., 1991; Beccari et al., 1999; Buttazzo and Stankovic, 1993; Koren and Shasha, 1992; Locke, 1986) or only provide heuristics without theoretical guarantees (Garyali, 2010).

Our paper uses the idea of "virtual time" from Zhang Zhang (1990) (as also used by Stoica et al. Stoica et al. (1996)), where job separation times are determined using a virtual clock that changes speeds with respect to the actual clock. In our work, we recover from overload by slowing down virtual time, effectively reducing the frequency of job releases. Unlike in Stoica et al. (1996), we never speed up virtual time relative to the normal underloaded system, so we avoid problems that have previously prevented virtual time from being used on a multiprocessor. To our knowledge, this work is the first to use virtual time in multiprocessor scheduling.

Some past work on recovering from PWCET overruns in mixed-criticality systems has used techniques similar to ours, albeit in the context of trying to meet all deadlines (Jan et al., 2013; Santy et al., 2012, 2013; Su and Zhu, 2013; Su et al., 2013). Our technique is also similar to reweighting techniques that modify task parameters such as periods. A detailed survey of several such techniques is provided by Block (2008). Dissipation bounds are a new contribution of our work.

[1] We note that MC^2 supports optional *budget enforcement* that ensures that tasks at level ℓ do not exceed their level-ℓ PWCETs. If this technique is used, then it can be guaranteed that no level-C task will overrun its level-C PWCET (though level-A and -B tasks by definition *can* overrun their level-C PWCETs). In this paper, we provide analysis for the more general case when budget enforcement is not assumed.

Organization. In Section 2, we describe the task model and scheduler, in addition to defining notation. Then, in Section 3, we show how to compute general response-time bounds. In Section 4, we leverage the results from Section 3 to show how to compute dissipation bounds, assuming that a transient overload has completed.

2 System Model

In this paper, we consider a generalized version of GEL scheduling, *GEL with virtual time* (*GEL-v*) scheduling, and a generalized version of the sporadic task model, called the *sporadic with virtual time and overload* (*SVO*) *model*. We assume that time is continuous.

In our analysis, we consider only the system at level C. In other words, we model level-A and -B tasks as supply that is unavailable to level C, rather than as explicit tasks. We consider a system $\tau = \{\tau_0, \tau_1, \ldots, \tau_{n-1}\}$ of n level-C tasks running on m processors $P = \{P_0, P_0, \ldots, P_{m-1}\}$. Each τ_i is composed of a (potentially infinite) series of jobs $\{\tau_{i,0}, \tau_{i,1}, \ldots\}$. The release time of $\tau_{i,k}$ is denoted as $r_{i,k}$. We assume that $\min_{\tau_i \in \tau} r_{i,0} = 0$. Each $\tau_{i,k}$ is prioritized on the basis of a *priority point* (*PP*), denoted $y_{i,k}$. The time when $\tau_{i,k}$ actually completes is denoted $t^c_{i,k}$. We define the following quantities that pertain to the execution time of $\tau_{i,k}$.

Definition 1. $e_{i,k}$ is the actual execution time of $\tau_{i,k}$.

Definition 2. $e^c_{i,k}(t)$ (*c*ompleted) is the amount of execution that $\tau_{i,k}$ completes before time t.

Definition 3. $e^r_{i,k}(t)$ (*r*emaining) is the amount of execution that $\tau_{i,k}$ compeletes after time t.

These quantities are related by the following property.

Property 1. *For arbitrary* $\tau_{i,k}$ *and time* t, $e^c_{i,k}(t) + e^r_{i,k}(t) = e_{i,k}$.

We also define what it means for $\tau_{i,k}$ to be "pending".

Definition 4. $\tau_{i,k}$ is defined to be *pending* at time t if $r_{i,k} \leq t \leq t^c_{i,k}$.

Under GEL scheduling and the conventional sporadic task model, each task is characterized by a per-job worst-case execution time (WCET) $C_i > 0$, a minimum separation $T_i > 0$ between releases, and a relative PP $Y_i \geq 0$. Using the above notation, the system is subject to the following constraints for every $\tau_{i,k}$:

$$e_{i,k} \leq C_i, \tag{1}$$

$$r_{i,k+1} \geq r_{i,k} + T_i, \tag{2}$$

$$y_{i,k} = r_{i,k} + Y_i. \tag{3}$$

Under the SVO model, we no longer assume a particular WCET (thus allowing overload). Therefore, (1) is no longer required to hold.

Under GEL-v scheduling and the SVO model, we use a notion of *virtual time* (as in Stoica et al. (1996)), and we define the minimum separation time and relative PP of a task with respect to virtual time after one of its job releases instead of actual time. The purpose of virtual time is depicted in Figure 2, which we now describe.

In Figure 2, we depict a system that only has level-A and level-C tasks, with one level-A task per CPU. For level-A tasks, we use the notation (T_i, C_i^C, C_i^A), where T_i is task τ_i's period, C_i^C is its level-C PWCET, and C_i^A is its level-A PWCET. For level-C tasks, we use the notation (T_i, Y_i, C_i), where all parameters are defined below. Using the analysis provided in this paper, response times for all jobs can be shown to be bounded in the absence of overload. However, even before the overload occurs at actual time 12, some jobs complete shortly after their PPs or after successor jobs are released.

Once an overload occurs, the system can respond by altering virtual time for level C. Virtual time is based on a global speed function $s(t)$. During normal operation of the system, $s(t)$ is always 1. This means that actual time and virtual time progress at the same rate. However, after an overload occurs, the scheduler may choose to select $0 < s(t) < 1$, at which point virtual time progresses more slowly than actual time. In Figure 2, the system chooses to use $s(t) = 0.5$ for $t \in [19, 29)$. As a result, virtual time progresses more slowly in this interval, and new releases of jobs are delayed. This allows the system to recover from the overload, so at actual time 29, $s(t)$ returns to 1. Observe that job response times are significantly increased after actual time 12 when the overload occurs, but after actual time 29, they are similar to before the overload. In fact, the arrival pattern of level A happens to result in better response times after recovery than before the overload, although this is not guaranteed under a sporadic release pattern.

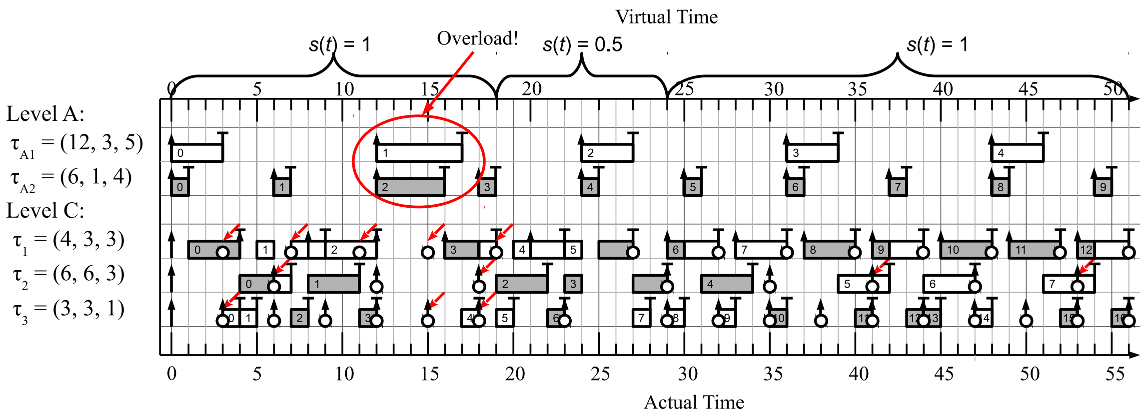


Figure 2: Example of a system that recovers after an overload.

An actual time t is converted to a virtual time using

$$v(t) \triangleq \int_0^t s(t)\,dt. \tag{4}$$

For example, in Figure 2, $v(25) = \int_0^{25} s(t)\,dt = \int_0^{19} 1\,dt + \int_{19}^{25} 0.5\,dt = 19 + 3 = 22$. This definition leads to the following property:

Property 2. *For arbitrary time instants t_0 and t_1,*

$$v(t_1) - v(t_0) = \int_{t_0}^{t_1} s(t)\,dt.$$

Unless otherwise noted, all instants (e.g., t, $r_{i,k}$, etc.) are specified in actual time, and all variables *except* T_i, Y_i, and U_i^v (all defined below) refer to quantities of actual time.

Under the SVO model, (2) generalizes to

$$v(r_{i,k+1}) \geq v(r_{i,k}) + T_i, \tag{5}$$

and under GEL-v scheduling, (3) generalizes to

$$v(y_{i,k}) = v(r_{i,k}) + Y_i. \tag{6}$$

For example, in Figure 2, $\tau_{1,0}$ is released at actual time 0, has its PP three units of (both actual and virtual) time later at actual time 3, and $\tau_{1,1}$ can be released four units of (both actual and virtual) time later at time 4. However, $\tau_{1,5}$ of the same task is released at actual time 21, shortly after the virtual clock slows down. Therefore, its PP is at actual time 27, which is three units of *virtual* time after its release, and the release of $\tau_{1,6}$ can be no sooner than actual time 29, which is four units of *virtual* time after the release of $\tau_{1,5}$. However, the execution time of $\tau_{1,5}$ is not affected by the slower virtual clock.

In light of (5), we denote as $b_{i,k}$ (*b*oundary) the earliest actual time that $\tau_{i,k+1}$ could be released, based on the release time of $\tau_{i,k}$. $b_{i,k}$ is indexed using k because it depends on the actual release time of $\tau_{i,k}$, not the actual release time of $\tau_{i,k+1}$.

Definition 5. $b_{i,k}$ is the actual time such that $v(b_{i,k}) = v(r_{i,k}) + T_i$.

Although it is possible to analyze systems where some $Y_i > T_i$, doing so increases proof complexity without providing any benefit to response-time or dissipation bounds. Therefore, we assume that for all i,

$$Y_i \leq T_i. \tag{7}$$

In our analysis, we will frequently refer to the total work that a task produces from jobs that have both releases and PPs in a certain interval. We therefore define a function for this quantity.

Definition 6.

$$D_i^e(t_0, t_1) = \sum_{\tau_{i,k} \in \omega} e_{i,k}$$

(*D*emand), where ω is the set of jobs with $t_0 \leq r_{i,k} \leq y_{i,k} \leq t_1$.

In order to model processor supply, we use a "service function" as in (Chakraborty et al., 2003; Leontyev and Anderson, 2010).

Definition 7. $\beta_p(t_0, t_1)$ is the total number of units of time during which P_p is available to level C within $[t_0, t_1)$.

We further characterize processor supply in two parts. First, we assign to each processor P_p a *nominal utilization* $\widehat{u_p}$, representing how much of its time we expect to be available to level C in the long term. Within an arbitrary $[t_0, t_1)$, we expect $\beta_p(t_0, t_1) \approx \widehat{u_p}(t_1 - t_0)$. For example, in Figure 2, P_0 is available whenever τ_{A1} is not running, so we choose $\widehat{u_0} = 1 - \frac{3}{12} = \frac{3}{4}$, since the utilization of τ_{A1} at level C is $\frac{3}{12}$. Similarly, P_1 is available whenever τ_{A2} is not running, so we choose $\widehat{u_1} = 1 - \frac{1}{6} = \frac{5}{6}$.

Over some intervals $[t_0, t_1]$, a CPU is available for less time than indicated by nominal utilization alone. For example, in Figure 2 over $[0, 3)$, P_0 is not available at all to level C. Thus, for our second characterization, we define a *supply restriction overload* function,

$$o_p(t_0, t_1) \triangleq \max\{0, \widehat{u_p} \cdot (t_1 - t_0) - \beta_p(t_0, t_1)\}. \tag{8}$$

This implies that

$$\beta_p(t_0, t_1) \geq \widehat{u_p} \cdot (t_1 - t_0) - o_p(t_0, t_1). \tag{9}$$

For example, consider $[0, 3)$ in Figure 2. By naïvely using nominal utilization, we would expect level C to receive $\widehat{u_0} \cdot (t_1 - t_0) = \frac{3}{4} \cdot (3 - 0) = \frac{9}{4}$ units of service on P_0, but it actually receives 0, so $o_0(t_0, t_1) = \frac{9}{4}$. In the absence of overload, there must be some constant σ_p such that, for all intervals $[t_0, t_1)$, $o_p(t_0, t_1) \leq \widehat{u_p}\sigma_p$, and our model reduces exactly to that used by Leontyev and Anderson (2010). However, our more general model can be used to account for arbitrary overloads, by allowing $o_p(t_0, t_1) > \sigma_p$ when overload occurs within $[t_0, t_1)$. We also define

$$u_{tot} = \sum_{P_p \in P} \widehat{u_p}, \tag{10}$$

which (when supply restriction overload is bounded) represents the total processing capacity available to the system at level C.

3 Response Time Analysis

In this section, we provide a *general* method for analyzing response times of a system at level C, under GEL-v scheduling and with most of the generality of the SVO model. Because we make few assumptions about overload, this method does not provide response-time bounds that apply to all jobs. In fact, it applies even in situations where such bounds *do not exist*. However, we will use these results, with additional assumptions, in Section 4 to provide dissipation bounds and long-term response-time bounds in the absence of overload.

Under GEL scheduling applied to ordinary sporadic task systems, Erickson et al. (2014) proved that $t^c_{i,k} \leq y_{i,k} + x_i + C_i$, where x_i is a per-task constant. Their proof works by analyzing the behavior of each $\tau_{i,k}$ after $y_{i,k}$, because no job with higher priority can be released after $y_{i,k}$. In the presence of overload a single per-task x_i may not exist. Furthermore, even in cases where such an x_i does exist, it must pessimistically bound all job releases, preventing any analysis of dissipation bounds. Therefore, we instead define a function of time $x_i(t) \geq 0$ so that $t^c_{i,k} \leq y_{i,k} + x_i(y_{i,k}) + e_{i,k}$. (We use $e_{i,k}$ in place of C_i because our analysis no longer assumes that $e_{i,k} \leq C_i$.) In our analysis, it is convenient to define $x_i(t)$ over all positive real numbers. Furthermore, it will be convenient to treat $x_i(t)$ as merely a safe upper bound. Therefore, we use the following definition.

Definition 8. $x_i(t)$ is x-sufficient if $x_i(t) \geq 0$ and for all $\tau_{i,k}$ with $y_{i,k} \leq t$,

$$t^c_{i,k} \leq t + x_i(t) + e_{i,k}.$$

Throughout our analysis both here and in Section 4, we will frequently use the following property, which follows immediately from Definition 8.

Property 3. *If $c_1 \geq c_0$ and $x_i(t_a) = c_0$ is x-sufficient, then $x_i(t_a) = c_1$ is x-sufficient.*

In the remainder of this section, we will provide an x-sufficient value for $x_i(t_a)$ for each τ_i and each time t_a (under *a*nalysis). We will exhaustively consider the cases depicted in Figure 3 for each t_a, in approximate order by increasing complexity. Note that Cases D and E reference terminology that will be defined later in this section.

We first consider Case A, which provides the value of $x_i(t_a)$ when $t_a < y_{i,0}$. This case is trivial.

Theorem 1. *If $t_a < y_{i,0}$, then $x_i(t_a) = 0$ is x-sufficient.*

Proof. This theorem results from the definition of x-sufficient in Definition 8. If $t_a < y_{i,0}$, then the condition in Definition 8 holds vacuously, because there are no jobs $\tau_{i,k}$ with $y_{i,k} \leq t_a$. □

We now consider Case B, in which $t_a = y_{i,k}$ for some k but $t^c_{i,k} \leq y_{i,k} + e_{i,k}$. This case is similarly trivial, and we analyze it separately from the cases with $t_a = y_{i,k}$ in order to simplify later proofs.

Theorem 2. *If $t_a = y_{i,k}$ for some k and $t^c_{i,k} \leq y_{i,k} + e_{i,k}$, then $x_i(t_a) = 0$ is x-sufficient.*

Proof. This lemma follows immediately from the definition of x-sufficient in Definition 8. □

A. $t_a < y_{i,0}$ (Theorem 1).

B. $t_a = y_{i,k}$ for some k and $t^c_{i,k} \leq y_{i,k} + e_{i,k}$ (Theorem 2).

C. $t_a \in (y_{i,k}, y_{i,k+1})$ for some k (Theorem 3).

D. $t_a = y_{i,k}$ for some k, $t^c_{i,k} > y_{i,k} + e_{i,k}$, and $\tau_{i,k}$ is f-dominant for L (Theorem 4).

E. $t_a = y_{i,k}$ for some k, $t^c_{i,k} > y_{i,k} + e_{i,k}$, and $\tau_{i,k}$ is m-dominant for L (Theorem 5).

Figure 3: Cases for which values of $x_i(t_a)$ are provided.

We next consider Case C, in which t_a lies between two consecutive PPs. In this case, our bound depends on having an x-sufficient value at the last PP before t_a of a job in τ_i. This can be computed using Case B, D, or E.

Theorem 3. *If $t_a \in (y_{i,k}, y_{i,k+1})$, then $x_i(t_a) = \max\{0, x_i(y_{i,k}) - (t_a - y_{i,k})\}$ is x-sufficient as long as $x_i(y_{i,k})$ is x-sufficient.*

Proof. This theorem results from the definition of x-sufficient in Definition 8. If $t_a \in (y_{i,k}, y_{i,k+1})$, then there are no jobs of τ_i with PPs in $(y_{i,k}, t_a)$, so $t^c_{i,k}$ is the latest completion of any job of τ_i with a PP before t_a. We have

$$\begin{aligned} t^c_{i,k} \leq& \{\text{By the definition of } x\text{-sufficient in Definition 8}\} \\ & y_{i,k} + x_i(y_{i,k}) + e_{i,k} \\ =& \{\text{Rearranging}\} \\ & t_a + x_i(y_{i,k}) - (t_a - y_{i,k}) + e_{i,k}. \end{aligned}$$

Therefore, combined with the requirement from Definition 8 that $x_i(t_a) \geq 0$, $x_i(t_a) = \max\{0, x_i(y_{i,k}) - (t_a - y_{i,k})\}$ is x-sufficient. □

We will next consider Cases D and E. In both of these cases, $t_a = y_{i,k}$ for some k. Before providing proofs, we will first provide a basic explanation for why the presence of supply restriction adds complexity to response-time analysis, and how we account for such complexity. This discussion motivates the structure of our proofs, and also motivates the separate consideration of Case D and Case E.

After $y_{i,k}$, $\tau_{i,k}$ can be delayed for two reasons: all processors can be occupied by either other work and/or supply restriction, or some predecessor job of $\tau_{i,k}$ within τ_i can be incomplete. We define work from $\tau_{j,\ell}$ as *competing* with $\tau_{i,k}$ if $y_{j,\ell} \leq y_{i,k}$ and $j \neq i$, and supply restriction as *competing* if it occurs before $t^c_{i,k}$. Note that, in order to account for carry-in work, we do not require that work or supply restriction happen before $r_{i,k}$ in order to say that it is "competing" with $\tau_{i,k}$.

We first describe the basic structure of previous analysis from Erickson et al. (2014) in the *absence* of supply restriction. Such analysis considers competing work remaining at $y_{i,k}$. Some example patterns for the completion of competing work are depicted in Figure 4. Figure 4(a) depicts the worst-case delay *due to competing work rather than a predecessor*, when all processors are occupied until $\tau_{i,k}$ can begin execution. Figure 4(b) depicts an alternative completion pattern for the same amount of work. Observe that before $t^c_{i,k-1}$, there are idle CPUs. Thus, this example depicts the situation where $\tau_{i,k}$ is delayed due to its predecessor.

If some processor is idle, then there must be fewer than m tasks with remaining work. Thus, in the absence of supply restriction, τ_i will run continuously until $\tau_{i,k}$ completes. This is why the worst-case completion pattern is the one with maximum parallelism, as depicted in Figure 4(a). For fixed $t^c_{i,k-1}$, any other completion pattern might allow $\tau_{i,k}$ to complete earlier (as happens in Figure 4(b)) or else does not change the completion time of $\tau_{i,k}$ (if the delay due to an incomplete predecessor already dominated). To summarize, in the absence of overload, either the delay due to an incomplete predecessor dominates, as in Figure 4(b), or the delay due to competing work dominates, as in Figure 4(a).

We now consider the effects of introducing supply restriction. Figure 5 depicts similar completion patterns as Figure 4. As before, $\tau_{i,k}$ can be delayed either because all processors are occupied by competing work or supply restriction, or because some predecessor of $\tau_{i,k}$ within τ_i is incomplete. However, as can be seen by comparing Figure 5(a) and Figure 5(b), having all competing work complete with maximum parallelism is no longer the worst case. This phenomenon occurs because supply restriction can now prevent the execution of τ_i even after some processor has become idle, by reducing the number of available processors below the number of tasks with remaining work. This increases the complexity of determining the interaction between delays caused by competing work and delays caused by an incomplete predecessor, as the simple dominance that occurred in the absence of supply restriction may not occur.

To determine an upper bound on $t^c_{i,k}$, we add to $y_{i,k}$ the sum of the lengths of three types of sub-intervals within $[y_{i,k}, t^c_{i,k})$, as depicted in Figure 5(b).

1. Sub-intervals during which τ_i does not run because all m processors are occupied by competing work or supply restriction.

2. Sub-intervals during which jobs of τ_i before $\tau_{i,k}$ execute.

3. Sub-intervals during which $\tau_{i,k}$ executes.

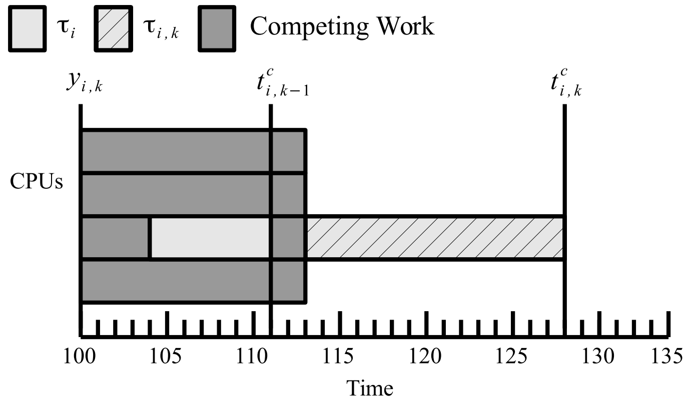
We will bound the total length of sub-intervals of Type 1 by bounding the total amount of competing work and supply restriction. We will now define the total length of sub-intervals of Type 2 as $e^p_{i,k}$; the total length of sub-intervals of Type 3 is simply $e^r_{i,k}(y_{i,k})$.

Definition 9. $e^p_{i,k}$ is the work remaining after $y_{i,k}$ due to jobs of τ_i prior to $\tau_{i,k}$.

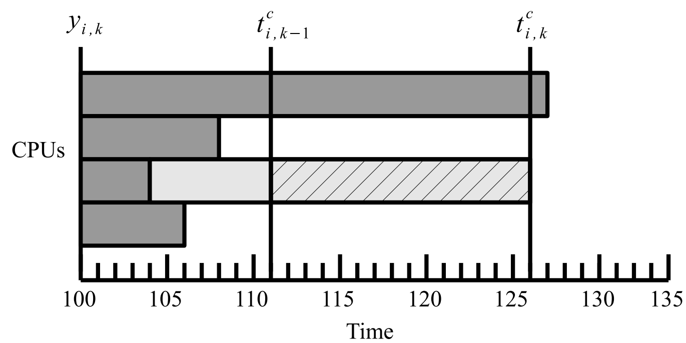
Let c denote a bound on the total amount of competing work after $y_{i,k}$ and competing supply restriction after $y_{i,k}$. The specific value of c will be derived in Lemma 6 below, but its exact expression is not relevant for the purposes of this introductory discussion. The total length of Type 1 sub-intervals of $[y_{i,k}, t^c_{i,k})$, where τ_i is not running, can be upper bounded by dividing c by m. However, this bound may be unnecessarily pessimistic, because some of the competing work and supply restriction may actually run concurrently with τ_i. For example, in Figure 5(b), some competing work and supply restriction runs within $[104, 120)$ even though this interval is composed of a sub-interval of Type 2 and a sub-interval of Type 3.

We now informally describe an optimization that allows us to reduce some of this pessimism. We will simplify our informal analysis by assuming that $k > 0$ and $t^c_{i,k-1} > y_{i,k}$. We will later discuss how to relax this assumption. Let v be an arbitrary integer with $0 \leq v < m$. We consider two cases.

Few Tasks Case. If there is some time within $[y_{i,k}, t^c_{i,k-1})$ such that at most v processors are occupied by work or supply restriction, then there are at most v tasks that have work remaining, or more CPUs would be occupied. Thus, in this case there are at most v tasks with competing work remaining after $t^c_{i,k-1}$, and $\tau_{i,k}$ can execute after $t^c_{i,k-1}$ whenever there are at least v processors available to level C. For example, in Figure 5(b), if $v = 2$, then because only v processors are occupied just before $t^c_{i,k-1}$, there are only v tasks with remaining work at this time, and $\tau_{i,k}$ can run after $t^c_{i,k-1}$ whenever at least $v = 2$ processors are available to level C. Therefore, rather than



(a) Maximum parallelism (worst-case) completion pattern.



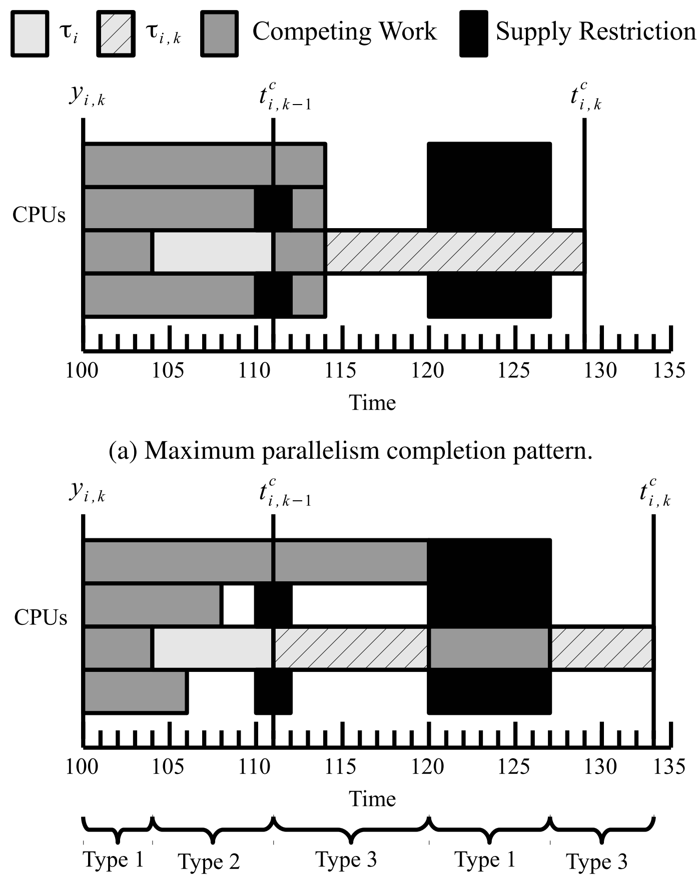
(b) Alternative completion pattern.

Figure 4: Example completion patterns for competing work in the absence of supply restriction.

summing the lengths of the intervals of each type, we can compute an upper bound on the time it takes for there to be $e_{i,k}$ time units with at least v processors available after $t^c_{i,k-1}$.

Many Tasks Case. If there are at least $v+1$ tasks with work remaining throughout $[y_{i,k}, t^c_{i,k-1})$, then at least v processors are occupied with competing work and/or supply restriction in all sub-intervals of Type 2 (in which jobs of τ_i prior to $\tau_{i,k}$ are running). For example, in Figure 5(b), this case holds with $v = 1$, because there is some other task executing on the first processor until $t^c_{i,k-1}$. By the definition of $e^p_{i,k}$ in Definition 9, the total length of Type 2 intervals is $e^p_{i,k}$. Thus, at least $v \cdot e^p_{i,k}$ units of competing work and supply restriction actually run in intervals of Type 2, so we can subtract $v \cdot e^p_{i,k}$ from the bound c to upper bound the amount of work and supply restriction running in Type 1 intervals.

As mentioned above, this informal analysis assumes that $\tau_{i,k-1}$ exists and that $t^c_{i,k-1} > y_{i,k}$. If this is not the case, then by the definition of $e^p_{i,k}$ in Definition 9, $e^p_{i,k} = 0$. Thus, the analysis from the



(a) Maximum parallelism completion pattern.

(b) Alternative completion pattern.

Figure 5: Example completion patterns for competing work in the presence of supply restriction.

Many Tasks Case will subtract $v \cdot e^p_{i,k} = 0$ units of competing work and supply restriction from the bound c for Type 1 intervals. Therefore, it is safe to use the analysis from the Many Tasks Case when $k = 0$ or $t^c_{i,k-1} \leq y_{i,k}$. Observe that for *any* v with $0 \leq v < m$, one of these two cases must hold. We therefore define a pair of properties, one for each case.

Definition 10. Let L be an arbitrary integer with $0 \leq L < m$. If $k > 0$, $t^c_{i,k-1} > y_{i,k}$, and there are at most $m - L - 1$ tasks that have work remaining at $t^c_{i,k-1}$, then $\tau_{i,k}$ is *f-dominant for* L (*f*ew tasks case).

$\tau_{i,k}$ is f-dominant for L if the Few Tasks Case applies.

Definition 11. Let L be an arbitrary integer with $0 \leq L < m$. If $\tau_{i,k}$ is not f-dominant for L, then $\tau_{i,k}$ is *m-dominant for* L (*m*any tasks case).

$\tau_{i,k}$ is m-dominant for L if the Many Tasks Case applies.

In Section 3.1 below, we will consider Case D, in which $\tau_{i,k}$ is f-dominant for L. Then, in Section 3.2, we will consider Case E, in which $\tau_{i,k}$ is m-dominant for L.

3.1 Case D: $t_a = y_{i,k}$ for some k and $\tau_{i,k}$ is f-dominant for L.

In this case, we use the Few Tasks Case with $v = m - L - 1$. Recall from the above discussion that in this case, $\tau_{i,k}$ runs after $t^c_{i,k-1}$ whenever there are at least v processors available to level C. Lemma 2 below provides a bound on $t^c_{i,k}$ in this case. Lemma 1 is used to prove Lemma 2.

Lemma 1. *For any integer $0 \leq v \leq m$, in any time interval $[t_0, t_1)$ there are at least*

$$(t_1 - t_0) - \sum_{P_p \in \zeta} ((1 - \widehat{u_p}) \cdot (t_1 - t_0) + o_p(t_0, t_1))$$

units of time during which at least v processors are available to level C, where ζ is the set of v processors that minimizes the sum.

Proof. We prove this lemma by induction. Without loss of generality, we fix t_0 and t_1 and assume that P is ordered by increasing $(1 - \widehat{u_p})(t_1 - t_0) + o_p(t_0, t_1)$. We prove the stronger condition that within $[t_0, t_1)$, there are $(t_1 - t_0) - \sum_{p=1}^{v}((1 - \widehat{u_p}(t_1 - t_0) + o_p(t_0, t_1))$ units time during which processors P_1 through P_v are available to level C.

As the base case, we consider $v = 0$. During any time instant, it is vacuously true that all processors in the empty set are available to level C, so there are $t_1 - t_0$ such units of time in $[t_0, t_1)$ and the lemma holds.

For the inductive case, assume that there are

$$(t_1 - t_0) - \sum_{p=1}^{v}((1 - \widehat{u_p}) \cdot (t_1 - t_0) + o_p(t_0, t_1)) \tag{11}$$

units of time in $[t_0, t_1)$ during which processors P_1 through P_v are available to level C.

By the definition of $\beta_{v+1}(t_0, t_1)$ in Definition 7, P_{v+1} is unavailable to level C in $[t_0, t_1)$ for

$$(t_1 - t_0) - \beta_{v+1}(t_0, t_1)$$

$$\leq \{\text{By (9)}\}$$

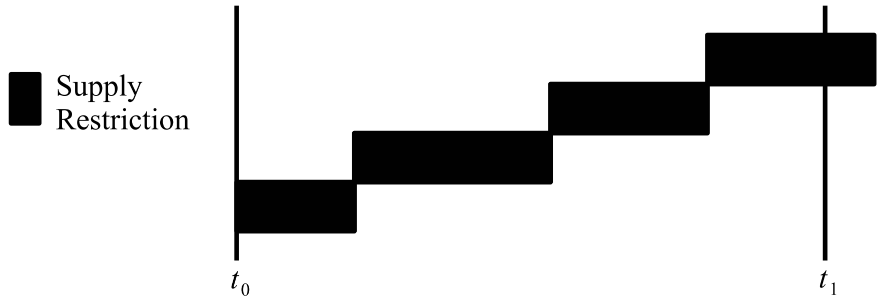


Figure 6: Worst-case pattern of supply restriction for the first four processors, in order to minimize the amount of time that all four processors are available. In this case, the four processors are never all available at the same time within $[t_0, t_1)$.

$$
\begin{aligned}
& (t_1 - t_0) - (\widehat{u_{v+1}} \cdot (t_1 - t_0) - o_{v+1}(t_0, t_1)) \\
= & \{\text{Rearranging}\} \\
& (1 - \widehat{u_{v+1}}) \cdot (t_1 - t_0) + o_{v+1}(t_0, t_1) \qquad (12)
\end{aligned}
$$

units of time.

In the worst case, as depicted in Figure 6, either all of P_{v+1}'s unavailable time occurs when processors P_1 through P_v are all available, or P_{v+1} is unavailable during all times when P_1 through P_v are available. In either case, the lemma holds by subtracting (12) from (11). □

We now use Lemma 1 to prove the next lemma, which bounds $t^c_{i,k}$ in the Few Tasks Case.

Lemma 2. *Let v be an integer with $0 \leq v < m$. Let*

$$
A_{i,k}(v) \triangleq \begin{cases} \frac{e_{i,k} + \sum_{P_p \in \Theta} o_p(t^c_{i,k-1}, t^c_{i,k})}{1 - v + \sum_{P_p \in \Theta} \widehat{u_p}} & \textit{If } 1 - v + \sum_{P_p \in \Theta} \widehat{u_p} > 0 \\ \infty & \textit{Otherwise,} \end{cases} \qquad (13)
$$

where Θ is the set of v processors that minimizes $A_{i,k}(v)$.

If $\tau_{i,k}$ can run after $t^c_{i,k-1}$ whenever there are at least v processors available to level C, then $t^c_{i,k} \leq t^c_{i,k-1} + A_{i,k}(v)$.

Proof. If $A_{i,k}(v)$ is infinite, then the lemma must hold by our assumption that the available supply is eventually infinite, and thus $\tau_{i,k}$ eventually completes. Thus, we assume that $A_{i,k}(v)$ is finite. Therefore, $1 - v + \sum_{P_p \in \Theta} \widehat{u_p} > 0$.

We use proof by contradiction. Suppose $t^c_{i,k} > t^c_{i,k-1} + A_{i,k}(v)$. Then, by Lemma 1, the number of time units in $[t^c_{i,k-1}, t^c_{i,k})$ with v processors available to level C is at least

$$
\begin{aligned}
&(t^c_{i,k} - t^c_{i,k-1}) - \sum_{P_p \in \zeta} ((1 - \widehat{u_p})(t^c_{i,k} - t^c_{i,k-1}) + o_p(t^c_{i,k-1}, t^c_{i,k})) \\
&\geq \{\text{By the definition of } \zeta \text{ in Lemma 1, because } \Theta \text{ has } v \text{ processors}\} \\
&(t^c_{i,k} - t^c_{i,k-1}) - \sum_{P_p \in \Theta} ((1 - \widehat{u_p})(t^c_{i,k} - t^c_{i,k-1}) + o_p(t^c_{i,k-1}, t^c_{i,k})) \\
&= \{\text{Rearranging}\} \\
&\left(1 - v + \sum_{P_p \in \Theta} \widehat{u_p}\right)(t^c_{i,k} - t^c_{i,k-1}) - \sum_{P_p \in \Theta} o_p(t^c_{i,k-1}, t^c_{i,k}) \\
&> \{\text{Because } t^c_{i,k} > t^c_{i,k-1} + A_{i,k}(v) \text{ and } 1 - v + \textstyle\sum_{P_p \in \Theta} \widehat{u_p} > 0\} \\
&\left(1 - v + \sum_{P_p \in \Theta} \widehat{u_p}\right) A_{i,k}(v) - \sum_{P_p \in \Theta} o_p(t^c_{i,k-1}, t^c_{i,k}) \\
&= \{\text{By (13), because } A_{i,k}(v) \text{ is finite}\} \\
&\left(1 - v + \sum_{P_p \in \Theta} \widehat{u_p}\right) \cdot \frac{e_{i,k} + \sum_{P_p \in \Theta} o_p(t^c_{i,k-1}, t^c_{i,k})}{1 - q + \sum_{P_p \in \Theta} \widehat{u_p}} - \sum_{P_p \in \Theta} o_p(t^c_{i,k-1}, t^c_{i,k}) \\
&= \{\text{Rearranging}\} \\
&e_{i,k}.
\end{aligned} \tag{14}
$$

However, because $\tau_{i,k}$ can run after $t^c_{i,k-1}$ whenever there are at least v processors available to level C, $\tau_{i,k}$ must have executed for longer than $e_{i,k}$ units. This is a contradiction. □

We now use this result to provide a bound on $x_i(t_a)$ to handle Case D.

Theorem 4. *If $t_a = y_{i,k}$ for some k and $\tau_{i,k}$ is f-dominant for L, then $x_i(t_a) = x^f_{i,k}$ is x-sufficient, where*

$$x^f_{i,k} \triangleq t^c_{i,k-1} - y_{i,k} + A_{i,k}(m - L - 1) - e_{i,k} \tag{15}$$

(few tasks).

Proof. By the definition of f-dominant for L in Definition 10, because no new competing work is released after $y_{i,k}$, throughout $(t^c_{i,k-1}, t^c_{i,k}]$, there are at most $m - L - 1$ tasks that have remaining competing work. Therefore, whenever at least $m - L - 1$ processors are available to level C within $(t^c_{i,k-1}, t^c_{i,k}]$, $\tau_{i,k}$ is running. Thus,

$$
\begin{aligned}
t^c_{i,k} \leq\ & \{\text{By Lemma 2}\} \\
& t^c_{i,k-1} + A_{i,k}(m - L - 1) \\
=\ & \{\text{Rearranging}\} \\
& y_{i,k} + (t^c_{i,k-1} - y_{i,k} + A_{i,k}(m - L - 1) - e_{i,k}) + e_{i,k} \\
=\ & \{\text{By the definition of } x^f_{i,k} \text{ in (15)}\} \\
& y_{i,k} + x^f_{i,k} + e_{i,k}.
\end{aligned}
$$

Thus, by the definition of x-sufficient in Definition 8, $x_i(y_{i,k}) = x^f_{i,k}$ is x-sufficient. Because $t_a = y_{i,k}$, the lemma follows. □

3.2 Case E: $t_a = y_{i,k}$ for some k and $\tau_{i,k}$ is m-dominant for L

The basic structure of our analysis of Case E is fundamentally similar to the analysis in Erickson et al. (2014). We will analyze the lateness of an arbitrary job $\tau_{i,k}$, ignoring all jobs that have PPs after $y_{i,k}$. We define an interval as *busy* if, for the entire interval, all processors are either unavailable or are executing tasks with PPs not after $y_{i,k}$. As depicted in Figure 7, we denote as $t^b_{i,k}$ the earliest time such that $[t^b_{i,k}, y_{i,k})$ is busy. We separately upper bound work (Lemma 4) and competing supply restriction (Lemma 5) after $t^b_{i,k}$, and then use those results to determine a full response-time bound.

We will first upper bound all remaining work at $t^b_{i,k}$, including both competing work and work due to τ_i. In order to do so, we first prove a lemma that bounds the amount of work after arbitrary time $t_0 \leq y_{i,k}$ contributed by a task τ_j with a pending job at t_0. This will allow us to bound the work by tasks that have pending jobs at $t^b_{i,k}$. (We use t_0 instead of $t^b_{i,k}$ because the same lemma will also be used in Section 4 with a different choice of t_0.)

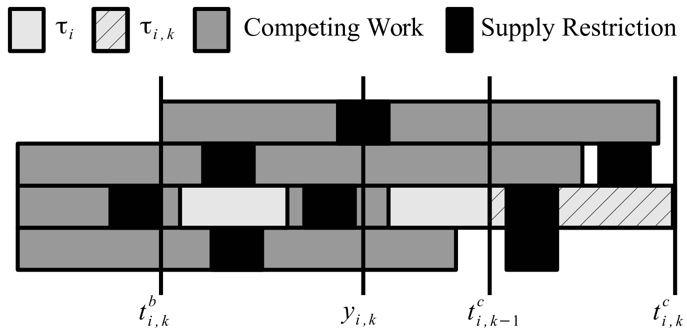


Figure 7: Example depicting $t^b_{i,k}$ when $m = 4$.

Lemma 3. *If there is a pending job of τ_j at* arbitrary *time $t_0 \leq y_{i,k}$, then denote as $\tau_{j,\ell}$ the earliest job of τ_j pending at t_0. The total remaining work at time t_0 for jobs of τ_j with PPs not later than $y_{i,k}$ is $e^r_{j,\ell}(t_0) + D^e_j(b_{j,\ell}, y_{i,k})$.*

Proof. By the definition of $b_{j,\ell}$ in Definition 5 and the definition of T_i in (5), any job of τ_j after $\tau_{j,\ell}$ must be released no sooner than $b_{j,\ell}$. Therefore, by Definition 6, the total work from such jobs with PPs not later than $y_{i,k}$ is $D^e_j(b_{j,\ell}, y_{i,k})$. Adding $e^r_{j,\ell}(t_0)$ for the remaining work due to $\tau_{j,\ell}$ yields the lemma. □

We now bound all remaining work at $t^b_{i,k}$, including that due to τ_i.

Lemma 4. *The remaining work at $t^b_{i,k}$ for jobs with PPs not later than $y_{i,k}$ is*

$$W_{i,k} \triangleq \sum_{\tau_{j,\ell} \in \theta_{i,k}} (e^r_{j,\ell}(t^b_{i,k}) + D^e_j(b_{j,\ell}, y_{i,k})) + \sum_{\tau_j \in \overline{\theta_{i,k}}} D^e_j(t^b_{i,k}, y_{i,k}). \tag{16}$$

where $\theta_{i,k}$ is the set of jobs $\tau_{j,\ell}$ such that $\tau_{j,\ell}$ is the earliest pending job of τ_j at $t^b_{i,k}$, $r_{j,\ell} < t^b_{i,k}$, and $y_{j,\ell} \leq y_{i,k}$, and $\overline{\theta_{i,k}}$ is the set of tasks that do not have jobs in $\theta_{i,k}$.

Proof. As before, we ignore any jobs that have PPs after $y_{i,k}$. We bound the work remaining for each task τ_j at $t^b_{i,k}$, depending on whether it has a pending job at $t^b_{i,k}$ with a release before $t^b_{i,k}$.

Case 1: τ_j has no pending job at $t^b_{i,k}$ with a release before $t^b_{i,k}$. If no job of τ_j with PP at or before $y_{i,k}$ is pending at $t^b_{i,k}$, or if the earliest pending job of τ_j at $t^b_{i,k}$ is released at $t^b_{i,k}$, then all relevant work remaining for τ_j at $t^b_{i,k}$ comes from jobs $\tau_{j,\ell}$ with $t^b_{i,k} \leq r_{j,\ell} \leq y_{j,\ell} \leq y_{i,k}$. Thus, by the definition of $D_j^e(t_{i,k}^b, y_{i,k})$ in Definition 6, there are

$$D_j^e(t_{i,k}^b, y_{i,k}) \tag{17}$$

units of such work. Furthermore, such a task is in $\overline{\theta_{i,k}}$ by the definition of $\overline{\theta_{i,k}}$ in the statement of the lemma.

Case 2: τ_j has a pending job at $t_{i,k}^b$ with a release before $t_{i,k}^b$. Denote as $\tau_{j,\ell}$ the earliest pending job of τ_j at $t_{i,k}^b$. By Lemma 3 (with $t_0 = t_{i,k}^b$), the remaining work for τ_j at $t_{i,k}^b$ is

$$e_{j,\ell}^r(t_{i,k}^b) + D_j^e(b_{j,\ell}, y_{i,k}). \tag{18}$$

Furthermore, $\tau_{j,\ell}$ is in $\theta_{i,k}$ by the definition of $\theta_{i,k}$ in the statement of the lemma.

Summing over all tasks, using (17) or (18) as appropriate, yields $W_{i,k}$ by (16). □

We next consider supply restriction, accounting for it as if it were competing work. There is one significant difference between supply restriction and competing work. Under GEL-v scheduling, once a job has reached its PP, no new competing work can arrive. However, new supply restriction can continue to be encountered until the job completes. Because we assume that the available supply is infinite when extended into the future, every job must eventually complete. Therefore, *by the definition of x-sufficient in Definition 8*, we have that each $\tau_{i,k}$ completes by time $y_{i,k} + x_i(y_{i,k}) + e_{i,k}$ for some x-sufficient $x_i(y_{i,k})$. We reference such a $x_i(y_{i,k})$ in the following lemma, and instantiate it to a specific value in Theorem 5 below.

Lemma 5. *For arbitrary job $\tau_{i,k}$ and x-sufficient $x_i(y_{i,k})$, at most*

$$(m - u_{tot})((y_{i,k} - t_{i,k}^b) + x_i(y_{i,k}) + e_{i,k}) + O_{i,k},$$

units of competing supply restriction exist after $t_{i,k}^b$, where

$$O_{i,k} \triangleq \sum_{P_p \in P} o_p(t_{i,k}^b, t_{i,k}^c). \tag{19}$$

Proof. By the definition of $\beta_p(t^b_{i,k}, t^c_{i,k})$ in Definition 7, the amount of time that P_p is *not* available to level C over $[t^b_{i,k}, t^c_{i,k})$ is

$$\begin{aligned}
&(t^c_{i,k} - t^b_{i,k}) - \beta_p(t^b_{i,k}, t^c_{i,k}) \\
\leq\ &\{\text{By (9)}\} \\
&(t^c_{i,k} - t^b_{i,k}) - \widehat{u_p} \cdot (t^c_{i,k} - t^b_{i,k}) + o_p(t^b_{i,k}, t^c_{i,k}). \\
=\ &\{\text{Rearranging}\} \\
&(1 - \widehat{u_p}) \cdot (t^c_{i,k} - t^b_{i,k}) + o_p(t^b_{i,k}, t^c_{i,k}).
\end{aligned}$$

This quantity upper bounds the competing supply restriction on P_p. Summing over all processors, the total amount of competing supply restriction on all processors is at most

$$\begin{aligned}
&\sum_{P_p \in P} ((1 - \widehat{u_p}) \cdot (t^c_{i,k} - t^b_{i,k}) + o_p(t^b_{i,k}, t^c_{i,k})) \\
=\ &\{\text{Rearranging}\} \\
&\left(\sum_{P_p \in P} 1 - \sum_{P_p \in P} \widehat{u_p} \right) \cdot (t^c_{i,k} - t^b_{i,k}) + \sum_{P_p \in P} o_p(t^b_{i,k}, t^c_{i,k}) \\
=\ &\{\text{Because there are } m \text{ processors in } P \text{, by the definition of } u_{tot} \text{ in (10), and by the definition} \\
&\text{of } O_{i,k} \text{ in (19)}\} \\
&(m - u_{tot}) \cdot (t^c_{i,k} - t^b_{i,k}) + O_{i,k} \\
\leq\ &\{\text{By the definition of } x\text{-sufficient in Definition 8}\} \\
&(m - u_{tot}) \cdot (y_{i,k} + x_i(y_{i,k}) + e_{i,k} - t^b_{i,k}) + O_{i,k} \\
=\ &\{\text{Rearranging}\} \\
&(m - u_{tot}) \cdot ((y_{i,k} - t^b_{i,k}) + x_i(y_{i,k}) + e_{i,k}) + O_{i,k}.
\end{aligned}$$

□

We now compute a lateness bound that accounts for both work and competing supply restriction. As discussed earlier, we will analyze the behavior of the system after $y_{i,k}$, when new job arrivals cannot preempt $\tau_{i,k}$.

We will now consider how to bound the total length of sub-intervals of Type 1 as described earlier, during which τ_i does not execute because all processors are occupied by competing work or supply restriction. We will do so by bounding the total amount of competing work and supply restriction over $[y_{i,k}, t^c_{i,k})$. Recall that in Lemmas 4–5, competing work and supply restriction were determined over $[t^b_{i,k}, t^c_{i,k})$ rather than $[y_{i,k}, t^c_{i,k})$. The following property will allow us to transition to reasoning about $[y_{i,k}, t^c_{i,k})$. It holds by the definition of $t^b_{i,k}$.

Property 4. *$m \cdot (y_{i,k} - t^b_{i,k})$ units of work and/or supply restriction complete in $[t^b_{i,k}, y_{i,k})$.*

We now bound the amount of competing work and supply restriction in $[y_{i,k}, t^c_{i,k})$.

Lemma 6. *For arbitrary $\tau_{i,k}$, at most*

$$W_{i,k} - R_{i,k} + (m - u_{tot})(x_i(y_{i,k}) + e_{i,k}) + O_{i,k} - e^r_{i,k}(y_{i,k}) - e^p_{i,k}$$

units of competing work and supply restriction remain at $y_{i,k}$, where

$$R_{i,k} \triangleq u_{tot}(y_{i,k} - t^b_{i,k}). \tag{20}$$

Proof. By Lemma 4, the total amount of remaining work at $t^b_{i,k}$ is $W_{i,k}$. Adding this to the bound on competing supply restriction in Lemma 5, there are at most

$$W_{i,k} + (m - u_{tot})((y_{i,k} - t^b_{i,k}) + x_i(y_{i,k}) + e_{i,k}) + O_{i,k}$$

units of work and supply restriction after $t^b_{i,k}$. Of this work and supply restriction, by Property 4, the amount remaining at $y_{i,k}$ is at most

$$\begin{aligned} & W_{i,k} + (m - u_{tot})((y_{i,k} - t^b_{i,k}) + x_i(y_{i,k}) + e_{i,k}) + O_{i,k} - m \cdot (t^b_{i,k} - y_{i,k}) \\ = & \{\text{Rearranging}\} \\ & W_{i,k} - u_{tot} \cdot (y_{i,k} - t^b_{i,k}) + (m - u_{tot}) \cdot (x_i(y_{i,k}) + e_{i,k}) + O_{i,k} \\ = & \{\text{By the definition of } R_{i,k} \text{ in (20)}\} \\ & W_{i,k} - R_{i,k} + (m - u_{tot}) \cdot (x_i(y_{i,k}) + e_{i,k}) + O_{i,k}. \end{aligned} \tag{21}$$

Of this remaining work and supply restriction, by the definition of $e^p_{i,k}$ in Definition 9, $e^p_{i,k}$ units are due to jobs of τ_i prior to $\tau_{i,k}$, and by the definition of $e^r_{i,k}(y_{i,k})$ in Definition 3, $e^r_{i,k}(y_{i,k})$ units are due to $\tau_{i,k}$ itself. The lemma follows immediately. $\square$

We now bound the completion time of $\tau_{i,k}$.

Lemma 7. *If $\tau_{i,k}$ is m-dominant for L and $x_i(y_{i,k})$ is x-sufficient, then*

$$t^c_{i,k} \leq y_{i,k} + \frac{W_{i,k} - R_{i,k} + (m - u_{tot})(x_i(y_{i,k}) + e_{i,k}) + O_{i,k} - e_{i,k} + Le^p_{i,k}}{m} + e_{i,k}.$$

Proof. By the definition of m-dominant for L in Definition 11, there are always at least $m - L - 1$ units of competing work or supply restriction that must run concurrently with τ_i whenever jobs of τ_i prior to $\tau_{i,k}$ are running after $y_{i,k}$. (This statement is vacuously true if no jobs of τ_i prior to $\tau_{i,k}$ run after $y_{i,k}$.) In other words, during any instant within any sub-interval of Type 2 (as depicted in Figure 5(b)), there are at least $m - L - 1$ processors executing competing work or supply restriction. Recall that, by the definition of "Type 2" and the definition of $e^p_{i,k}$ in Definition 9, the total length of such intervals is $e^p_{i,k}$.

By Lemma 6 there can be at most

$$\begin{aligned}
c \triangleq\ & W_{i,k} - R_{i,k} + (m - u_{tot})(x_i(y_{i,k}) + e_{i,k}) + O_{i,k} - e^r_{i,k}(y_{i,k}) \\
& - e^p_{i,k} - (m - L - 1)e^p_{i,k} \\
=\ & \{\text{Rearranging}\} \\
& W_{i,k} - R_{i,k} + (m - u_{tot})(x_i(y_{i,k}) + e_{i,k}) + O_{i,k} - e^r_{i,k}(y_{i,k}) + (L - m)e^p_{i,k} \qquad (22)
\end{aligned}$$

units of computing work and supply restriction after $y_{i,k}$ that do not run concurrently with jobs of τ_i prior to $\tau_{i,k}$. This bound includes all work and/or supply restriction in sub-intervals of Type 1. All m processors are occupied by work or supply restriction in such sub-intervals, so the total length of such sub-intervals is at most c/m.

Recall from Definition 9 that the total length of Type 2 sub-intervals (in which jobs of τ_i prior to $\tau_{i,k}$ execute) is defined to be $e^p_{i,k}$, and the total length of Type 3 sub-intervals (in which $\tau_{i,k}$ runs) is $e^r_{i,k}(y_{i,k})$.

Therefore,

$$
\begin{aligned}
t^c_{i,k} \leq\ & \{\text{Adding the total length of each type of sub-interval to } y_{i,k}\} \\
& y_{i,k} + \frac{c}{m} + e^p_{i,k} + e^r_{i,k}(y_{i,k}) \\
=\ & \{\text{By (22)}\} \\
& y_{i,k} + \frac{W_{i,k} - R_{i,k} + (m - u_{tot})(x_i(y_{i,k}) + e_{i,k}) + O_{i,k} - e^r_{i,k}(y_{i,k}) + (L - m)e^p_{i,k}}{m} \\
& + e^p_{i,k} + e^r_{i,k}(y_{i,k}) \\
=\ & \{\text{Rearranging}\} \\
& y_{i,k} + \frac{W_{i,k} - R_{i,k} + (m - u_{tot})(x_i(y_{i,k}) + e_{i,k}) + O_{i,k} + Le^p_{i,k}}{m} + \frac{m-1}{m} \cdot e^r_{i,k}(y_{i,k}) \\
\leq\ & \{\text{Because } e^r_{i,k}(y_{i,k}) \leq e_{i,k} \text{ and } m \geq 1\} \\
& y_{i,k} + \frac{W_{i,k} - R_{i,k} + (m - u_{tot})(x_i(y_{i,k}) + e_{i,k}) + O_{i,k} + Le^p_{i,k}}{m} + \frac{m-1}{m} \cdot e_{i,k} \\
=\ & \{\text{Rearranging}\} \\
& y_{i,k} + \frac{W_{i,k} - R_{i,k} + (m - u_{tot})(x_i(y_{i,k}) + e_{i,k}) + O_{i,k} - e_{i,k} + Le^p_{i,k}}{m} + e_{i,k}.
\end{aligned}
$$

□

The next lemma provides the actual bound on $x_i(t_a)$.

Theorem 5. *If $t_a = y_{i,k}$ for some k, $t^c_{i,k} > y_{i,k} + e_{i,k}$ and $\tau_{i,k}$ is m-dominant for L, then $x_i(t_a) = x^m_{i,k}$ is x-sufficient, where*

$$x^m_{i,k} \triangleq \frac{W_{i,k} - R_{i,k} + (m - u_{tot} - 1)e_{i,k} + O_{i,k} + Le^p_{i,k}}{u_{tot}} \tag{23}$$

*(*many *tasks).*

Proof. We let

$$x^t_{i,k} \triangleq t^c_{i,k} - e_{i,k} - y_{i,k} \tag{24}$$

(*t*ight). Rearranging,

$$t^c_{i,k} = y_{i,k} + x^t_{i,k} + e_{i,k}. \tag{25}$$

Because $t^c_{i,k} > y_{i,k} + e_{i,k}$, by (24)–(25) and the definition of x-sufficient in Definition 8, $x_i(y_{i,k}) = x^t_{i,k}$ is x-sufficient.

Therefore, by Lemma 7 with $x_i(y_{i,k}) = x^t_{i,k}$ and (25),

$$x^t_{i,k} \leq \frac{W_{i,k} - R_{i,k} + (m - u_{tot}) \cdot (x^t_{i,k} + e_{i,k}) + O_{i,k} - e_{i,k} + L \cdot e^p_{i,k}}{m}.$$

We solve for $x^t_{i,k}$. First, we will add $\frac{u_{tot}-m}{m} \cdot x^t_{i,k}$ to both sides, which yields

$$\frac{u_{tot}}{m} \cdot x^t_{i,k} \leq \frac{W_{i,k} - R_{i,k} + (m - u_{tot}) \cdot e_{i,k} + O_{i,k} - e_{i,k} + L \cdot e^p_{i,k}}{m}.$$

We then multiply both sides by $\frac{u_{tot}}{m}$. Because $u_{tot} > 0$ and $m > 0$,

$$\begin{aligned}
x^t_{i,k} &\leq \frac{W_{i,k} - R_{i,k} + (m - u_{tot}) \cdot e_{i,k} + O_{i,k} - e_{i,k} + L \cdot e^p_{i,k}}{u_{tot}} \\
&= \{\text{Rearranging}\} \\
&\quad \frac{W_{i,k} - R_{i,k} + (m - u_{tot} - 1) \cdot e_{i,k} + O_{i,k} + L \cdot e^p_{i,k}}{u_{tot}} \\
&= \{\text{By the definition of } x^m_{i,k} \text{ in (23)}\} \\
&\quad x^m_{i,k}.
\end{aligned}$$

Because $x_i(y_{i,k}) = x^t_{i,k}$ is x-sufficient, by Property 3 with $c_0 = x^t_{i,k}$ and $c_1 = x^m_{i,k}$, $x_i(y_{i,k}) = x^m_{i,k}$ is x-sufficient. Because $t_a = y_{i,k}$, the lemma follows. □

4 Dissipation Bounds

The response-time analysis provided in Section 3 is very general, in order to provide an accurate characterization of the behavior in overload situations. In particular, it can even be used to analyze the behavior of systems where no per-task bound on response times exists. In this section, we consider systems that have per-task response time bounds in the absence of overload. In other words, each task has some constant $x^s_i(1)$ such that, if $s(t) = 1$ for all t, then $x_i(t) = x^s_i(1)$ is x-sufficient for all τ_i and time t. (The reason for the "1" argument will be described later.) In this section, we analyze a system where an overload actually does occur, but the overload is transient. This situation is similar

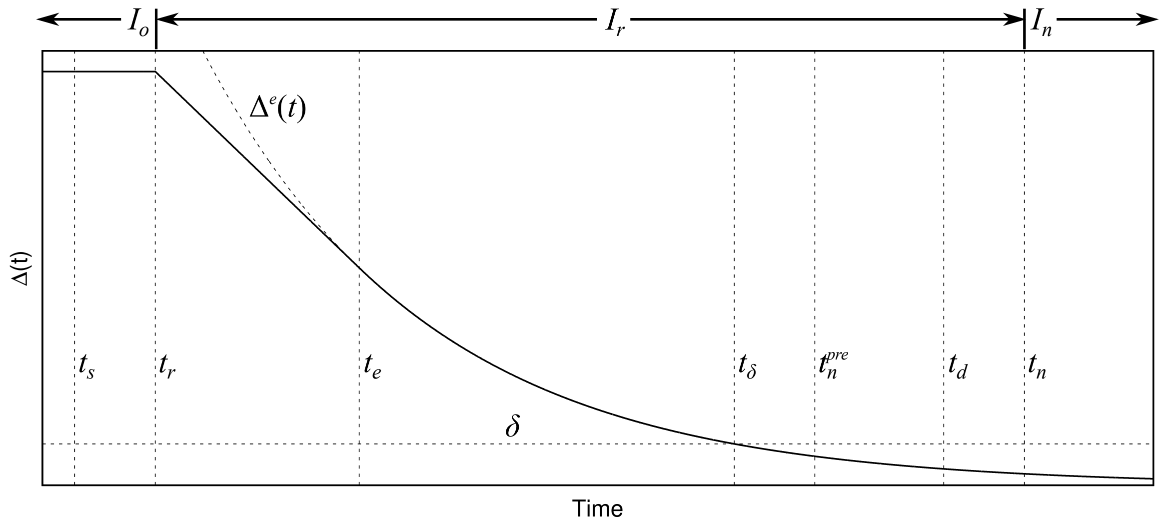


Figure 8: Graph of $\Delta(t)$, marked with various terms used in its definition and analysis.

to that depicted in Figure 2, where both τ_{A1} and τ_{A2} have jobs starting at actual time 12 that run for longer than their level-C PWCETs, but no later jobs that do so. Under ideal analysis, $x_i(t) = x_i^s(1)$ would no longer be x-sufficient for $t > 12$, although this may not be the case under the analysis presented here due to its pessimism. For illustration purposes, we consider a system under ideal analysis. We would like to return the system to a state where $x_i(t) = x_i^s(1)$ is x-sufficient for all τ_i and all t greater than some t_n (*n*ormal operation). In this section, we demonstrate a method to provide such a guarantee, *provided that we use the analytically-derived* $x_i^s(1)$ *described herein.*

In Figure 8, we depict many details of our analysis of dissipation bounds. The first of these details is depicted at the top of the figure: the three intervals into which we divide time. The first is the *overload* interval I_o, from the beginning of the schedule until after a transient overload has passed. If we had ideal analysis, this interval would occur from actual time 0 to actual time 19 in Figure 2. We make very few assumptions about the behavior of the system in I_o, primarily using the analysis from Section 3. This allows us to account for any overload condition allowed by our general model. The second considered interval is the *recovery* interval I_r, during which the virtual time clock operates at a slower rate in order to recover from the overload. Under ideal analysis, this would occur from actual time 19 to actual time 29 in Figure 2. The final interval we consider is the *normal* interval I_n, when the system operates normally. The virtual time clock executes at full speed during the normal interval. Under ideal analysis, this would occur from actual time 29 onwards in Figure 2.

In order to provide boundaries between these intervals, we define several variables. t_s is defined to be the time at which the virtual clock actually slows. t_n will be defined as the time when the virtual clock can be returned to a normal speed. We note that the virtual clock can be returned to a normal speed at a *later* time without compromising correctness. We assume that the virtual clock is slowed to a constant speed s_r from t_s to t_n, as specified in the following property.

Property 5. *For all $t \in [t_s, t_n)$, $s(t) = s_r < 1$.*

In Figure 2, $s_r = 0.5$. Similarly, the following property describes the behavior of the virtual clock after the system has returned to normal.

Property 6. *If $t \in I_n$, then $s(t) = 1$.*

Because the speed of the virtual clock is determined by the operating system, it is always possible to ensure that both properties hold.

Although the virtual clock is actually slowed at time t_s, for our analysis within I_r, it will often be convenient to assume that that the virtual clock has been operating at a constant rate for a period of time. Furthermore, we will also need to assume that overload does not occur in the recovery interval in order to make guarantees, even though unexpected overload could continue to occur even after t_s. Therefore, we define the start of the recovery interval, denoted t_r, as the earliest time that satisfies all of the following properties.

Property 7. *If any $\tau_{i,k}$ is pending at t_r, then $y_{i,k} \geq t_s$.*

Property 8. *Each task τ_i has a constant $C_i \leq T_i$ such that for any $\tau_{i,k}$, if $t^c_{i,k} \geq t_r$, then $e_{i,k} \leq C_i$.*

Property 9. *For each P_p, there is some constant σ_p such that if $t_r \leq t_0 \leq t_1$, then $o_p(t_0, t_1) \leq \widehat{u_p}\sigma_p$.*

Property 8 states that C_i is the worst-case execution time for any job of τ_i that influences our analysis within $I_r \cup I_n$. Property 9 eliminates some of the generality of our supply model from t_r onward, so that our supply model becomes identical to that used in Leontyev and Anderson (2010) holds from t_r onward, in $I_r \cup I_n$. In light of Property 8, we define a task's *base utilization* (with respect to virtual time)

$$U^v_i = \frac{C_i}{T_i} \tag{26}$$

and its I_r utilization (with respect to actual time in I_r)

$$U_i^r = U_i^v \cdot s_r. \tag{27}$$

Observe that the utilization of τ_i with respect to actual time in I_n is simply U_i^v, because $s(t) = 1$ for all $t \in I_n$.

With these definitions in place, we formally define the extent of each interval.

$$I_o \triangleq [0, t_r), \tag{28}$$

$$I_r \triangleq [t_r, t_n), \tag{29}$$

$$I_n \triangleq [t_n, \infty). \tag{30}$$

If we can guarantee that $x_i(t) = x_i^s(1)$ is x-sufficient for $t \in I_n$ under Properties 5–9, then we define a *dissipation bound* as the length of I_r, i.e., $t_n - t_r$.

Whenever $s(t)$ remains constant over an interval (as it does over I_r and I_n), it is possible to correctly choose $x_i(t)$ such that it asymptotically approaches a constant value. We will below define this (task-dependent) constant value as $x_i^s(s_I)$, where s_I is the constant value of $s(t)$ (s_r in I_r and 1 in I_n). We will then define a task-independent function $\Delta(t)$ that guarantees that $x_i(t) = x_i^s(s_r) + \Delta(t)$ is x-sufficient for every τ_i and time $t \in I_r$. $\Delta(t)$ is graphed in Figure 8.

Recall that, in Section 3, L was arbitrary for each $\tau_{i,k}$. Our analysis will require us to choose a particular L for each *task*, so in Section 4.1 below, we discuss how to make this choice. In Section 4.2, we then turn our attention to formally defining $x_i^s(s_I)$ and $\Delta(t)$. Then, in Section 4.3, we formally prove that $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient for $t_a \in I_r$. In Section 4.4 we then upper bound t_n. Finally, in Section 4.5, we formally prove that $x_i(t) = x_i^s(1)$ is x-sufficient for $t \in I_n$.

4.1 Choosing L

In Section 3, L was arbitrary for any $\tau_{i,k}$. In this subsection, we will choose a specific per-task L_i that will take the place of L in several of our bounds. Because L_i will appear in our definition of $x_i^s(s_I)$, we first describe its selection here. We will then define $x_i^s(s_I)$ and $\Delta(t)$ in Section 4.2.

The choice of L appears in the definition of $x_{i,k}^m$ in (23), in the term $Le_{i,k}^p$, and in the definition of $x_{i,k}^f$ in (15), in the argument to $A_{i,k}(m-L-1)$. In order to analyze $x_{i,k}^f$, we first upper bound $A_{i,k}(m-L-1)$ in the case that will be relevant to our choice of $x_i^s(s_I)$. The following lemma does so, using arbitrary $v = m - L - 1$ to match the notation used in Lemma 2.

Lemma 8. *Let v be an integer with $0 \leq v < m$, and let*

$$A_i^{rn}(v) \triangleq \begin{cases} \frac{C_i + \sum_{P_p \in \Theta_{rn}} \widehat{u_p}\sigma_p}{1 - v + \sum_{P_p \in \Theta_{rn}} \widehat{u_p}} & \text{If } 1 - v + \sum_{P_p \in \Theta_{rn}} \widehat{u_p} > 0 \\ \infty & \text{Otherwise,} \end{cases} \tag{31}$$

(for I_r and I_n) where Θ_{rn} is the set of v processors that minimizes $A_i^{rn}(v)$. Then, if $k \geq 0$ and $t_{i,k-1}^c > t_r$. $A_{i,k}(v) \leq A_i^{rn}(v)$ and $A_{i,k}(v) - e_{i,k} \leq A_i^{rn}(v) - C_i$.

Proof. If $A_i^{rn}(v) = \infty$, then the lemma holds. Furthermore, if $A_{i,k}(v) = \infty$, then by (13), for any choice of v processors Θ, $1 - v + \sum_{P_p \in \Theta} \widehat{u_p} \leq 0$. Therefore, $A_i^{rn}(v) = \infty$, and the lemma holds. Thus, we assume that $A_i^{rn}(v)$ is finite, implying by (31) that

$$1 - v + \sum_{P_p \in \Theta_{rn}} \widehat{u_p} > 0, \tag{32}$$

and that $A_{i,k}(v)$ is finite, implying by (13) that

$$1 - v + \sum_{P_p \in \Theta} \widehat{u_p} > 0. \tag{33}$$

Additionally,

$$\begin{aligned} 1 - v + \sum_{P_p \in \Theta_{rn}} \widehat{u_p} &\leq \{\text{Because each } \widehat{u_p} \leq 1\} \\ & 1 - v + \sum_{P_p \in \Theta_{rn}} 1 \\ &= \{\text{Because there are } v \text{ processors in } \Theta_{rn}\} \\ & 1. \end{aligned} \tag{34}$$

We have

$$
\begin{aligned}
A_i^{rn}(v) &= \{\text{By the definition of } A_i^{rn}(v) \text{ in (31) and by (32)}\} \\
&\frac{C_i + \sum_{P_p \in \Theta_{rn}} \widehat{u_p}\sigma_p}{1 - v + \sum_{P_p \in \Theta_{rn}} \widehat{u_p}} \\
&\geq \{\text{By Property 9 and (32)}\} \\
&\frac{C_i + \sum_{P_p \in \Theta_{rn}} o_i(t_{i,k-1}^c, t_{i,k}^c)}{1 - v + \sum_{P_p \in \Theta_{rn}} \widehat{u_p}} \\
&\geq \{\text{By Property 8 and (32)}\} \\
&\frac{e_{i,k} + \sum_{P_p \in \Theta_{rn}} o_i(t_{i,k-1}^c, t_{i,k}^c)}{1 - v + \sum_{P_p \in \Theta_{rn}} \widehat{u_p}} \\
&\geq \{\text{Because } \Theta \text{ (as defined in Lemma 2) is chosen to minimize } A_{i,k}(v)\text{, and by (32)}\} \\
&A_{i,k}(v).
\end{aligned} \tag{35}
$$

Similarly,

$$
\begin{aligned}
A_i^{rn}(v) - C_i &= \{\text{By the definition of } A_i^{rn}(v) \text{ in (31) and by (32)}\} \\
&\frac{C_i + \sum_{P_p \in \Theta_{rn}} \widehat{u_p}\sigma_p}{1 - v + \sum_{P_p \in \Theta_{rn}} \widehat{u_p}} - C_i \\
&\geq \{\text{By Property 9 and (32)}\} \\
&\frac{C_i + \sum_{P_p \in \Theta_{rn}} o_i(t_{i,k-1}^c, t_{i,k}^c)}{1 - v + \sum_{P_p \in \Theta_{rn}} \widehat{u_p}} - C_i \\
&\geq \{\text{By Property 8 and (32) and (34)}\} \\
&\frac{e_{i,k} + \sum_{P_p \in \Theta_{rn}} o_i(t_{i,k-1}^c, t_{i,k}^c)}{1 - v + \sum_{P_p \in \Theta_{rn}} \widehat{u_p}} - e_{i,k} \\
&\geq \{\text{Because } \Theta \text{ (as defined in Lemma 2) is chosen to minimize } A_{i,k}(v)\text{, and by (32)}\} \\
&A_{i,k}(v) - e_{i,k}.
\end{aligned} \tag{36}
$$

□

We now define our choice of L_i.

Definition 12. For each τ_i, L_i is the smallest integer such that $0 \leq L_i < m$ and $A_i^{rn}(m - L_i - 1) \leq T_i$.

Such an integer must exist, because

$$\begin{aligned}
A_i^{rn}(m - (m-1) - 1) &= \{\text{Rearranging}\} \\
&\quad A_i^{rn}(0) \\
&= \{\text{By the definition of } A_i^{rn}(0) \text{ in (31)}\} \\
&\quad C_i \\
&\leq \{\text{By Property 8}\} \\
&\quad T_i.
\end{aligned}$$

4.2 Defining $x_i^s(s_I)$ and $\Delta(t)$

In this subsection, we define $x_i^s(s_I)$ and $\Delta(t)$. We will prove in Section 4.3 below that they can be used to obtain x-sufficient bounds.

We first define $x_i^s(s_I)$. Its definition is implicit — $x_i^s(s_I)$ appears on both sides of (37) below. In Appendix B, we discuss how to use linear programming to determine the specific value of $x_i^s(s_I)$ if it exists. In Appendix B, we also show that if $x_i^s(1)$ exists, then $x_i^s(s_I)$ must exist for all $s_I \leq 1$.

Definition 13.

$$\begin{aligned}
x_i^s(s_I) \triangleq \max\Bigg\{0, \Bigg(&\sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^s(s_I) - S_j) + \sum_{\tau_j \in \tau} S_j + (m - u_{tot} - 1)C_i \\
&+ O^{rn} + L_i \cdot U_i^v \cdot s_I \cdot x_i^s(s_I)\Bigg)/u_{tot}\Bigg\}. \qquad (37)
\end{aligned}$$

where

$$S_i \triangleq C_i\left(1 - \frac{Y_i}{T_i}\right), \qquad (38)$$

and

$$O^{rn} \triangleq \sum_{P_p \in P} \widehat{u_p}\sigma_p \qquad (39)$$

(for intervals I_r and I_n).

In order to prove that $x_i(t_a) = x_i^s(1)$ is x-sufficient for $t_a \in I_n$, it is necessary that $x_i^s(1)$ exist. In Appendix B, we show that this occurs if the provided linear program is feasible, and that the following condition is sufficient for feasibility.

Property 10.

$$\sum_{m-1 \text{ largest}} U_j^v + \max_{\tau_i \in \tau} L_i \cdot U_i^v < u_{tot}.$$

If Property 10 is not satisfied, then the methods provided in this paper cannot provide dissipation bounds. Furthermore, our analysis also assumes the following property, without which bounded response times cannot be guaranteed even in the absence of overload.

Property 11.

$$\sum_{\tau_j \in \tau} U_j^v \leq u_{tot}.$$

We next define $\Delta(t)$. The definition of $\Delta(t)$ uses several upper bounds of quantities from Section 3. We will justify the correctness of these upper bounds in Section 4.3. We will describe each segment of $\Delta(t)$, as depicted in Figure 8, from left to right. We will provide necessary definitions as we proceed.

Observe in Figure 8 that for $t \leq t_r$, $\Delta(t)$ is constant. We will denote this constant value as λ, and we will define λ below. First, we describe a function closely related to $x_i(t)$ that will be used in defining λ. Observe in Definition 8 that the provided equation must hold for all $\tau_{i,k}$ with $y_{i,k} \leq t$. We define $\dot{x}_i(t)$ by changing this precondition to a strict inequality, in order to handle an edge case in our analysis.

Definition 14. $\dot{x}_i(t)$ is *$\dot{x}$-sufficient* if $\dot{x}_i(t) \geq 0$ and for all $\tau_{i,k}$ with $y_{i,k} < t$,

$$t_{i,k}^c \leq t + \dot{x}_i(t) + e_{i,k}.$$

With this definition in place, we now define λ.

$$\lambda \triangleq \max\Big\{ \max_{\tau_i \in \tau} \dot{x}_i(t_r) - x_i^s(s_r) + A_i^{rn}(m - L_i - 1),$$

$$\delta,$$

$$\max_{\tau_{i,k}\in\psi}\left(\frac{W^o_{i,k}-R^o_{i,k}+(m-u_{tot}-1)e_{i,k}+O^o_{i,k}+O^{rn}+L_i\cdot U^r_i\cdot x^s_i(s_r)}{u_{tot}-L_i\cdot U^r_i}\right),$$

$$\max_{\tau_{i,k}\in\kappa}(x_i(y_{i,k})-x^s_i(s_r)),$$

$$0\Big\}, \tag{40}$$

where each $\dot{x}_i(t_r)$ is $\dot{x}$-sufficient,

$$\delta \triangleq \min_{\tau_i\in\tau} x^s_i(1)-x^s_i(s_r), \tag{41}$$

ψ is the set of jobs with $y_{i,k}\in I_r\cup I_n$ and $t^b_{i,k}\in I_o$, κ is the set of jobs with $y_{i,k}\in I_o$ and $t^c_{i,k}\in I_o\cup I_r$, each $x_i(y_{i,k})$ is x-sufficient,

$$W^o_{i,k}\triangleq W_{i,k}-\sum_{\tau_j\in\tau}D^e_j(t_r,y_{i,k})+\sum_{\tau_j\in\tau}S_j \tag{42}$$

(for jobs with $t^b_{i,k}\in I_o$), and

$$R^o_{i,k}\triangleq u_{tot}\cdot(t_r-t^b_{i,k}) \tag{43}$$

$$O^o_{i,k}\triangleq\sum_{P_p\in P}o_p(t^b_{i,k},t_r) \tag{44}$$

(each for I_o).

Observe in Figure 8 that, from t_r to t_e (switch to *e*xponential), $\Delta\left(t\right)$ is linear. We define this segment as its own function

$$\Delta^\ell\left(t\right)\triangleq\phi\cdot(t-t_r)+\lambda \tag{45}$$

(*l*inear), where

$$\phi\triangleq\max\left\{\max_{\tau_j\in\tau}\left(\frac{s_r\cdot A^{rn}_j(m-L_j-1)}{T_i}-1\right),\frac{\sum_{\tau_j\in\tau}U^r_j-u_{tot}}{u_{tot}}\right\}. \tag{46}$$

As can be seen in Figure 8, from t_e onward, $\Delta\left(t\right)$ decays exponentially. We will also define this segment as its own function,

$$\Delta^e\left(t\right)\triangleq\Delta^\ell\left(t_e\right)\cdot q^{\frac{t-t_e}{\rho}} \tag{47}$$

A. $t_a < y_{i,0}$ (Lemma 14).

B. $t_a = y_{i,k}$ for some k and $t^c_{i,k} \leq y_{i,k} + e_{i,k}$ (Lemma 15).

C. $t_a \in (y_{i,k}, y_{i,k+1})$ for some k (Lemma 21).

D. $t_a = y_{i,k}$ for some k, $t^c_{i,k} > y_{i,k} + e_{i,k}$, and $\tau_{i,k}$ is f-dominant for L_i (Lemma 25).

E. $t_a = y_{i,k}$ for some k, $t^c_{i,k} > y_{i,k} + e_{i,k}$, and $\tau_{i,k}$ is m-dominant for L_i (Lemma 49).

Figure 9: Cases considered when proving that $x_i(t_a) = x^s_i(s_r) + \Delta(t_a)$ is x-sufficient for $t_a \in I_r$

(*e*xponential), where

$$t_e \triangleq \begin{cases} t_r & \text{If } \lambda \leq \phi \cdot \frac{\rho}{\ln q} \\ t_r + \frac{\rho}{\ln q} - \frac{\lambda}{\phi} & \text{Otherwise} \end{cases} \tag{48}$$

(switch to *e*xponential),

$$q \triangleq \frac{\sum_{m-1 \text{ largest}} U^r_j + \max_{\tau_j \in \tau} L_j \cdot U^v_j \cdot s_r}{u_{tot}}, \tag{49}$$

and

$$\rho \triangleq \max_{\tau_j \in \tau} (x^s_j(s_r) + \lambda + C_j). \tag{50}$$

Finally, we fully define $\Delta(t)$ for all t.

$$\Delta(t) = \begin{cases} \lambda & \text{If } t \in (-\infty, t_r) \\ \Delta^\ell(t) & \text{If } t \in [t_r, t_e) \\ \Delta^e(t) & \text{If } t \in [t_e, \infty). \end{cases} \tag{51}$$

4.3 Proving that $x_i(t_a) = x^s_i(s_r) + \Delta(t)$ is x-sufficient for $t_a \in I_r$

In this subsection, we provide a x-sufficient choice of $x_i(t_a)$ for each τ_i and $t_a \in I_r$. For each such combination of t_a and τ_i, we will exhaustively consider the cases depicted in Figure 9 to show that $x_i(t_a) = x^s_i(s_r) + \Delta(t_a)$ is x-sufficient. We will prove this result by induction on its correctness for smaller choices of t_a.

Furthermore, a necessary condition in Definition 8 for $x_i(t)$ to be x-sufficient is that $x_i(t) \geq 0$. Lemma 13 below establishes that this is the case for $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ for arbitrary t_a. By the definition of $x_i^s(s_r)$ in (37), $x_i^s(s_r)$ is nonnegative. Therefore, showing that $\Delta(t_a)$ is also nonnegative for arbitrary t_a will be sufficient to prove Lemma 13.

By the definition of $\Delta(t_a)$ in (51), we will consider the three intervals $(-\infty, t_r)$, $[t_r, t_e)$, and $[t_e, \infty)$. $\Delta(t_a)$ is nonnegative for $t_a \in (-\infty, t_r)$, because by the definition of λ in (40), $\lambda \geq 0$. We thus consider $t_a \in [t_r, t_e)$. In order to prove that $\Delta(t_a)$ is nonnegative in this case, we will show (in Lemma 9) that $\Delta^\ell(t)$ is decreasing over $[t_r, t_e)$, and (in Lemma 12) that $\Delta^\ell(t_e)$ is nonnegative. The result that $\Delta(t_e)$ is nonnegative will also be used to prove that $\Delta(t_a)$ is nonnegative for $t_a \in [t_e, \infty)$.

By the definition of $\Delta^\ell(t)$ in (45), $\Delta^\ell(t)$ is decreasing if and only if $\phi < 0$. We now prove that this is implied by Property 11.

Lemma 9. $\phi < 0$.

Proof. By the definition of ϕ in (46), either $\phi = \frac{s_r \cdot A_j^{rn}(m-L_j-1)}{T_j} - 1$ for some τ_j, or $\phi = \frac{\sum_{\tau_j \in \tau} U_j^r - u_{tot}}{u_{tot}}$. We consider each of these cases.

Case 1: $\phi = \frac{s_r \cdot A_j^{rn}(m-L_j-1)}{T_j} - 1$ **for some** τ_j**.** In this case, we have

$$\begin{aligned}\phi &= \frac{s_r \cdot A_j^{rn}(m-L_j-1)}{T_j} - 1 \\ &\leq \{\text{By the definition of } L_j \text{ in Definition 12}\} \\ &\quad \frac{s_r \cdot T_j}{T_j} - 1 \\ &= \{\text{Cancelling}\} \\ &\quad s_r - 1 \\ &< \{\text{By Property 5}\} \\ &\quad 0.\end{aligned}$$

Case 2: $\phi = \frac{\sum_{\tau_j \in \tau} U_j^r - u_{tot}}{u_{tot}}$**.** In this case, we have

$$\phi = \frac{\sum_{\tau_j \in \tau} U_j^r - u_{tot}}{u_{tot}}$$

$$
\begin{aligned}
&= \{\text{By the definition of } U_j^r \text{ in (27)}\} \\
&\quad \frac{\sum_{\tau_j \in \tau}(U_j^v \cdot s_r) - u_{tot}}{u_{tot}} \\
&< \{\text{By Property 5}\} \\
&\quad \frac{\sum_{\tau_j \in \tau} U_j^v - u_{tot}}{u_{tot}} \\
&\leq \{\text{By Property 11}\} \\
&\quad \frac{u_{tot} - u_{tot}}{u_{tot}} \\
&= \{\text{Simplifying}\} \\
&\quad 0.
\end{aligned}
$$

□

We need to show that $\Delta^{\ell}(t_e)$ is nonnegative. By the definition of t_e in (48), the value of t_e is dependent on $\ln q$. Thus, we first characterize the value of q.

Lemma 10. $0 < q < 1$.

Proof. We first show that $0 < q$. All variables that appear in the definition of q in (49) are nonnegative, and U_j^r for each τ_j is strictly positive. Therefore, $0 < q$.

We now show that $q < 1$. We have

$$
\begin{aligned}
q &= \{\text{By the definition of } q \text{ in (49)}\} \\
&\quad \frac{\sum_{m-1 \text{ largest}} U_j^r + \max_{\tau_j \in \tau} L_j \cdot U_j^v \cdot s_r}{u_{tot}} \\
&= \{\text{By the definition of } U_j^r \text{ in (27)}\} \\
&\quad \frac{\sum_{m-1 \text{ largest}}(U_j^v \cdot s_r) + \max_{\tau_j \in \tau} L_j \cdot U_j^v \cdot s_r}{u_{tot}} \\
&< \{\text{By Property 5}\} \\
&\quad \frac{\sum_{m-1 \text{ largest}} U_j^v + \max_{\tau_j \in \tau} L_j \cdot U_j^v}{u_{tot}} \\
&\leq \{\text{By Property 10}\} \\
&\quad \frac{u_{tot}}{u_{tot}}
\end{aligned}
$$

$$= \{\text{Simplifying}\}$$
$$1.$$

□

Also by the definition of t_e in (48), the value of t_e is also dependent on ρ. Thus, we also characterize the value of ρ.

Lemma 11. $\rho > 0$.

Proof. We have

$$\begin{aligned}
\rho &= \{\text{By the definition of } \rho \text{ in (50)}\} \\
&\quad \max_{\tau_j \in \tau}(x_j^s(s_r) + \lambda + C_j) \\
&\geq \{\text{By the definition of } x_j^s(s_r) \text{ in (37) and the definition of } \lambda \text{ in (40)}\} \\
&\quad \max_{\tau_j \in \tau}(C_j) \\
&> \{\text{Because each } C_j > 0\} \\
&\quad 0.
\end{aligned}$$

□

We finally show that $\Delta^{\ell}(t_e)$ is nonnegative. Furthermore, the value of $\Delta^{\ell}(t_e)$ and the identical values of $\Delta(t_e)$ and $\Delta^{e}(t_e)$ are used in later proofs. For convenience, we consider all of these terms in a single lemma.

Lemma 12.

$$\Delta(t_e) = \Delta^{e}(t_e) = \Delta^{\ell}(t_e) = \min\left\{\lambda, \phi \cdot \frac{\rho}{\ln q}\right\} \geq 0.$$

Proof. We will demonstrate the equalities in the order they appear in the statement of the lemma.

First, we have

$$\begin{aligned}
\Delta(t_e) &= \{\text{By the definition of } \Delta(t_e) \text{ in (51)}\} \\
&\quad \Delta^{e}(t_e)
\end{aligned}$$

$$
\begin{aligned}
&= \{\text{By the definition of } \Delta^{e}\left(t_e\right) \text{ in (47)}\} \\
&\quad \Delta^{\ell}\left(t_e\right) \cdot q^{\frac{t_e - t_e}{\rho}} \\
&= \{\text{Simplifying}\} \\
&\quad \Delta^{\ell}\left(t_e\right).
\end{aligned}
$$

We now establish that $\Delta^{\ell}\left(t_e\right) = \min\left\{\lambda, \phi \cdot \frac{\rho}{\ln q}\right\}$ by considering two cases.

Case 1: $\lambda \leq \phi \cdot \frac{\rho}{\ln q}$**.** In this case,

$$
\begin{aligned}
\Delta^{\ell}\left(t_e\right) &= \{\text{By the definition of } t_e \text{ in (48)}\} \\
&\quad \Delta^{\ell}\left(t_r\right) \\
&= \{\text{By the definition of } \Delta^{\ell}\left(t_r\right) \text{ in (45)}\} \\
&\quad \phi \cdot \left(t_r - t_r\right) + \lambda \\
&= \{\text{Simplifying}\} \\
&\quad \lambda \\
&= \{\text{By the case we are considering}\} \\
&\quad \min\left\{\lambda, \phi \cdot \frac{\rho}{\ln q}\right\}.
\end{aligned}
$$

Case 2: $\lambda > \phi \cdot \frac{\rho}{\ln q}$**.** In this case,

$$
\begin{aligned}
\Delta^{\ell}\left(t_e\right) &= \{\text{By the definition of } t_e \text{ in (48)}\} \\
&\quad \Delta^{\ell}\left(t_r + \frac{\rho}{\ln q} - \frac{\lambda}{\phi}\right) \\
&= \{\text{By the definition of } \Delta^{\ell}\left(t_r + \tfrac{\rho}{\ln q} - \tfrac{\lambda}{\phi}\right) \text{ in (45)}\} \\
&\quad \phi \cdot \left(t_r + \frac{\rho}{\ln q} - \frac{\lambda}{\phi} - t_r\right) + \lambda \\
&= \{\text{Simplifying}\} \\
&\quad \phi \cdot \frac{\rho}{\ln q} \\
&= \{\text{By the case we are considering}\}
\end{aligned}
$$

$$\min\left\{\lambda, \phi \cdot \frac{\rho}{\ln q}\right\}.$$

Finally, we demonstrate that $\min\left\{\lambda, \phi \cdot \frac{\rho}{\ln q}\right\} \geq 0$. By the definition of λ in (40), $\lambda \geq 0$. Furthermore, because $\phi < 0$ by Lemma 9, $0 < q < 1$ by Lemma 10, and $\rho > 0$ by Lemma 11, $\phi \cdot \frac{\rho}{\ln q} > 0$. Therefore, $\min\left\{\lambda, \phi \cdot \frac{\rho}{\ln q}\right\} \geq 0$. □

We are now ready to establish that $x_i^s(s_r) + \Delta(t_a) \geq 0$. By Definition 8, this is a necessary condition for $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ to be x-sufficient.

Lemma 13. *For all* t_a, $x_i^s(s_r) + \Delta(t_a) \geq 0$.

Proof. We first establish that $\Delta(t_a) \geq 0$. We consider three cases, depending on the value of t_a.

Case 1: $t_a \in (-\infty, t_r)$**.** In this case,

$$\begin{aligned}
\Delta(t_a) &= \{\text{By the definition of } \Delta(t_a) \text{ in (51)}\} \\
&\quad \lambda \\
&\geq \{\text{By the definition of } \lambda \text{ in (40)}\} \\
&\quad 0.
\end{aligned}$$

Case 2: $t_a \in [t_r, t_e)$**.** In this case,

$$\begin{aligned}
\Delta(t_a) &= \{\text{By the definitions of } \Delta(t_a) \text{ in (51) and of } \Delta^\ell(t_a) \text{ in (45)}\} \\
&\quad \phi \cdot (t_a - t_r) + \lambda \\
&= \{\text{Rearranging}\} \\
&\quad \phi \cdot (t_e - t_r) + \lambda + \phi \cdot (t_a - t_e) \\
&= \{\text{By the definition of } \Delta^\ell(t_e) \text{ in (45)}\} \\
&\quad \Delta^\ell(t_e) + \phi \cdot (t_a - t_e) \\
&> \{\text{Because } \phi < 0 \text{ by Lemma 9 and } t_a < t_e\} \\
&\quad \Delta^\ell(t_e) \\
&\geq \{\text{By Lemma 12}\}
\end{aligned}$$

0.

Case 3: $t_a \in [t_e, \infty)$**.** In this case,

$$
\begin{aligned}
\Delta(t_a) &= \{\text{By the definitions of } \Delta(t_a) \text{ in (51) and of } \Delta^e(t_a) \text{ in (47)}\} \\
&\quad \Delta^\ell(t_e) \cdot q^{\frac{t_a - t_e}{\rho}} \\
&\geq \{\text{Because } \Delta^\ell(t_e) \geq 0 \text{ by Lemma 12 and } q > 0 \text{ by Lemma 10}\} \\
&\quad 0.
\end{aligned}
$$

In any of the above cases, because $x_i^s(s_r) \geq 0$ by the definition of $x_i^s(s_r)$ in (37), $x_i^s(s_r) + \Delta(t_a) \geq 0$. □

We now show that $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient by considering all the cases depicted in Figure 9, which match those in Figure 3 in Section 3.

We first consider Case A in Figure 9, in which $t_a < y_{i,0}$.

Lemma 14. *If $t_a < y_{i,0}$, then $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient.*

Proof. If $t_a < y_{i,0}$, then by Theorem 1, $x_i(t_a) = 0$ is x-sufficient. Furthermore, by Lemma 13, $x_i^s(s_r) + \Delta(t_a) \geq 0$. Therefore, by Property 3 with $c_0 = 0$ and $c_1 = x_i^s(s_r) + \Delta(t_a)$, $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient. □

The analysis of Case B in Figure 9 is simple.

Lemma 15. *If $t_a = y_{i,k}$ for some k and $t_{i,k}^c \leq y_{i,k} + e_{i,k}$, then $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient.*

Proof. The lemma follows immediately from Theorem 2, Lemma 13, and Property 3 with $c_0 = 0$ and $c_1 = x_i^s(s_r) + \Delta(t_a)$. □

We next consider Case C in Figure 9, in which $t_a \in (y_{i,k}, y_{i,k+1})$ for some k. For this case, as well as most subsequent cases, correctness is proved from the fact that the value of $\Delta(t)$ decreases sufficiently slowly as t increases. Therefore, we will prove Lemma 20 below, which explicitly bounds the decrease between two values of $\Delta(t)$. Our analysis will be based on the following form of the Fundamental Theorem of Calculus (FTC).

Fundamental Theorem of Calculus (FTC). *If $f(t)$ is continuous over $[t_0, t_1]$ and $f'(t)$ is the derivative of $f(t)$ at all but finitely many points within $[t_0, t_1]$, then*

$$f(t_1) = f(t_0) + \int_{t_0}^{t_1} f'(t)\, dt.$$

In order to use the FTC, we will demonstrate in Lemma 17 below that $\Delta(t)$ is continuous over all real numbers, and in Lemma 19 below we will provide a function $\Delta'(t)$ that is equal to the derivative of $\Delta(t)$ at all but finitely many points. Lemma 19 also provides bounds on $\Delta'(t)$ that are used to prove Lemma 20. In several parts of our analysis throughout this section, the value of $\Delta(t_r)$ and/or $\Delta^\ell(t_r)$ will be used. For convenience, we provide this value now as a separate lemma.

Lemma 16. $\Delta(t_r) = \Delta^\ell(t_r) = \lambda$.

Proof. We first establish that $\Delta(t_r) = \Delta^\ell(t_r)$. By (48), either $t_r = t_e$ or $t_r < t_e$. We consider each of these two cases.

Case 1: $t_e = t_r$**.** In this case,

$$\begin{aligned}
\Delta(t_r) &= \Delta(t_e) \\
&= \{\text{By Lemma 12}\} \\
&\quad \Delta^\ell(t_e) \\
&= \{\text{Because } t_e = t_r\} \\
&\quad \Delta^\ell(t_r).
\end{aligned}$$

Case 2: $t_r < t_e$**.** In this case, by the definition of $\Delta(t_r)$ in (51), $\Delta(t_r) = \Delta^\ell(t_r)$.

To conclude the proof, note that

$$\begin{aligned}
\Delta^\ell(t_r) &= \{\text{By the definition of } \Delta^\ell(t_r) \text{ in (45)}\} \\
&\quad \phi \cdot (t_r - t_r) + \lambda
\end{aligned}$$

$$= \{\text{Simplifying}\}$$

$$\lambda.$$

□

The following property is a standard result in real analysis. (If $t_2 \leq t_1$, then it holds vacuously.)

Property 12. *For arbitrary t_0, t_1, t_2, and continuous functions $f(t)$ and $g(t)$, if $t_1 > t_0$, $\Delta(t) = f(t)$ for $t \in (t_0, t_1)$, $\Delta(t) = g(t)$ for $t \in [t_1, t_2)$, and $f(t_1) = g(t_1)$, then $\Delta(t)$ is continuous over $[t_1, t_2)$.*

We now use this property to prove that $\Delta(t)$ is continuous over the reals, so that we can use the FTC with $f(t) = \Delta(t)$.

Lemma 17. *$\Delta(t)$ is continuous over all real numbers.*

Proof. We first observe that, by the definition of $\Delta(t)$ in (51), $\Delta(t)$ is constant (and therefore continuous) over $(-\infty, t_r)$.

To prove that $\Delta(t)$ is continuous over $[t_r, \infty)$, we consider two cases, depending on the relationship between λ and $\phi \cdot \frac{\rho}{\ln q}$.

Case 1: $\lambda \leq \phi \cdot \frac{\rho}{\ln q}$. We will use Property 12 with $t_0 = -\infty$, $t_1 = t_r$, $t_2 = \infty$, $f(t) = \lambda$, and $g(t) = \Delta^e(t)$. It is trivially the case that $t_1 > t_0$.

In this case, by the definition of t_e in (48),

$$t_r = t_e. \tag{52}$$

Therefore, by the definition of $\Delta(t)$ in (51), $\Delta(t) = \lambda = f(t)$ for $t \in (t_0, t_1) = (-\infty, t_r)$ and $\Delta(t) = \Delta^e(t) = g(t)$ for $t \in [t_1, t_2) = [t_r, \infty)$, as desired.

$f(t)$ is continuous because it is constant. $g(t)$ is continuous by the definition of $\Delta^e(t)$ in (47), because exponential functions are continuous.

Finally, we show that

$$g(t_1) = \Delta^e(t_r)$$

$$= \{\text{By (52)}\}$$

$$
\begin{aligned}
&\Delta^e(t_e) \\
&= \{\text{By Lemma 12 and the case we are considering}\} \\
&\quad \lambda \\
&= f(t_1).
\end{aligned} \tag{53}
$$

Therefore, all the preconditions for Property 12 are met, so $\Delta(t)$ is continuous over $[t_r, \infty)$.

Case 2: $\lambda \leq \phi \cdot \frac{\rho}{\ln q}$**.** In this case, by the definition of t_e in (48), we have

$$
t_r < t_e. \tag{54}
$$

We first prove that $\Delta(t)$ is continuous over $[t_r, t_e)$, using Property 12 with $f(t) = \lambda$, $g(t) = \Delta^\ell(t)$, $t_0 = -\infty$, $t_1 = t_r$, and $t_2 = t_e$. It is trivially the case that $t_1 > t_0$. By the definition of $\Delta(t)$ in (51), $\Delta(t) = \lambda = f(t)$ for $t \in (t_0, t_1) = (-\infty, t_r)$ and $\Delta(t) = \Delta^\ell(t) = g(t)$ for $t \in [t_1, t_2) = [t_r, t_e)$. $f(t)$ is continuous because it is constant. $g(t)$ is continuous by the definition of $\Delta^\ell(t)$ in (45), because linear functions are continuous. Furthermore,

$$
\begin{aligned}
g(t_1) &= \Delta^\ell(t_r) \\
&= \{\text{By Lemma 16}\} \\
&\quad \lambda \\
&= f(t_1).
\end{aligned}
$$

Therefore, all the preconditions for Property 12 are met, so $\Delta(t)$ is continuous over $[t_r, t_e)$.

We next prove that $\Delta(t)$ is continuous over $[t_e, \infty)$ using Property 12 with $f(t) = \Delta^\ell(t)$, $g(t) = \Delta^e(t)$, $t_0 = t_r$, $t_1 = t_e$, and $t_2 = \infty$. By (54), $t_1 > t_0$. By the definition of $\Delta(t)$ in (51), $\Delta(t) = \Delta^\ell(t) = f(t)$ for $t \in (t_0, t_1) \subset [t_r, t_e)$ and $\Delta(t) = \Delta^e(t) = g(t)$ for $t \in [t_1, t_2) = [t_e, \infty)$. $f(t)$ is continuous by the definition of $\Delta^\ell(t)$ in (45), because linear functions are continuous. $g(t)$ is continuous by the definition of $\Delta^e(t)$ in (47), because exponential functions are continuous.

Furthermore,

$$\begin{aligned}
g(t_1) &= \Delta^e(t_e) \\
&= \{\text{By Lemma 12}\} \\
&\quad \Delta^\ell(t_e) \\
&= f(t_1).
\end{aligned}$$

Therefore, all the preconditions for Property 12 are met, so $\Delta(t)$ is also continuous over $[t_e, \infty)$. We reasoned above that $\Delta(t)$ is continuous over $[t_r, t_e)$, so $\Delta(t)$ is continuous over $[t_r, \infty)$. □

In order to use the FTC, we must also provide a function $\Delta'(t)$ that is equal to the derivative of $\Delta(t)$ at all but finitely many points. Both for the purposes of determining $\Delta'(t)$, and for a later proof, we must reason about the derivative of $\Delta^e(t)$. The following lemma provides a necessary property of that derivative.

Lemma 18. *Let $\Delta^{e\prime}(t)$ be the derivative of $\Delta^e(t)$ with respect to t. If $t \geq t_e$, then $0 \geq \Delta^{e\prime}(t) \geq \phi$.*

Proof. If $t \geq t_e$, then we have

$$\begin{aligned}
\Delta^{e\prime}(t) &= \{\text{By the definition of } \Delta^e(t) \text{ in (47) and differentiation}\} \\
&\quad \frac{\ln q}{\rho} \cdot \Delta^\ell(t_e) \cdot q^{\frac{t-t_e}{\rho}} \\
&\geq \{\text{By Lemma 12, because } 0 < q < 1 \text{ by Lemma 10 so that } \ln q < 0\} \\
&\quad \frac{\ln q}{\rho} \cdot \phi \cdot \frac{\rho}{\ln q} \cdot q^{\frac{t-t_e}{\rho}} \\
&= \{\text{Rearranging}\} \\
&\quad \phi \cdot q^{\frac{t-t_e}{\rho}}. \qquad (55)
\end{aligned}$$

The lemma follows from (55), because $\phi < 0$ by Lemma 9, $0 < q < 1$ by Lemma 10, and $t \geq t_e$. □

We finally provide the derivative of $\Delta(t)$, except at $t = t_r$ and $t = t_e$ (finitely many points), and provide bounds on the resulting $\Delta'(t)$.

Lemma 19. *Let*

$$\Delta'(t) \triangleq \begin{cases} 0 & \textit{If } t \in (-\infty, t_r) \\ \phi & \textit{If } t \in [t_r, t_e) \\ \Delta^{e\prime}(t) & \textit{If } t \in [t_e, \infty). \end{cases} \tag{56}$$

$\Delta'(t)$ is the derivative of $\Delta(t)$ everywhere except at $t = t_r$ and at $t = t_e$. Furthermore, for all t, $\phi \leq \Delta'(t) \leq 0$.

Proof. We prove the lemma for arbitrary t, considering each of the three intervals that occur in the definition of $\Delta(t)$ in (51) and in the definition of $\Delta'(t)$ in (56).

Case 1: $t \in (-\infty, t_r)$**.** In this case, by the definition of $\Delta'(t)$ in (56), $\Delta'(t) = 0$. Therefore, by Lemma 9, $\phi < \Delta'(t) = 0$. By the definition of $\Delta(t)$ in (51), for $t \in (-\infty, t_r)$, $\Delta(t) = \lambda$. Thus, the derivative of $\Delta(t)$ at t is $\Delta'(t) = 0$.

Case 2: $t \in [t_r, t_e)$**.** In this case, by the definition of $\Delta'(t)$ in (56), $\Delta'(t) = \phi$. Therefore, by Lemma 9, $\Delta'(t) = \phi < 0$. By the definition of $\Delta(t)$ in (51), for $t \in [t_r, t_e)$, $\Delta(t) = \Delta^{\ell}(t)$. Therefore, by the definition of $\Delta^{\ell}(t)$ in (45), if $t \neq t_r$, then the derivative of $\Delta(t)$ at t is $\phi = \Delta'(t)$.

Case 3: $t \in [t_e, \infty)$**.** In this case, by the definition of $\Delta'(t)$ in (56), $\Delta'(t) = \Delta^{e\prime}(t)$. By Lemma 18, $\phi \leq \Delta^{e\prime}(t) < 0$. Therefore, $\phi \leq \Delta'(t) < 0$. By the definition of $\Delta(t)$ in (51), for $t \in [t_e, \infty)$, $\Delta(t) = \Delta^{e}(t)$. Therefore, by the definition of $\Delta^{e}(t)$ in (47) and the definition of $\Delta^{e\prime}(t)$ as the derivative of $\Delta^{e}(t)$, if $t \neq t_e$, then the derivative of $\Delta(t)$ at t is $\Delta^{e\prime}(t) = \Delta'(t)$. □

We can now provide bounds on the value of $\Delta(t_1)$ relative to $\Delta(t_0)$ for arbitrary $t_0 \leq t_1$, as required for the proof of Lemma 21 and several later lemmas.

Lemma 20. *For arbitrary $t_0 \leq t_1$,*

$$\Delta(t_0) \geq \Delta(t_1) \geq \Delta(t_0) + \phi \cdot (t_1 - t_0).$$

Proof. We have

$$\Delta(t_1) = \{\text{By the FTC with } f(t) = \Delta(t), \text{ and by Lemmas 17 and 19}\}$$

$$\Delta(t_0) + \int_{t_0}^{t_1} \Delta'(t)\ dt$$

$$\leq \{\text{By Lemma 19}\}$$

$$\Delta(t_0) + \int_{t_0}^{t_1} 0\, dt$$

$$= \{\text{Rearranging}\}$$

$$\Delta(t_0).$$

Also,

$$\Delta(t_1) = \{\text{By the FTC with } f(t) = \Delta(t)\text{, and by Lemmas 17 and 19}\}$$

$$\Delta(t_0) + \int_{t_0}^{t_1} \Delta'(t)\ dt$$

$$\geq \{\text{By Lemma 19}\}$$

$$\Delta(t_0) + \int_{t_0}^{t_1} \phi\, dt$$

$$= \{\text{Rearranging}\}$$

$$\Delta(t_0) + \phi \cdot (t_1 - t_0).$$

□

We now provide the lemma that addresses Case C in Figure 9.

Lemma 21. *If $t_a \in I_r$, $t_a \in (y_{i,k}, y_{i,k+1})$ for some k, and $x_i(y_{i,\ell}) = x_i^s(s_r) + \Delta(y_{i,\ell})$ is x-sufficient for all $\tau_{i,\ell}$ such that $y_{i,\ell} \in [t_r, t_a)$, then $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient.*

Proof. Observe that $\tau_{i,k}$ is the last job of τ_i released prior to t_a. We consider two cases, depending on the location of $y_{i,k}$.

Case 1: $y_{i,k} < t_r$**.** Let $\dot{x}_i(t_r)$ be the value used in the definition of λ in (40). We have

$$t_{i,k}^c \leq \{\text{By the definition of } \dot{x}\text{-sufficient in Definition 14}\}$$

$$t_r + \dot{x}_i(t_r) + e_{i,k}$$

$$\leq \{\text{Rearranging}\}$$

$$
\begin{aligned}
& t_r + x_i^s(s_r) + \dot{x}_i(t_r) - x_i^s(s_r) + e_{i,k} \\
\leq & \{\text{By the definition of } \lambda \text{ in (40)}\} \\
& t_r + x_i^s(s_r) + \lambda + e_{i,k} \\
= & \{\text{By Lemma 16}\} \\
& t_r + x_i^s(s_r) + \Delta(t_r) + e_{i,k} \\
= & \{\text{Rearranging}\} \\
& t_a + x_i^s(s_r) + \Delta(t_r) - (t_a - t_r) + e_{i,k} \\
\leq & \{\text{By the definition of } \phi \text{ in (46)}\} \\
& t_a + x_i^s(s_r) + \Delta(t_r) + \phi \cdot (t_a - t_r) + e_{i,k} \\
\leq & \{\text{By Lemma 20 with } t_0 = t_r \text{ and } t_1 = t_a\} \\
& t_a + x_i^s(s_r) + \Delta(t_a) + e_{i,k}. \qquad (57)
\end{aligned}
$$

Therefore, by the definition of x-sufficient in Definition 8, $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient.

Case 2: $y_{i,k} \geq t_r$**.** We have

$$
\begin{aligned}
x_i^s(s_r) + \Delta(t_a) \geq & \{\text{By Lemma 20 with } t_0 = y_{i,k} \text{ and } t_1 = t_a\} \\
& x_i^s(s_r) + \Delta(y_{i,k}) + \phi \cdot (t_a - y_{i,k}) \\
\geq & \{\text{By the definition of } \phi \text{ in (46)}\} \\
& x_i^s(s_r) + \Delta(y_{i,k}) - (t_a - y_{i,k}).
\end{aligned}
$$

By the statement of the lemma, $x_i(y_{i,k}) = x_i^s(s_r) + \Delta(y_{i,k})$ is x-sufficient, so by Theorem 3, $x_i(t_a) = x_i^s(s_r) + \Delta(y_{i,k}) - (t_a - y_{i,k})$ is x-sufficient. Therefore, by Property 3 with $c_0 = x_i^s(s_r) + \Delta(y_{i,k}) - (t_a - y_{i,k})$ and $c_1 = x_i^s(s_r) + \Delta(t_a)$, $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient. □

We now consider Case D in Figure 9, where $t_a = y_{i,k}$ for some k, $t_{i,k}^c > y_{i,k} + e_{i,k}$, and $\tau_{i,k}$ is f-dominant for L_i. In this case, by the definition of f-dominant for L_i in Definition 10, $k > 0$, and thus $\tau_{i,k-1}$ exists. In Lemma 22, we consider the case that $y_{i,k-1} \in I_o$, and in Lemma 24, we consider the case that $y_{i,k-1} \in I_r$.

Lemma 22. *If $t_a \in I_r$, $t_a = y_{i,k}$ for some $\tau_{i,k}$, $\tau_{i,k}$ is f-dominant for L_i, and $y_{i,k-1} \in I_o$, then $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient.*

Proof. We have

$$
\begin{aligned}
& x_i^s(s_r) + \Delta(t_a) \\
= & \{\text{Because } t_a = y_{i,k}\} \\
& x_i^s(s_r) + \Delta(y_{i,k}) \\
\geq & \{\text{By Lemma 20 with } t_0 = t_r \text{ and } t_1 = y_{i,k}\} \\
& x_i^s(s_r) + \Delta(t_r) + \phi \cdot (y_{i,k} - t_r) \\
= & \{\text{By Lemma 16}\} \\
& x_i^s(s_r) + \lambda + \phi \cdot (y_{i,k} - t_r) \\
\geq & \{\text{By the definition of } \lambda \text{ in (40)}\} \\
& x_i^s(s_r) + \dot{x}_i(t_r) - x_i^s(s_r) + A_i^{rn}(m - L_i - 1) + \phi \cdot (y_{i,k} - t_r) \\
= & \{\text{Rearranging}\} \\
& t_r + \dot{x}_i(t_r) + e_{i,k-1} + A_i^{rn}(m - L_i - 1) - e_{i,k-1} - t_r + \phi \cdot (y_{i,k} - t_r) \\
\geq & \{\text{By the definition of } \dot{x}\text{-sufficient in Definition 14, because } y_{i,k-1} < t_r\} \\
& t_{i,k-1}^c + A_i^{rn}(m - L_i - 1) - e_{i,k-1} - t_r + \phi \cdot (y_{i,k} - t_r) \\
\geq & \{\text{By the definition of } \phi \text{ in (46)}\} \\
& t_{i,k-1}^c + A_i^{rn}(m - L_i - 1) - e_{i,k-1} - t_r + \left(\frac{s_r \cdot A_i^{rn}(m - L_i - 1)}{T_i} - 1 \right) \cdot (y_{i,k} - t_r) \\
= & \{\text{Rearranging}\} \\
& t_{i,k-1}^c - y_{i,k} + A_i^{rn}(m - L_i - 1) - e_{i,k-1} + \frac{s_r \cdot A_i^{rn}(m - L_i - 1)}{T_i} \cdot (y_{i,k} - t_r) \\
\geq & \{\text{Because } s_r \geq 0, A_i^{rn}(m - L_i - 1) > 0, T_i > 0, \text{ and } y_{i,k} > t_r\} \\
& t_{i,k-1}^c - y_{i,k} + A_i^{rn}(m - L_i - 1) - e_{i,k-1} \\
\geq & \{\text{By the definition of f-dominant for } L_i \text{ in Definition 10 and Property 8, because} \\
& t_{i,k-1}^c > y_{i,k} \geq t_r\} \\
& t_{i,k-1}^c - y_{i,k} + A_i^{rn}(m - L_i - 1) - C_i
\end{aligned}
$$

$$
\begin{aligned}
&\geq \{\text{By Lemma 8}\} \\
&\quad t^c_{i,k-1} - y_{i,k} + A_{i,k}(m - L_i - 1) - e_{i,k} \\
&= \{\text{By the definition of } x^f_{i,k} \text{ in (15)}\} \\
&\quad x^f_{i,k}. \qquad (58)
\end{aligned}
$$

Furthermore, by the preconditions of the lemma and Theorem 4, $x_i(t_a) = x^f_{i,k}$ is x-sufficient. Therefore, by (58) and Property 3 with $c_0 = x^f_{i,k}$ and $c_1 = x^s_i(s_r) + \Delta(t_a)$, the lemma holds. □

We next consider the case that $y_{i,k-1} \in I_r$ in Lemma 24. We will explicitly consider the difference between $y_{i,k-1}$ and $y_{i,k}$, based on the following lemma, which will also be used in Section 4.5.

Lemma 23. *For $k > 0$, $v(y_{i,k}) \geq v(y_{i,k-1}) + T_i$.*

Proof. We have

$$
\begin{aligned}
v(y_{i,k}) &= \{\text{By the definition of } Y_i \text{ in (6)}\} \\
&\quad v(r_{i,k}) + Y_i \\
&\geq \{\text{By the definition of } T_i \text{ in (5)}\} \\
&\quad v(r_{i,k-1}) + T_i + Y_i \\
&= \{\text{By the definition of } Y_i \text{ in (6)}\} \\
&\quad v(y_{i,k-1}) + T_i.
\end{aligned}
$$

□

We now consider Case D when $y_{i,k-1} \in I_r$.

Lemma 24. *If $t_a \in I_r$, $t_a = y_{i,k}$ for some $\tau_{i,k}$, $\tau_{i,k}$ is f-dominant for L_i, $y_{i,k-1} \in I_r$, and $x_i(y_{i,\ell}) = x^s_i(s_r) + \Delta(y_{i,\ell})$ is x-sufficient for all $\tau_{j,\ell}$ with $t_r \leq y_{i,\ell} < y_{i,k}$, then $x_i(t_a) = x^s_i(s_r) + \Delta(t_a)$ is x-sufficient.*

Proof. In Lemma 23 we considered the difference between $y_{i,k}$ and $y_{i,k-1}$ in *virtual* time. We now consider their difference in *actual* time.

$$
\begin{aligned}
(y_{i,k} - y_{i,k-1}) \cdot s_r &= \{\text{By Lemma 35}\} \\
&\quad v(y_{i,k}) - v(y_{i,k-1}) \\
&\geq \{\text{By Lemma 23}\} \\
&\quad v(y_{i,k-1}) + T_i - v(y_{i,k-1}) \\
&= \{\text{Rearranging}\} \\
&\quad T_i.
\end{aligned}
$$

Rearranging,

$$
y_{i,k-1} \leq y_{i,k} - \frac{T_i}{s_r}. \tag{59}
$$

Because $x_i(y_{i,\ell}) + x_i^s(s_r) + \Delta\left(y_{i,\ell}\right)$ is x-sufficient for all $y_{i,\ell} < y_{i,k}$, by (59) and Lemma 21 with $t_a = y_{i,k} - T_i/s_r$, $x_i(y_{i,k} - T_i/s_r) = x_i^s(s_r) + \Delta\left(y_{i,k} - T_i/s_r\right)$ is x-sufficient. Thus,

$$
\begin{aligned}
&\quad x_i^s(s_r) + \Delta\left(t_a\right) \\
&= \{\text{Because } t_a = y_{i,k}\} \\
&\quad x_i^s(s_r) + \Delta\left(y_{i,k}\right) \\
&\geq \{\text{By Lemma 20 with } t_0 = y_{i,k} - \tfrac{T_i}{s_r} \text{ and } t_1 = y_{i,k}\} \\
&\quad x_i^s(s_r) + \Delta\left(y_{i,k} - \frac{T_i}{s_r}\right) + \phi \cdot \frac{T_i}{s_r} \\
&= \{\text{Rearranging}\} \\
&\quad y_{i,k} - \frac{T_i}{s_r} + x_i^s(s_r) + \Delta\left(y_{i,k} - \frac{T_i}{s_r}\right) + e_{i,k-1} - y_{i,k} + \frac{T_i}{s_r} + \phi \cdot \frac{T_i}{s_r} - e_{i,k-1} \\
&\geq \{\text{By the definition of } x\text{-sufficient in Definition 8, and by (59)}\} \\
&\quad t_{i,k-1}^c - y_{i,k} + \frac{T_i}{s_r} + \phi \cdot \frac{T_i}{s_r} - e_{i,k-1} \\
&\geq \{\text{By the definition of } \phi \text{ in (46)}\} \\
&\quad t_{i,k-1}^c - y_{i,k} + \frac{T_i}{s_r} + \left(\frac{s_r \cdot A_i^{rn}(m - L_i - 1)}{T_i} - 1\right) \cdot \frac{T_i}{s_r} - e_{i,k-1}
\end{aligned}
$$

$= \{\text{Simplifying}\}$

$t^c_{i,k-1} - y_{i,k} + A^{rn}_i(m - L_i - 1) - e_{i,k-1}$

$\geq \{\text{By Definition 10 and Property 8, because } t^c_{i,k-1} \geq y_{i,k} \geq t_r\}$

$t^c_{i,k-1} - y_{i,k} + A^{rn}_i(m - L_i - 1) - C_i$

$\geq \{\text{By Lemma 8}\}$

$t^c_{i,k-1} - y_{i,k} + A_{i,k}(m - L_i - 1) - e_{i,k}$

$= \{\text{By the definition of } x^f_{i,k} \text{ in (15)}\}$

$x^f_{i,k}.$

Furthermore, by the preconditions of the lemma and Theorem 4, $x_i(t_a) = x^f_{i,k}$ is x-sufficient. Therefore, by Property 3 with $c_0 = x^f_{i,k}$ and $c_1 = x^s_i(s_r) + \Delta(t_a)$, the lemma holds. □

We now provide a combined lemma that addresses Case D.

Lemma 25. *If $t_a \in I_r$, $t_a = y_{i,k}$ for some $\tau_{i,k}$, $\tau_{i,k}$ is f-dominant for L_i, and $x_i(y_{i,\ell}) = x^s_i(s_r) + \Delta(y_{i,\ell})$ is x-sufficient for all $\tau_{i,\ell}$ with $y_{i,\ell} \in [t_r, t_a)$, then $x_i(t_a) = x^s_i(s_r) + \Delta(t_a)$ is x-sufficient.*

Proof. If $y_{i,k-1} \in I_o$, then the lemma follows from Lemma 22. Otherwise, it follows from Lemma 24. □

We now address Case E in Figure 9, where $t_a = y_{i,k}$ for some k, $t^c_{i,k} > y_{i,k} + e_{i,k}$, and $\tau_{i,k}$ is m-dominant for L_i. In this case, by Theorem 5 with $L = L_i$, $x^m_{i,k}$ is x-sufficient. Observe that $e^p_{i,k}$, the total amount of work remaining at $y_{i,k}$ from jobs of τ_i prior to $\tau_{i,k}$, appears in the expression for $x^m_{i,k}$ in (23). Thus, we must bound $e^p_{i,k}$. In order to do so, we will use the result of Lemma 3 with $t_0 = y_{i,k}$.

Observe in Lemma 3 the presence of $D^e_j(b_{j,\ell}, y_{i,k})$, and in the definition of $D^e_j(b_{j,\ell}, y_{i,k})$ in Definition 6 the presence of $e_{i,k}$. Although when accounting for overloads it was necessary to account for specific parameters of specific jobs, we want to develop general dissipation bounds that do not require such parameters. We will eliminate all such job references by deriving upper bounds using Properties 5–9 and the inductive x-sufficiency of $x_i(t) = x^s_i(s_r) + \Delta(t)$ for some smaller values of t.

Similarly, observe that the definition of $W_{i,k}$ in (16) involves $D^e_j(b_{j,\ell}, y_{i,k})$ and $D^e_j(t^b_{i,k}, y_{i,k})$. Therefore, many of the same lemmas used when analyzing $e^p_{i,k}$ will be additionally used when analyzing $W_{i,k}$.

We first provide a general upper bound to $D^e_j(t_0, t_1)$ (defined in Definition 6) when Property 8 applies.

Definition 15.

$$D^C_i(t_0, t_1) = \sum_{\tau_{i,k} \in \omega} C_i,$$

where ω is the set of jobs with $t_0 \le r_{i,k} \le y_{i,k} \le t_1$.

Observe that the definition of $D^C_i(t_0, t_1)$ in Definition 15 differs from that of $D^e_i(t_0, t_1)$ in Definition 6 only in that it uses C_i in place of $e_{i,k}$. The following lemma justifies this definition. However, in some cases, the replacement of some $e_{i,\ell}$ with C_i creates excessive pessimism. Thus, the lemma also provides a version of the bound that eliminates this pessimism for one particular job.

Lemma 26. *If, for all jobs $\tau_{i,k}$ with $r_{i,k} \ge t_0$, $t^c_{i,k} \ge t_r$, then*

$$D^e_i(t_0, t_1) \le D^C_i(t_0, t_1).$$

If, furthermore, there is a $\tau_{i,\ell}$ such that $t_0 \le r_{i,\ell} \le y_{i,\ell} \le t_1$, then

$$D^e_i(t_0, t_1) \le D^C_i(t_0, t_1) + e_{i,\ell} - C_i.$$

Proof. The definitions of ω in Definition 6 and in Definition 15 are identical: ω is the set of jobs $\tau_{i,k}$ with $t_0 \le r_{i,k} \le y_{i,k} \le t_1$. By the statement of the lemma, $t^c_{i,k} \ge t_r$ for each such $\tau_{i,k}$. Thus, by Property 8, $e_{i,k} \le C_i$ for all $\tau_{i,k} \in \omega$. Therefore,

$$\begin{aligned} D^e_i(t_0, t_1) &= \{\text{By the definition of } D^e_i(t_0, t_1) \text{ in Definition 6}\} \\ &\quad \sum_{\tau_{i,k} \in \omega} e_{i,k} \\ &\le \{\text{Because } e_{i,k} \le C_i \text{ for all } \tau_{i,k} \in \omega\} \\ &\quad \sum_{\tau_{i,k} \in \omega} C_i \end{aligned}$$

$$= \{\text{By the definition of } D_i^C(t_0, t_1) \text{ in Definition 6}\}$$

$$D_i^C(t_0, t_1).$$

If, furthermore, there is a $\tau_{i,\ell}$ such that $t_0 \le r_{i,\ell} \le y_{i,\ell} \le t_1$, then $\tau_{i,\ell} \in \omega$ by the definition of ω in Definitions 6 and 15. Therefore,

$$\begin{aligned}
D_i^e(t_0, t_1) &= \{\text{By the definition of } D_i^e(t_0, t_1) \text{ in Definition 6}\} \\
&\quad \sum_{\tau_{i,k} \in \omega} e_{i,k} \\
&= \{\text{Rearranging}\} \\
&\quad \sum_{\tau_{i,k} \in \omega} C_i + \sum_{\tau_{i,k} \in \omega \setminus \{\tau_{i,\ell}\}} (e_{i,k} - C_i) + (e_{i,\ell} - C_i) \\
&\le \{\text{Because } e_{i,k} \le C_i \text{ for } \tau_{i,k} \in \omega\} \\
&\quad \sum_{\tau_{i,k} \in \omega} C_i + (e_{i,\ell} - C_i) \\
&= \{\text{By the definition of } D_i^C(t_0, t_1) \text{ in Definition 15}\} \\
&\quad D_i^C(t_0, t_1) + e_{i,\ell} - C_i.
\end{aligned}$$

□

We now provide a general upper bound on $D_i^C(t_0, t_1)$ that will be used in conjunction with Lemma 26. This upper bound uses a result from Erickson et al. (2014).

Lemma 27. *If $t_0 \le t_1$, then $D_j^C(t_0, t_1) \le U_j^v \cdot (v(t_1) - v(t_0)) + S_j$.*

Proof. When considering virtual time instead of actual time for the purpose of job separation and PPs, and given the use of C_i in Definition 15 in place of $e_{i,k}$ in Definition 6, GEL-v scheduling under the SVO model reduces to traditional GEL scheduling under the ordinary sporadic task model. Thus, after translating t_0 and t_1 from actual time to virtual time, the lemma is identical to Lemma 2 from (Erickson et al., 2014). □

In Lemma 3 and in the definition of $W_{i,k}$ in (16), $D_j^e(b_{j,\ell}, y_{i,k})$ appears for some $\tau_{j,\ell}$. Using Lemma 26, this term can be upper bounded using $D_j^C(b_{j,\ell}, y_{i,k})$ and possibly an extra term to reduce pessimism. We more specifically characterize $D_j^C(b_{j,\ell}, y_{i,k})$ in the following lemma. Using the definition of x-sufficient in Definition 8, we will be able to reason about $y_{j,\ell}$ using an inductive assumption about the x-sufficiency of $x_i(t) = x_i^s(s_r) + \Delta(t)$ for some smaller values of t. Therefore, on the right hand side in Lemma 28 we use $y_{j,\ell}$ instead of $b_{j,\ell}$. As we show in the lemma, the analysis required to do so effectively cancels out the S_j term that would appear using Lemma 27.

Lemma 28. *If $\tau_{j,\ell}$ is pending at time $t_2 \geq t_r$, then*

$$D_j^C(b_{j,\ell}, y_{i,k}) \leq \max\{0, U_j^v \cdot (v(y_{i,k}) - v(y_{j,\ell}))\}.$$

Proof. We consider two cases, depending on the relative values of $b_{j,\ell}$ and $y_{i,k}$.

Case 1: $b_{j,\ell} > y_{i,k}$**.** If $b_{j,\ell} > y_{i,k}$, then there are no jobs $\tau_{j,v}$ such that $b_{j,\ell} \leq r_{j,v} \leq y_{j,v} \leq y_{i,k}$. Therefore, by the definition of $D_j^C(b_{j,\ell}, y_{i,k})$ in Definition 15, $D_j^C(b_{j,\ell}, y_{i,k}) = 0 \leq \max\{0, U_j^v \cdot (v(y_{i,k}) - v(y_{j,\ell}))\}$.

Case 2: $b_{j,\ell} \leq y_{i,k}$**.** We first relate $v(b_{j,\ell})$ to $v(y_{j,\ell})$.

$$\begin{aligned} v(b_{j,\ell}) &= \{\text{By the definition of } b_{j,\ell} \text{ in Definition 5}\} \\ &\quad v(r_{j,\ell}) + T_j \\ &= \{\text{Rearranging}\} \\ &\quad v(r_{j,\ell}) + Y_j + T_j - Y_j \\ &= \{\text{By (6)}\} \\ &\quad v(y_{j,\ell}) + T_j - Y_j. \end{aligned} \tag{60}$$

Furthermore, because $\tau_{j,\ell}$ is pending at $t_2 \geq t_r$, $t_{j,v}^c \geq t_r$ for $v \geq \ell$. We have

$$\begin{aligned} D_j^C(b_{j,\ell}, y_{i,k}) &\leq \{\text{By Lemma 27}\} \\ &\quad U_j^v \cdot (v(y_{i,k}) - v(b_{i,j})) + S_j \\ &= \{\text{By (60)}\} \\ &\quad U_j^v \cdot (v(y_{i,k}) - v(y_{j,\ell}) - T_j + Y_j) + S_j \end{aligned}$$

$$
\begin{aligned}
&= \{\text{Rearranging}\} \\
&\quad U_j^v \cdot (v(y_{i,k}) - v(y_{j,\ell})) - U_j^v \cdot (T_j - Y_j) + S_j \\
&= \{\text{By the definition of } U_j^v \text{ in (26) and the definition of } S_j \text{ in (38)}\} \\
&\quad U_j^v \cdot (v(y_{i,k}) - v(y_{j,\ell})) - C_j \cdot \left(1 - \frac{Y_j}{T_j}\right) + C_j \cdot \left(1 - \frac{Y_j}{T_j}\right) \\
&= \{\text{Cancelling}\} \\
&\quad U_j^v \cdot (v(y_{i,k}) - v(y_{j,\ell})) \\
&\leq \{\text{By the definition of ``}\max\text{''}\} \\
&\quad \max\{0, U_j^v \cdot (v(y_{i,k}) - v(y_{j,\ell}))\}.
\end{aligned}
$$

□

We will provide in Lemma 30 a lower bound on $y_{j,\ell}$, using an inductive assumption about the x-sufficiency of $x_j(t) = x_j^s(s_r) + \Delta(t)$ for some smaller values of t. In a similar manner to how we defined a notion of x-sufficient for a value of the function $x_i(t)$ for a particular i and t, we define a notion of x^p*-sufficient* for a function $x_i^p(t)$ (*p*ending). We will show that $x_i^p(t)$ is closely related to $x_i(t)$, hence the similar notation.

Definition 16. $x_i^p(t)$ is x^p*-sufficient* if $x_i^p(t) \geq 0$ and, for all $\tau_{i,k}$ pending at t,

$$
y_{i,k} \geq t - (x_i^p(t) + e_{i,k}^c(t)).
$$

In Lemma 30 below, we will provide a specific x^p-sufficient choice of $x_i^p(t_2)$ for an arbitrary τ_i and $t_2 \in I_r$. (We use t_2 in place of t_0 to avoid later conflicts in notation.) That choice will be based on the simple observation in the following lemma. Comparing this lemma to Definition 16 shows the reason for the similar notation between $x_i(t)$ and $x_i^p(t)$. (A different choice of $x_i^p(t)$, also based on Lemma 29, will be used in Section 4.5.)

Lemma 29. *If $\tau_{j,\ell}$ is pending at t_2 and $x_j(y_{j,\ell})$ is x-sufficient, then $y_{j,\ell} \geq t_2 - (x_j(y_{j,\ell}) + e_{j,\ell}^c(t_2))$.*

Proof. We use proof by contradiction. Suppose that $x_j(y_{j,\ell})$ is x-sufficient, but

$$
y_{j,\ell} < t_2 - (x_j(y_{j,\ell}) + e_{j,\ell}^c(t_2)). \tag{61}
$$

Then,

$$
\begin{aligned}
t^c_{j,\ell} \leq &\{\text{By Definition 8}\} \\
&y_{j,\ell} + x_j(y_{j,\ell}) + e_{j,\ell} \\
< &\{\text{By (61)}\} \\
&t_2 - (x_j(y_{j,\ell}) + e^c_{j,\ell}(t_2)) + x_j(y_{j,\ell}) + e_{j,\ell} \\
= &\{\text{Simplifying, and by Property 1}\} \\
&t_2 + e^r_{j,\ell}(t_2). \qquad (62)
\end{aligned}
$$

If $e^r_{j,\ell}(t_2) = 0$, then (62) contradicts the assumption that $\tau_{j,\ell}$ is pending at t_2. Otherwise, (62) contradicts the definition of $e^r_{j,\ell}(t_2)$ in Definition 3. □

We now define a x^p-sufficient choice of $x^p_i(t_2)$ for $t_2 \in I_r$.

Lemma 30. *Suppose that for each job $\tau_{j,\ell}$ pending at time $t_2 \in I_r$, if $y_{j,\ell} \in [t_r, t_2)$, then $x_j(y_{j,\ell}) = x^s_j(s_r) + \Delta(y_{j,\ell})$ is x-sufficient. Then, $x^p_j(t_2) = x^{p_r}_j(t_2)$ is x^p-sufficient, where*

$$x^{p_r}_j(t_2) \triangleq x^s_j(s_r) + \Delta(t_2 - \rho). \qquad (63)$$

Proof. By the definition of $x^{p_r}_j(t_2)$ in (63) and by Lemma 13 with $t_a = t_2 - \rho$, $x^{p_r}_j(t_2) \geq 0$. To show the remaining condition for $x^p_j(t_2) = x^{p_r}_j(t_2)$ to be x^p-sufficient, we consider an arbitrary job $\tau_{j,\ell}$ pending at t_2. By showing that $y_{j,\ell} \geq t_2 - (x^{p_r}_j(t_2) + e^c_{j,\ell}(t_2))$ for such an *arbitrary* job, we show that $x^p_j(t_2) = x^{p_r}_j(t_2)$ is x^p-sufficient. We consider three cases, depending on the value of $y_{j,\ell}$.

Case 1: $y_{j,\ell} \in (-\infty, t_r)$. We first bound the value of t_2 for this case to apply.

$$
\begin{aligned}
t_2 \leq &\{\text{By the definition of “pending” in Definition 4}\} \\
&t^c_{j,\ell} \\
\leq &\{\text{By the definition of } \dot{x}\text{-sufficient in Definition 14}\} \\
&t_r + \dot{x}_j(t_r) + e_{j,\ell} \\
\leq &\{\text{By Property 8, because } \tau_{j,\ell} \text{ is pending at } t_2 \geq t_r\}
\end{aligned}
$$

$$
\begin{aligned}
& t_r + \dot{x}_j(t_r) + C_j \\
< & \{\text{Because } A_j^{rn}(m - L_j - 1) > 0\} \\
& t_r + \dot{x}_j(t_r) + A_j^{rn}(m - L_j - 1) + C_j \\
= & \{\text{Rearranging}\} \\
& t_r + x_j^s(s_r) + (\dot{x}_j(t_r) - x_j^s(s_r) + A_j^{rn}(m - L_j - 1)) + C_j \\
\leq & \{\text{By the definition of } \lambda \text{ in (40)}\} \\
& t_r + x_j^s(s_r) + \lambda + C_j \\
\leq & \{\text{By the definition of } \rho \text{ in (50)}\} \\
& t_r + \rho.
\end{aligned}
$$

Therefore, $t_2 - \rho < t_r$, so by the definition of $\Delta\,(t_2 - \rho)$ in (51),

$$\Delta\,(t_2 - \rho) = \lambda. \tag{64}$$

Thus, we have

$$
\begin{aligned}
x_j^{p_r}(t_2) = & \{\text{By the definition of } x_j^{p_r}(t_2) \text{ in (63)}\} \\
& x_j^s(s_r) + \Delta\,(t_2 - \rho) \\
= & \{\text{By (64)}\} \\
& x_j^s(s_r) + \lambda \\
\geq & \{\text{By the definition of } \lambda \text{ in (40)}\} \\
& x_j^s(s_r) + (x_j(y_{j,\ell}) - x_j^s(s_r)) \\
= & \{\text{Cancelling}\} \\
& x_j(y_{j,\ell})
\end{aligned} \tag{65}
$$

for some x-sufficient choice of $x_j(y_{j,\ell})$. Therefore, by Property 3 with c_0 defined to be that choice and $c_1 = x_j^{p_r}(t_2)$, $x_j(y_{j,\ell}) = x_j^{p_r}(t_2)$ is x-sufficient. Therefore, by Lemma 29, $y_{j,\ell} \geq t_2 - (x_j^{p_r}(t_2) + e_{j,\ell}^c(t_2))$.

Case 2: $y_{j,\ell} \in [t_r, t_2)$**.** As in the previous case, we again bound the value of t_2 in order for this case to apply.

$$
\begin{aligned}
t_2 \leq\ & \{\text{By the definition of "pending" in Definition 4}\} \\
& t^c_{j,\ell} \\
& \leq \{\text{By the definition of } x\text{-sufficient in Definition 8 and the statement of the lemma}\} \\
& y_{j,\ell} + x^s_j(s_r) + \Delta\left(y_{j,\ell}\right) + e_{j,\ell} \\
& \leq \{\text{By Property 8, because } \tau_{j,\ell} \text{ is pending at } t_2 > t_r\} \\
& y_{j,\ell} + x^s_j(s_r) + \Delta\left(y_{j,\ell}\right) + C_j \\
& \leq \{\text{By Lemma 20 with } t_0 = t_r \text{ and } t_1 = y_{j,\ell}\} \\
& y_{j,\ell} + x^s_j(s_r) + \Delta\left(t_r\right) + C_j \\
& = \{\text{By Lemma 16}\} \\
& y_{j,\ell} + x^s_j(s_r) + \lambda + C_j \\
& \leq \{\text{By the definition of } \rho \text{ in (50)}\} \\
& y_{j,\ell} + \rho.
\end{aligned}
$$

Rearranging,

$$
t_2 - \rho \leq y_{j,\ell}. \tag{66}
$$

Thus, we have

$$
\begin{aligned}
x^{p_r}_j\left(t_2\right) =\ & \{\text{By the definition of } x^{p_r}_j\left(t_2\right) \text{ in (63)}\} \\
& x^s_j(s_r) + \Delta\left(t_2 - \rho\right) \\
& \geq \{\text{By Lemma 20 with } t_0 = t_2 - \rho \text{ and } t_1 = y_{j,\ell}\text{, and by (66)}\} \\
& x^s_j(s_r) + \Delta\left(y_{j,\ell}\right).
\end{aligned}
$$

Therefore, by Property 3 with $c_0 = x^s_j(s_r) + \Delta\left(y_{i,k}\right)$ and $c_1 = x^{p_r}_j\left(t_2\right)$, $x_j(y_{j,\ell}) = x^{p_r}_j\left(t_2\right)$ is x-sufficient. Therefore, by Lemma 29, $y_{j,\ell} \geq t_2 - (x^{p_r}_j\left(t_2\right) + e^c_{j,\ell}(t_2))$.

Case 3: $t_2 \leq y_{j,\ell}$**.** In this case,

$$\begin{aligned}
y_{j,\ell} &\geq t_2 \\
&\geq \{\text{By Lemma 13 with } t_a = t_2 - \rho\text{, and because } e^c_{j,\ell}(t_2) \geq 0\} \\
&\quad t_2 - (x^s_j(t_r) + \Delta\,(t_2 - \rho) + e^c_{j,\ell}(t_2)) \\
&= \{\text{By the definition of } x^{p_r}_j\,(t_2) \text{ in (63)}\} \\
&\quad t_2 - (x^{p_r}_j\,(t_2) + e^c_{j,\ell}(t_2)). \qquad (67)
\end{aligned}$$

□

In Lemma 32 below, we will bound an expression that occurs in Lemma 3 and in the definition of $W_{i,k}$ in (16). Observe in Lemma 28 that virtual times are used, in the form of $v(y_{j,\ell})$ and $v(y_{i,k})$. However, in Lemma 30, actual times are used. In order to combine the results of these two lemmas, we will need to characterize as Lemma 31 the behavior of $v(t)$, using Property 2. We will consider $v(t_1) - v(t_0)$ for arbitrary $t_1 \geq t_0$, as Lemma 31 will also be used elsewhere in our analysis.

If we were only concerned with the analysis when $t_0 \geq t_s$ and $t_1 \in I_r$, then by Property 5 we could assume that $s(t) = s_r$ for $t \in [t_0, t_1)$. However, Lemma 32 is general enough to be used in Section 4.5 in analysis that involves I_n, during which $s(t) = 1$. Therefore, we instead define as s_{ub} (*u*pper *b*ound) an upper bound on $s(t)$ for $t \in [t_0, t_1)$. In this section, we will use $s_{ub} = s_r$ for the just-noted reason, and in Section 4.5 we will use $s_{ub} = 1$, which is always valid by the definition of $s(t)$.

Lemma 31. *If* $t_1 \geq t_0$ *and* $s(t) \leq s_{ub}$ *for* $t \in [t_0, t_1)$*, then* $v(t_1) - v(t_0) \leq s_{ub} \cdot (t_1 - t_0)$*.*

Proof. We have

$$\begin{aligned}
v(t_1) - v(t_0) &= \{\text{By Property 2}\} \\
&\quad \int_{t_0}^{t_1} s(t)\,dt \\
&\leq \{\text{By the statement of the lemma}\} \\
&\quad \int_{t_0}^{t_1} s_{ub}\,dt
\end{aligned}$$

$$= \{\text{Rearranging}\}$$

$$s_{ub} \cdot (t_1 - t_0).$$

□

We now provide a result that allows us to upper bound expressions that appear in Lemma 3 and in the definition of $W_{i,k}$ in (16). We use the general $x_j^p(t_2)$ in place of $x_j^{p_r}(t_2)$ so that we can reuse this lemma in Section 4.5.

Lemma 32. *If $\tau_{j,\ell}$ is pending at $t_2 \in [t_r, y_{i,k}]$, $s_{ub} \in (0, 1]$, $s(t) \leq s_{ub}$ for $t \in [y_{j,\ell}, t_2)$, and $x_j^p(t_2)$ is x^p-sufficient, then*

$$e^r_{j,\ell}(t_2) + D^e_j(b_{j,\ell}, t_3) \leq C_j + U^v_j \cdot s_{ub} \cdot x^p_j(t_2) + U^v_j \cdot (v(t_3) - v(t_2)).$$

If furthermore $j = i$ and $k \geq \ell$, then

$$e^r_{j,\ell}(t_2) + D^e_j(b_{j,\ell}, t_3) \leq e_{i,k} + U^v_i \cdot s_{ub} \cdot x^p_i(t_2) + U^v_j \cdot (v(t_3) - v(t_2)).$$

Proof. We first provide reasoning that will address both the cases present in the statement of the lemma, which correspond to the two cases present in Lemma 26. We will then apply specific reasoning for each case. We have

$$e^r_{j,\ell}(t_2) + D^C_j(b_{j,\ell}, t_3)$$

$$\leq \{\text{By Lemma 28}\}$$

$$e^r_{j,\ell}(t_2) + \max\{0, U^v_j \cdot (v(t_3) - v(y_{j,\ell}))\}$$

$$= \{\text{Rearranging}\}$$

$$e^r_{j,\ell}(t_2) + \max\{0, U^v_j \cdot (v(t_2) - v(y_{j,\ell})) + U^v_j \cdot (v(t_3) - v(t_2))\}$$

$$\leq \{\text{Because } t_2 \leq t_3\}$$

$$e^r_{j,\ell}(t_2) + \max\{0, U^v_j \cdot (v(t_2) - v(y_{j,\ell}))\} + U^v_j \cdot (v(t_3) - v(t_2))$$

$$\leq \{\text{By Lemma 31 if } y_{i,\ell} < t_2\text{, or by the 0 term in the “max” otherwise}\}$$

$$e^r_{j,\ell}(t_2) + \max\{0, U^v_j \cdot s_{ub} \cdot (t_2 - y_{j,\ell})\} + U^v_j \cdot (v(t_3) - v(t_2))$$

$$\leq \{\text{By the definition of } x^p\text{-sufficient in Definition 16 with } t = t_2\}$$

$$e^r_{j,\ell}(t_2) + \max\{0, U^v_j \cdot s_{ub} \cdot (x^p_j(t_2) + e^c_{j,\ell}(t_2))\} + U^v_j \cdot (v(t_3) - v(t_2))$$

$$= \{\text{Because } x^p_j(t_2) \geq 0 \text{ (by Definition 16) and } e^c_{j,\ell}(t_2) \geq 0\}$$

$$e^r_{j,\ell}(t_2) + U^v_j \cdot s_{ub} \cdot (x^p_j(t_2) + e^c_{j,\ell}(t_2)) + U^v_j \cdot (v(t_3) - v(t_2))$$

$$\leq \{\text{By Property 1, and because } U^v_j \leq 1 \text{ and } s_{ub} \leq 1\}$$

$$e_{j,\ell} + U^v_j \cdot s_{ub} \cdot x^p_j(t_2) + U^v_j \cdot (v(t_3) - v(t_2)). \tag{68}$$

Furthermore,

$$e^r_{j,\ell}(t_2) + D^e_j(b_{j,\ell}, t_3)$$

$$\leq \{\text{By Lemma 26, because } \tau_{j,\ell} \text{ is pending at } t_2 \geq t_r\}$$

$$e^r_{j,\ell}(t_2) + D^C_j(b_{j,\ell}, t_3)$$

$$\leq \{\text{By (68)}\}$$

$$e_{j,\ell} + U^v_j \cdot s_{ub} \cdot x^p_j(t_2) + U^v_j \cdot (v(t_3) - v(t_2)) \tag{69}$$

$$\leq \{\text{By Property 8, because } \tau_{j,\ell} \text{ is pending at } t_2 \geq t_r\}$$

$$C_j + U^v_j \cdot s_{ub} \cdot x^p_j(t_2) + U^v_j \cdot (v(t_3) - v(t_2)).$$

We divide the more specific case, when $j = i$ and $k \geq \ell$, into two subcases. If $k = \ell$, then

$$e^r_{j,\ell}(t_2) + D^e_j(b_{j,\ell}, t_3)$$

$$\leq \{\text{By (69) with } j = i \text{ and } \ell = k\}$$

$$e_{i,k} + U^v_i \cdot s_{ub} \cdot x^p_i(t_2) + U^v_i \cdot (v(t_3) - v(t_2)).$$

If $k > \ell$, then by the definition of $b_{i,\ell}$ in Definition 5, $r_{i,k} \geq b_{i,\ell}$. Therefore,

$$e^r_{i,\ell}(t_2) + D^e_i(b_{i,\ell}, t_3)$$

$$\leq \{\text{By Lemma 26, because } \tau_{i,\ell} \text{ is pending at } t_2 \geq t_r\}$$

$$e^r_{i,\ell}(t_2) + D^C_i(b_{i,\ell}, t_3) + e_{i,k} - C_i$$

$$
\begin{aligned}
&\leq \{\text{By (68) with } j = i\} \\
&\quad e_{i,\ell} + U_i^v \cdot s_{ub} \cdot x_i^p(t_2) + U_i^v \cdot (v(t_3) - v(t_2)) + e_{i,k} - C_i \\
&\leq \{\text{By Property 8, because } \tau_{i,\ell} \text{ is pending at } t_2 \geq t_r\} \\
&\quad C_i + U_i^v \cdot s_{ub} \cdot x_i^p(t_2) + U_i^v \cdot (v(t_3) - v(t_2)) + e_{i,k} - C_i \\
&= \{\text{Simplifying}\} \\
&\quad e_{i,k} + U_i^v \cdot s_{ub} \cdot x_i^p(t_2) + U_i^v \cdot (v(t_3) - v(t_2)).
\end{aligned}
$$

□

We first use Lemma 32 to bound $e^p_{i,k}$ when $y_{i,k} \geq t_r$.

Lemma 33. *Suppose $\tau_{i,\ell}$ is the earliest pending job of τ_i at $y_{i,k} \geq t_r$. If $s(t) \leq s_{ub}$ for all $t \in [y_{i,\ell}, y_{i,k})$, $s_{ub} \in (0, 1]$, and $x_i^p(y_{i,k})$ is x^p-sufficient, then $e^p_{i,k} \leq U_i^v \cdot s_{ub} \cdot x_i^p(y_{i,k})$.*

Proof. We consider two cases.

Case 1: $\tau_{i,k}$ is the earliest pending job of τ_i at $y_{i,k}$, or there is no pending job of τ_i at $y_{i,k}$. In this case,

$$
\begin{aligned}
e^p_{i,k} &= \{\text{By the case we are considering}\} \\
&\quad 0 \\
&\leq \{\text{Because } x_i^p(y_{i,k}) \geq 0 \text{ by the definition of } x^p\text{-sufficient in Definition 16}\} \\
&\quad U_i^v \cdot s_{ub} \cdot x_i^p(y_{i,k}).
\end{aligned}
$$

Case 2: $\tau_{i,\ell}$ with $\ell < k$ is the earliest pending job of τ_i at $y_{i,k}$. By Lemma 3 with $t_0 = y_{i,k}$, the *total* remaining work from τ_i at $y_{i,k}$ is at most

$$
\begin{aligned}
&\quad e^r_{i,\ell}(y_{i,k}) + D^e_i(b_{i,\ell}, y_{i,k}) \\
&\leq \{\text{By Lemma 32 with } t_2 = y_{i,k} \text{ and } t_3 = y_{i,k}\} \\
&\quad e_{i,k} + U_i^v \cdot s_{ub} \cdot x_i^p(y_{i,k}) + U_i^v \cdot (v(y_{i,k}) - v(y_{i,k})) \\
&= \{\text{Rearranging}\}
\end{aligned}
$$

$$e_{i,k} + U_i^v \cdot s_{ub} \cdot x_i^p\left(y_{i,k}\right). \tag{70}$$

Of this work, $e_{i,k}$ units are from $\tau_{i,k}$ itself. Thus, subtracting $e_{i,k}$ yields the lemma. □

When we upper bound $e_{i,k}^p$ for a $\tau_{i,k}$ with $t_{i,k}^b < t_r$, we will need to consider $x_i^{p_r}(t)$. Furthermore, we will need a general upper bound on $x_i^{p_r}(t)$. The next lemma provides such an upper bound.

Lemma 34. *For all τ_i and $t \leq t_n$, $x_i^{p_r}(t) \leq x_i^s(s_r) + \lambda$.*

Proof. We consider two cases, depending on the value of t.

Case 1: $t < t_r + \rho$. In this case, $t - \rho < t_r$. Therefore,

$$\begin{aligned}
x_i^s(s_r) + \lambda &= \{\text{By the definition of } \Delta(t) \text{ in (51)}\} \\
&\quad x_i^s(s_r) + \Delta(t-\rho) \\
&= \{\text{By the definition of } x_i^{p_r}(t) \text{ in (63)}\} \\
&\quad x_i^{p_r}(t).
\end{aligned}$$

Case 2: $t \geq t_r + \rho$. In this case, $t - \rho \geq t_r$. Therefore,

$$\begin{aligned}
x_i^s(s_r) + \lambda &= \{\text{By Lemma 16}\} \\
&\quad x_i^s(s_r) + \Delta(t_r) \\
&\geq \{\text{By Lemma 20 with } t_0 = t_r \text{ and } t_1 = t - \rho\} \\
&\quad x_i^s(s_r) + \Delta(t-\rho) \\
&= \{\text{By the definition of } x_i^{p_r}(t) \text{ in (63)}\} \\
&\quad x_i^{p_r}(t).
\end{aligned}$$

□

We will next consider the remaining terms that appear in the definition of $x_{i,k}^m$ in (23). We will continue to need to translate between differences in virtual time and differences in actual time. Lemma 31 above provided an upper bound on $v(t_1) - v(t_0)$ when $t_1 \geq t_0$, but sometimes an exact value is needed. The next lemma provides an exact value instead of an upper bound when $t_0 \in [t_s, t_n)$ and $t_1 \in [t_s, t_n)$.

Lemma 35. *If $t_0 \in [t_s, t_n)$ and $t_1 \in [t_s, t_n)$, then $v(t_1) - v(t_0) = s_r \cdot (t_1 - t_0)$.*

Proof. We have

$$
\begin{aligned}
v(t_1) - v(t_0) &= \{\text{By Property 2}\} \\
&\int_{t_0}^{t_1} s(t)\, dt \\
&= \{\text{By Property 5}\} \\
&\int_{t_0}^{t_1} s_r\, dt \\
&= \{\text{Rearranging}\} \\
&s_r \cdot (t_1 - t_0).
\end{aligned}
$$

□

We will first consider in Lemmas 36–40 the case that $t^b_{i,k} \in I_o$, and then in Lemmas 41–48 we will consider the case that $t^b_{i,k} \in I_r$. In Lemma 49 we will prove that $x_i(t_a) = x^s_i(s_r) + \Delta(t_a)$ is x-sufficient for either value of $t^b_{i,k}$.

First, we upper bound $W_{i,k}$ when $t^b_{i,k} \in I_o$.

Lemma 36. *If $t^b_{i,k} \in I_o$ and $y_{i,k} \in I_r$, then*

$$W_{i,k} \leq W^o_{i,k} + \sum_{\tau_j \in \tau} U^r_j \cdot (y_{i,k} - t_r).$$

Proof. We have

$$
\begin{aligned}
&W^o_{i,k} + \sum_{\tau_j \in \tau} U^r_j \cdot (y_{i,k} - t_r) \\
&= \{\text{By Lemma 35 and the definition of } U^r_j \text{ in (27)}\} \\
&W^o_{i,k} + \sum_{\tau_j \in \tau} U^v_j \cdot (v(y_{i,k}) - v(t_r))
\end{aligned}
$$

$$
\begin{aligned}
&= \{\text{By the definition of } W^o_{i,k} \text{ in (42)}\} \\
&\quad W_{i,k} - \sum_{\tau_j \in \tau} D^e_j(t_r, y_{i,k}) + \sum_{\tau_j \in \tau} S_j + \sum_{\tau_j \in \tau} U^v_j \cdot (v(y_{i,k}) - v(t_r)) \\
&= \{\text{Rearranging}\} \\
&\quad W_{i,k} - \sum_{\tau_j \in t_r} D^e_j(t_r, y_{i,k}) + \sum_{\tau_j \in \tau} (U^v_j \cdot (v(y_{i,k}) - v(t_r)) + S_j) \\
&\geq \{\text{By Lemma 26}\} \\
&\quad W_{i,k} - \sum_{\tau_j \in t_r} D^C_j(t_r, y_{i,k}) + \sum_{\tau_j \in \tau} (U^v_j \cdot (v(y_{i,k}) - v(t_r)) + S_j) \\
&\geq \{\text{By Lemma 27}\} \\
&\quad W_{i,k}.
\end{aligned}
\tag{71}
$$

□

In a similar manner, we next consider the value of $R_{i,k}$ when $t^b_{i,k} \in I_o$.

Lemma 37. *If $t^b_{i,k} \in I_o$ and $y_{i,k} \in I_r$, then $R^o_{i,k} + u_{tot} \cdot (y_{i,k} - t_r) = R_{i,k}$.*

Proof. We have

$$
\begin{aligned}
&\quad R^o_{i,k} + u_{tot} \cdot (y_{i,k} - t_r) \\
&= \{\text{By the definition of } R^o_{i,k} \text{ in (43)}\} \\
&\quad u_{tot} \cdot (t_r - t^b_{i,k}) + u_{tot} \cdot (y_{i,k} - t_r) \\
&= \{\text{Rearranging}\} \\
&\quad u_{tot} \cdot (y_{i,k} - t^b_{i,k}) \\
&= \{\text{By the definition of } R_{i,k} \text{ in (20)}\} \\
&\quad R_{i,k}.
\end{aligned}
$$

□

We now provide an upper bound on $O_{i,k}$. This lemma uses Property 9, which does not depend on $s(t)$, so it can also be applied in a straightforward manner in Section 4.5 where $y_{i,k} \in I_n$ as well.

Lemma 38. *If $t^b_{i,k} \in I_o$, $y_{i,k} \in I_r \cup I_n$, and $t^c_{i,k} > y_{i,k}$, then $O^o_{i,k} + O^{rn} \geq O_{i,k}$.*

Proof. We first observe that, by the definition of $\beta_p(t_0, t_1)$ in Definition 7,

$$\beta_p(t^b_{i,k}, t^c_{i,k}) = \beta_p(t^b_{i,k}, t_r) + \beta_p(t_r, t^c_{i,k}). \tag{72}$$

We have

$$\begin{aligned}
O^o_{i,k} + O^{rn} &= \{\text{By the definition of } O^o_{i,k} \text{ in (44) and the definition of } O^{rn} \text{ in (39)}\}\\
&\quad \sum_{P_p \in P} o_p(t^b_{i,k}, t_r) + \sum_{P_p \in P} \widehat{u_p}\sigma_p\\
&\geq \{\text{By Property 9}\}\\
&\quad \sum_{P_p \in P} o_p(t^b_{i,k}, t_r) + \sum_{P_p \in P} o_p(t_r, t^c_{i,k})\\
&= \{\text{Rearranging}\}\\
&\quad \sum_{P_p \in P} (o_p(t^b_{i,k}, t_r) + o_p(t_r, t^c_{i,k}))\\
&= \{\text{By the definition of } o_p(t_0, t_1) \text{ in (8)}\}\\
&\quad \sum_{P_p \in P} (\max\{0, \widehat{u_p}(t_r - t^b_{i,k}) - \beta_p(t^b_{i,k}, t_r)\} + \max\{0, \widehat{u_p}(t^c_{i,k} - t_r)\\
&\quad - \beta_p(t_r, y_{i,k})\})\\
&\geq \{\text{By the definition of ``}max\text{''}\}\\
&\quad \sum_{P_p \in P} (\max\{0, \widehat{u_p}(t_r - t^b_{i,k}) - \beta_p(t^b_{i,k}, t_r) + \widehat{u_p}(y_{i,k} - t_r) - \beta_p(t_r, y_{i,k})\})\\
&= \{\text{Rearranging, and by (72)}\}\\
&\quad \sum_{P_p \in P} (\max\{0, \widehat{u_p}(y_{i,k} - t^b_{i,k}) - \beta_p(t^b_{i,k}, y_{i,k})\})\\
&= \{\text{By the definition of } o_p(t^b_{i,k}, y_{i,k}) \text{ in (8)}\}\\
&\quad \sum_{P_p \in P} o_p(t^b_{i,k}, y_{i,k})
\end{aligned}$$

$= \{$By the definition of $O_{i,k}$ in (19)$\}$

$O_{i,k}$.

□

When considering $t^b_{i,k}$ in I_o, we will consider the value of λ. We now provide a lower bound on λ that closely resembles the bound we desire.

Lemma 39. *For arbitrary $\tau_{i,k}$ such that $t^b_{i,k} \in I_o$ and $y_{i,k} \in I_r \cup I_n$,*

$$\lambda \geq \frac{W^o_{i,k} - R^o_{i,k} + (m - u_{tot} - 1)e_{i,k} + O^o_{i,k} + O^{rn} + L_i \cdot U^r_i \cdot (x^s_i(s_r) + \lambda)}{u_{tot}}.$$

Proof. By the definition of λ in (40), we have

$$\lambda \geq \frac{W^o_{i,k} - R^o_{i,k} + (m - u_{tot} - 1)e_{i,k} + O^o_{i,k} + O^{rn} + L_i \cdot U^r_i \cdot x^s_i(s_r)}{u_{tot} - L_i \cdot U^r_i}.$$

Adding $\lambda \cdot \frac{L_i \cdot U^r_i}{u_{tot} - L_i \cdot U^r_i}$ to both sides yields

$$\lambda \cdot \frac{u_{tot}}{u_{tot} - L_i \cdot U^r_i} \geq \frac{W^o_{i,k} - R^o_{i,k} + (m - u_{tot} - 1)e_{i,k} + O^o_{i,k} + O^{rn} + L_i \cdot U^r_i \cdot (x^s_i(j) + \lambda)}{u_{tot} - L_i \cdot U^r_i}.$$

By Property 11, the definition of U^r_i in (27), and the restriction that $s_r < 1$ in Property 5, $L_i \cdot U^r_i < u_{tot}$. Therefore, $\frac{u_{tot} - L_i \cdot U^r_i}{u_{tot}} > 0$. Thus, multiplying both sides by $\frac{u_{tot} - L_i \cdot U^r_i}{u_{tot}}$ yields the lemma. □

We now consider the $t^b_{i,k} \in I_o$ subcase of Case E.

Lemma 40. *If $t_a \in I_r$, $t_a = y_{i,k}$ for some k, $\tau_{i,k}$ is m-dominant for L_i, and $t^b_{i,k} \in I_o$, then $x_i(t_a) = x^s_i(s_r) + \Delta(t_a)$ is x-sufficient.*

Proof. We have

$x^s_i(s_r) + \Delta(t_a)$

$\geq \{$By Lemma 20 with $t_0 = t_r$ and $t_1 = t_a = y_{i,k}\}$

$x^s_i(s_r) + \Delta(t_r) + \phi \cdot (y_{i,k} - t_r)$

$= \{\text{By Lemma 16}\}$

$$x_i^s(s_r) + \lambda + \phi \cdot (y_{i,k} - t_r)$$

$\geq \{\text{By Lemma 39}\}$

$$x_i^s(s_r) + \frac{W_{i,k}^o - R_{i,k}^o + (m - u_{tot} - 1)e_{i,k} + O_{i,k}^o + O^{rn} + L_i \cdot U_i^r \cdot (x_i^s(s_r) + \lambda)}{u_{tot}} - x_i^s(s_r)$$

$$+ \phi \cdot (y_{i,k} - t_r)$$

$= \{\text{Rearranging}\}$

$$(W_{i,k}^o + R_{i,k}^o + u_{tot} \cdot \phi \cdot (y_{i,k} - t_r) + (m - u_{tot} - 1)e_{i,k} + O_{i,k}^o + O^{rn}$$

$$+ L_i \cdot U_i^r \cdot (x_i^s(s_r) + \lambda))/u_{tot}$$

$\geq \{\text{By Lemma 34}\}$

$$\frac{W_{i,k}^o + R_{i,k}^o + u_{tot} \cdot \phi \cdot (y_{i,k} - t_r) + (m - u_{tot} - 1)e_{i,k} + O_{i,k}^o + O^{rn} + L_i \cdot U_i^r \cdot x_i^{p_r}(y_{i,k})}{u_{tot}}$$

$\geq \{\text{By Lemma 38}\}$

$$\frac{W_{i,k}^o + R_{i,k}^o + u_{tot} \cdot \phi \cdot (y_{i,k} - t_r) + (m - u_{tot} - 1)e_{i,k} + O_{i,k} + L_i \cdot U_i^r \cdot x_i^{p_r}(y_{i,k})}{u_{tot}}$$

$\geq$ {By Lemma 33 with $x_i^p(y_{i,k}) = x_i^{p_r}(y_{i,k})$, $s_{ub} = s_r$, and the definition of U_i^r in (27)}

$$\frac{W_{i,k}^o + R_{i,k}^o + u_{tot} \cdot \phi \cdot (y_{i,k} - t_r) + (m - u_{tot} - 1)e_{i,k} + O_{i,k} + L_i \cdot e_{i,k}^p}{u_{tot}}. \tag{73}$$

For simplicity, we now consider some of the terms separately.

$$W_{i,k}^o - R_{i,k}^o + u_{tot} \cdot \phi \cdot (y_{i,k} - t_r)$$

$\geq$ {By the definition of ϕ in (46)}

$$W_{i,k}^o - R_{i,k}^o + \left(\sum_{\tau_j \in \tau} U_j^r - u_{tot}\right) \cdot (y_{i,k} - t_r)$$

$= \{\text{Rearranging}\}$

$$W_{i,k}^o + \sum_{\tau_j \in \tau} U_j^r \cdot (y_{i,k} - t_r) - R_{i,k}^o - u_{tot} \cdot (y_{i,k} - t_r)$$

$\geq \{\text{By Lemmas 36 and 37}\}$

$$W_{i,k} - R_{i,k}. \tag{74}$$

Combining, we have

$$
\begin{aligned}
x^s_i(s_r) + \Delta\left(t_a\right) \geq\ & \{\text{By (73) and (74)}\} \\
& \frac{W_{i,k} - R_{i,k} + (m - u_{tot} - 1)e_{i,k} + O_{i,k} + L_i \cdot e^p_{i,k}}{u_{tot}} \\
=\ & \{\text{By the definition of } x^m_{i,k} \text{ in (23)}\} \\
& x^m_{i,k}.
\end{aligned}
\tag{75}
$$

By Theorem 5, $x_i(t_a) = x^m_{i,k}$ is x-sufficient. Therefore, by Property 3 with $c_0 = x^m_{i,k}$ and $c_1 = x^s_i(s_r) + \Delta\left(t_a\right)$, $x_i(t_a) = x^s_i(s_r) + \Delta\left(t_a\right)$ is x-sufficient. □

In Lemmas 41–48, we now turn our attention to the case when $t^b_{i,k} \in I_r$. In order to facilitate use in the next subsection, some of these lemmas also apply when $t^b_{i,k} \in I_n$. Observe in the definition of $x^m_{i,k}$ in (23) the presence of the term $(m - u_{tot} - 1)e_{i,k}$. This term was previously accounted for by its explicit inclusion in the definition of λ in (40). However, the value of λ is only directly relevant for $\tau_{i,k}$ with $t^b_{i,k} \in I_r$. In most lemmas in this section, terms with $e_{i,k}$ can be upper bounded by using C_i instead. However, depending on the size of u_{tot}, $m - u_{tot} - 1$ can be as small as -1 or as big as $m - 1$. If $(m - u_{tot} - 1) \in [0, m - 1)$, then $(m - u_{tot} - 1)C_i \geq (m - u_{tot} - 1)e_{i,k}$. However, if $(m - u_{tot} - 1) \in [-1, 0)$, then $(m - u_{tot} - 1)C_i$ could be smaller than $(m - u_{tot} - 1)e_{i,k}$ by as much as $C_i - e_{i,k}$.

As we will show, we can use the less general cases in Lemmas 26 and 32 while bounding $W_{i,k}$ in order to cancel out this discrepancy. Our reasoning will depend on whether τ_i has a job in $\theta_{i,k}$ or is in $\overline{\theta_{i,k}}$. To handle these cases, in the next lemma we define an indicator variable $\Omega_{i,k}(j)$ that will be used for all tasks in either case, but that will be nonzero only when $j = i$. We first consider how to handle a τ_j with a job in $\theta_{i,k}$, based on the expression that appears in the sum for $\theta_{i,k}$ in the definition of $W_{i,k}$ in (16).

Lemma 41. *If $\tau_{j,\ell} \in \theta_{i,k}$, $t^b_{i,k} \in I_r \cup I_n$, $s_{ub} \in (0, 1]$ for $t \in [y_{j,\ell}, t^b_{i,k})$, and $x^p_j\left(t^b_{i,k}\right)$ is x^p-sufficient, then*

$$e^r_{j,\ell}(t^b_{i,k}) + D^e_j(b_{j,\ell}, y_{i,k}) \leq C_j + U^v_j \cdot s_{ub} \cdot x^p_j\left(t^b_{i,k}\right) + U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) + \Omega_{i,k}(j),$$

where

$$\Omega_{i,k}(j) \triangleq \begin{cases} e_{i,k} - C_i & \text{If } j = i \\ 0 & \text{If } j \neq i. \end{cases} \tag{76}$$

Proof. We consider two cases, depending whether $j = i$.

Case 1: $j = i$. In this case, by the definition of $\theta_{i,k}$ in Lemma 4, $y_{i,\ell} \leq y_{i,k}$. Therefore, $k \geq \ell$. Thus,

$$\begin{aligned}
& e^r_{j,\ell}(t^b_{i,k}) + D^e_j(b_{j,\ell}, y_{i,k}) \\
\leq & \{\text{By Lemma 32 with } t_2 = t^b_{i,k} \text{ and } t_3 = y_{i,k}\} \\
& e_{i,k} + U^v_j \cdot s_{ub} \cdot x^p_j\left(t^b_{i,k}\right) + U^v_i \cdot (v(y_{i,k}) - v(t^b_{i,k})) \\
= & \{\text{Rearranging}\} \\
& C_i + U^v_j \cdot s_{ub} \cdot x^p_j\left(t^b_{i,k}\right) + U^v_i \cdot (v(y_{i,k}) - v(t^b_{i,k})) + e_{i,k} - C_i \\
= & \{\text{By the definition of } \Omega_{i,k}(j) \text{ in (76)}\} \\
& C_i + U^v_j \cdot s_{ub} \cdot x^p_j\left(t^b_{i,k}\right) + U^v_i \cdot (v(y_{i,k}) - v(t^b_{i,k})) + \Omega_{i,k}(j).
\end{aligned}$$

Case 2: $j \neq i$.

$$\begin{aligned}
& e^r_{j,\ell}(t^b_{i,k}) + D^e_j(b_{j,\ell}, y_{i,k}) \\
\leq & \{\text{By Lemma 32 with } t_2 = t^b_{i,k} \text{ and } t_3 = y_{i,k}\} \\
& C_i + U^v_i \cdot s_{ub} \cdot x^p_i\left(t^b_{i,k}\right) + U^v_i \cdot (v(y_{i,k}) - v(t^b_{i,k})) \\
= & \{\text{By the definition of } \Omega_{i,k}(j) \text{ in (76)}\} \\
& C_i + U^v_i \cdot s_{ub} \cdot x^p_i\left(t^b_{i,k}\right) + U^v_i \cdot (v(y_{i,k}) - v(t^b_{i,k})) + \Omega_{i,k}(j).
\end{aligned}$$

□

The next lemma is similar, but handles the expression that appears in the sum for $\overline{\theta_{i,k}}$ in the definition of $W_{i,k}$ in (16).

Lemma 42. *If* $\tau_j \in \overline{\theta_{i,k}}$, $t^b_{i,k} \in I_r \cup I_n$, *and* $t^c_{i,k} > y_{i,k}$, *then*

$$D^e_j(t^b_{i,k}, y_{i,k}) \leq U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) + S_j + \Omega_{i,k}(j).$$

Proof. We consider two cases, depending whether $j = i$.

Case 1: $j = i$**.** In this case, because $t^c_{i,k} > y_{i,k} \geq t^b_{i,k}$, $\tau_{i,k}$ must not be complete at $t^b_{i,k}$. If $r_{i,k} < t^b_{i,k}$, then by the definition of $\theta_{i,k}$ in Lemma 4, $\tau_{i,k}$ or some predecessor of $\tau_{i,k}$ must be in $\theta_{i,k}$. By the definition of $\overline{\theta_{i,k}}$ in Lemma 4, this contradicts the precondition that $\tau_i = \tau_j \in \overline{\theta_{i,k}}$. Therefore, $r_{i,k} \geq t^b_{i,k}$.

Thus, we have

$$\begin{aligned} D^e_j(t^b_{i,k}, y_{i,k}) \leq &\{\text{By Lemma 26}\} \\ & D^C_j(t^b_{i,k}, y_{i,k}) + e_{i,k} - C_i \\ = &\{\text{By the definition of } \Omega_{i,k}(j) \text{ in (76)}\} \\ & D^C_j(t^b_{i,k}, y_{i,k}) + \Omega_{i,k}(j) \\ \leq &\{\text{By Lemma 27}\} \\ & U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) + S_j + \Omega_{i,k}(j). \end{aligned}$$

Case 2: $j \neq i$**.** In this case, we have

$$\begin{aligned} D^e_j(t^b_{i,k}, y_{i,k}) \leq &\{\text{By Lemma 26}\} \\ & D^C_j(t^b_{i,k}, y_{i,k}) \\ = &\{\text{By the definition of } \Omega_{i,k}(j) \text{ in (76)}\} \\ & D^C_j(t^b_{i,k}, y_{i,k}) + \Omega_{i,k}(j) \\ \leq &\{\text{By Lemma 27}\} \\ & U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) + S_j + \Omega_{i,k}(j). \end{aligned}$$

□

We now combine these results to provide a bound on $W_{i,k}$ when $y_{i,k} \in I_r \cup I_n$. This bound is valid both when $t^b_{i,k} \in I_r$ and when $t^b_{i,k} \in I_n$.

Lemma 43. *If $t^c_{i,k} > y_{i,k}$, $t^b_{i,k} \in I_r \cup I_n$, $s_{ub} \in (0,1]$, and $s(t) \leq s_{ub}$ for $t \in [y_{j,\ell}, t^b_{i,k})$ for each $\tau_{j,\ell}$ in $\theta_{i,k}$, and each $x^p_j\left(t^b_{i,k}\right)$ is x^p-sufficient, then*

$$W_{i,k} \leq \sum_{m-1 \text{ largest}} (C_j + U^v_j \cdot s_{ub} \cdot x^p_j\left(t^b_{i,k}\right) - S_j) + \sum_{\tau_j \in \tau} S_j + \sum_{\tau_j \in \tau} U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) + e_{i,k} - C_i.$$

Proof. We have

$$\begin{aligned}
W_{i,k} &= \{\text{By the definition of } W_{i,k} \text{ in (16)}\}\\
&\quad \sum_{\tau_{j,\ell} \in \theta_{i,k}} (e^r_{j,\ell}(t^b_{i,k}) + D^e_j(b_{j,\ell}, y_{i,k})) + \sum_{\tau_j \in \overline{\theta_{i,k}}} D^e_j(t^b_{i,k}, y_{i,k})\\
&\leq \{\text{By Lemmas 41 and 42}\}\\
&\quad \sum_{\tau_{j,\ell} \in \theta_{i,k}} (C_j + U^v_j \cdot s_{ub} \cdot x^p_j\left(t^b_{i,k}\right) + U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) + \Omega_{i,k}(j))\\
&\quad + \sum_{\tau_j \in \overline{\theta_{i,k}}} (U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) + S_j + \Omega_{i,j}(k))\\
&= \{\text{Rearranging, and by the definitions of } \theta_{i,k} \text{ and } \overline{\theta_{i,k}} \text{ in Lemma 4}\}\\
&\quad \sum_{\tau_{j,\ell} \in \theta_{i,k}} (C_j + U^v_j \cdot s_{ub} \cdot x^p_j\left(t^b_{i,k}\right) - S_j) + \sum_{\tau_j \in \tau} S_j + \sum_{\tau_j \in \tau} U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k}))\\
&\quad + \sum_{\tau_j \in \tau} \Omega_{i,k}(j). \qquad (77)
\end{aligned}$$

By the definition of $\theta_{i,k}$ in Lemma 4, any τ_j with a $\tau_{j,\ell} \in \theta_{i,k}$ must be executing immediately before $t^b_{i,k}$, because $\tau_{j,\ell}$ was released before $t^b_{i,k}$, $\tau_{j,\ell}$ is still pending at $t^b_{i,k}$, and there is an idle processor just before $t^b_{i,k}$. Therefore, there can be at most $m-1$ tasks with jobs in $\theta_{i,k}$. Furthermore, it is more pessimistic to assume that a task has a job in $\theta_{i,k}$ than that it is in $\overline{\theta_{i,k}}$, because

$$\begin{aligned}
&C_j + U^v_j \cdot s_{ub} \cdot x^p_j\left(t^b_{i,k}\right)\\
&\geq \{\text{Because } U^r_j > 0\text{, and by the definition of } x^p\text{-sufficient in Definition 16}\}
\end{aligned}$$

$$
\begin{aligned}
&C_j \\
&\geq \{\text{Because } Y_i \geq 0 \text{ and } T_i > 0\} \\
&C_j \cdot \left(1 - \frac{Y_i}{T_i}\right) \\
&= \{\text{By the definition of } S_j \text{ in (38)}\} \\
&S_i. \qquad (78)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
W_{i,k} \leq &\{\text{By (77) and the above reasoning}\} \\
&\sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_{ub} \cdot x_j^p \left(t_{i,k}^b\right) - S_j) + \sum_{\tau_j \in \tau} S_j + \sum_{\tau_j \in \tau} U_j^v \cdot (v(y_{i,k}) - v(t_{i,k}^b)) \\
&+ \sum_{\tau_j \in \tau} \Omega_{i,j}(k) \\
= &\{\text{By the definition of } \Omega_{i,k}(j) \text{ in (76)}\} \\
&\sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_{ub} \cdot x_j^p \left(t_{i,k}^b\right) - S_j) + \sum_{\tau_j \in \tau} S_j + \sum_{\tau_j \in \tau} U_j^v \cdot (v(y_{i,k}) - v(t_{i,k}^b)) \\
&+ e_{i,k} - C_i.
\end{aligned}
$$

□

We now offer a bound that accounts for the $-e_{i,k}$ term that occurs as part of the $(m - u_{tot} - 1)e_{i,k}$ term in the definition of $x_{i,k}^m$ in (23). This lemma plays a similar role to Lemma 36 above, but for the case that $t_{i,k}^b \in I_r$.

Lemma 44. *If $t_{i,k}^c > y_{i,k}$, $t_{i,k}^b \in I_r$ and $y_{i,k} \in I_o$, then*

$$
W_{i,k}^r + \sum_{\tau_j \in \tau} U_j^r \cdot (y_{i,k} - t_{i,k}^b) - C_i \geq W_{i,k} - e_{i,k}.
$$

where

$$
W_{i,k}^r \triangleq \sum_{m-1 \text{ largest}} (C_j + U_j^r \cdot x_j^{p_r} \left(t_{i,k}^b\right) - S_j) + \sum_{\tau_j \in \tau} S_j. \qquad (79)
$$

Proof. Consider arbitrary $\tau_{j,\ell}$ pending at $t^b_{i,k}$. By Property 7, $y_{j,\ell} \geq t_s$. Therefore, if $y_{j,\ell} < t^b_{i,k}$, then for all $t \in [y_{j,\ell}, t^b_{i,k})$, $s(t) = s_r$. Because $\tau_{j,\ell}$ was arbitrary, we can use Lemma 43 with $s_{ub} = s_r$. Furthermore, by Lemma 30, we can use $x^p_j\left(t^b_{i,k}\right) = x^{p_r}_j\left(t^b_{i,k}\right)$. We have

$$
\begin{aligned}
& W^r_{i,k} + \sum_{\tau_j \in \tau} U^r_j \cdot (y_{i,k} - t^b_{i,k}) - C_i \\
= & \{\text{By Lemma 35 and the definition of } U^r_i \text{ in (27)}\} \\
& W^r_{i,k} + \sum_{\tau_j \in \tau} U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) - C_i \\
= & \{\text{By the definition of } W^r_{i,k} \text{ in (79)}\} \\
& \sum_{m-1 \text{ largest}} (C_j + U^r_j \cdot x^{p_r}_j\left(t^b_{i,k}\right) - S_j) + \sum_{\tau_j \in \tau} S_j + \sum_{\tau_j \in \tau} U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) - C_i \\
\geq & \{\text{By Lemma 43 with } x^p_j\left(t^b_{i,k}\right) = x^{p_r}_j\left(t^b_{i,k}\right) \text{ and } s_{ub} = s_r\text{, and by the definition of } U^r_j \text{ in (27)}\} \\
& W_{i,k} - e_{i,k}.
\end{aligned}
$$

□

We next bound $O_{i,k}$. This lemma plays the same role as Lemma 38, but for the case that $t^b_{i,k} \in I_r \cup I_n$.

Lemma 45. *If $t^b_{i,k} \in I_r \cup I_n$, then $O^{rn} \geq O_{i,k}$.*

Proof. We have

$$
\begin{aligned}
O^{rn} = & \{\text{By the definition of } O^{rn} \text{ in (39)}\} \\
& \sum_{P_p \in P} \widehat{u_p} \sigma_p \\
\geq & \{\text{By Property 9}\} \\
& \sum_{P_p \in P} o_p(t^b_{i,k}, y_{i,k}) \\
= & \{\text{By the definition of } O_{i,k} \text{ in (19)}\} \\
& O_{i,k}.
\end{aligned}
$$

□

Many previous lemmas in this section were based on Lemma 20, which essentially states that $\Delta(t)$ decreases sufficiently slowly in an *additive* sense. In Lemma 47 below, we will bound the value of $x_i^s(s_r) + \Delta\left(t_{i,k}^b\right)$ by using the fact that $\Delta(t)$ decreases sufficiently slowly in a *multiplicative* sense. In Lemma 48, we will then use Lemma 20 to bound the value of $x_i^s(s_r) + \Delta(y_{i,k})$.

The next lemma shows that $\Delta(t)$ decreases sufficiently slowly in this multiplicative sense, in a similar manner to how Lemma 20 shows that $\Delta(t)$ decreases sufficiently slowly in an additive sense.

Lemma 46. *For all* t_2, $\Delta(t_2) \geq q \cdot \Delta(t_2 - \rho)$.

Proof. We will consider a function

$$g(t) \triangleq \Delta(t_2) \cdot q^{\frac{t-t_2}{\rho}}. \tag{80}$$

(Recall that, by Lemma 11, $\rho > 0$.) Observe that

$$\begin{aligned} g(t_2) = &\{\text{By (80)}\} \\ &\Delta(t_2) \cdot q^{\frac{t_2-t_2}{\rho}} \\ = &\{\text{Simplifying}\} \\ &\Delta(t_2). \end{aligned} \tag{81}$$

Also,

$$\begin{aligned} g(t_2) = &\{\text{By (81)}\} \\ &\Delta(t_2) \\ = &\{\text{Rewriting}\} \\ &\Delta(t_2) \cdot q^{\frac{t_2-\rho-t_2}{\rho}} \cdot q^{\frac{-t_2+\rho+t_2}{\rho}} \\ = &\{\text{By the definition of } g(t_2-\rho) \text{ in (80)}\} \\ &g(t_2-\rho) \cdot q^{\frac{-t_2+\rho+t_2}{\rho}} \\ = &\{\text{Simplifying}\} \end{aligned}$$

$$q \cdot g(t_2 - \rho).$$

Therefore, we prove the lemma by establishing that $g(t_2 - \rho) \geq \Delta(t_2 - \rho)$. In order to simplify the analysis in one case, we will prove the more general result that $g(t_0) \geq \Delta(t_0)$ for $t_0 \leq t_2$. In order to do so, we consider the derivative of $g(t)$, denoted $g'(t)$, for $t \in [t_0, t_2)$. For such t, we have

$$\begin{aligned} g'(t) &= \{\text{By (80) and differentiation}\} \\ &\quad \frac{\ln q}{\rho} \cdot \Delta(t_2) \cdot q^{\frac{t-t_2}{\rho}} \\ &\leq \{\text{Because } t < t_2 \text{ and } 0 < q < 1 \text{ by Lemma 10}\} \\ &\quad \frac{\ln q}{\rho} \cdot \Delta(t_2). \end{aligned} \tag{82}$$

By the FTC with $f(t) = \Delta(t)$,

$$\Delta(t_2) = \Delta(t_0) + \int_{t_0}^{t_2} \Delta'(t)\ dt.$$

Rearranging,

$$\Delta(t_0) = \Delta(t_2) - \int_{t_0}^{t_2} \Delta'(t)\ dt. \tag{83}$$

By identical reasoning,

$$\begin{aligned} g(t_0) &= g(t_2) - \int_{t_0}^{t_2} g'(t)\, dt \\ &= \{\text{By (81)}\} \\ &\quad \Delta(t_2) - \int_{t_0}^{t_2} g'(t)\, dt. \end{aligned} \tag{84}$$

We consider three cases, depending on the value of t_2. We use intervals closed on the right in order to reduce the number of edge cases we must consider.

Case 1: $t_2 \in (-\infty, t_r]$**.** In this case,

$$\Delta(t_2) = \{\text{By the definition of } \Delta(t) \text{ in (51), or by Lemma 16}\}$$

$$\begin{aligned}
&\lambda \\
\geq &\{\text{By the definition of } \lambda \text{ in (40)}\} \\
&0.
\end{aligned} \tag{85}$$

Therefore, by (82), because $0 < q < 1$ by Lemma 10, and because $\rho > 0$ by Lemma 11,

$$g'(t) \leq 0 \tag{86}$$

for $t \in [t_0, t_2)$. We have

$$\begin{aligned}
g(t_0) = &\{\text{By (84)}\} \\
&\Delta(t_2) - \int_{t_0}^{t_2} g'(t)\, dt \\
\geq &\{\text{By (86)}\} \\
&\Delta(t_2) - \int_{t_0}^{t_2} 0\, dt \\
= &\{\text{By the definition of } \Delta'(t) \text{ in (56)}\} \\
&\Delta(t_2) - \int_{t_0}^{t_2} \Delta'(t)\; dt \\
= &\{\text{By (83)}\} \\
&\Delta(t_0).
\end{aligned}$$

Therefore, by the reasoning at the beginning of the proof, the lemma holds.

Case 2: $t_2 \in (t_r, t_e]$**.** (t_r, t_e) cannot be empty, because it contains t_2. Thus, $t_r < t_e$. Therefore, by the definition of t_e in (48), $\lambda > \phi \cdot \frac{\rho}{\ln q}$. Thus,

$$\begin{aligned}
\Delta(t_2) \geq &\{\text{By Lemma 20 with } t_0 = t_2 \text{ and } t_1 = t_e\} \\
&\Delta(t_e) \\
= &\{\text{By Lemma 12}\}
\end{aligned}$$

$$\phi \cdot \frac{\rho}{\ln q}.$$

Therefore, by (82),

$$g'(t) \leq \phi \tag{87}$$

for $t \in [t_0, t_2)$. We have

$$\begin{aligned}
g(t_0) = &\{\text{By (84)}\} \\
&\Delta(t_2) - \int_{t_0}^{t_2} g'(t)\, dt \\
\geq &\{\text{By (87)}\} \\
&\Delta(t_2) - \int_{t_0}^{t_2} \phi\, dt \\
\geq &\{\text{By Lemma 19}\} \\
&\Delta(t_2) - \int_{t_0}^{t_2} \Delta'(t)\ dt \\
= &\{\text{By (83)}\} \\
&\Delta(t_0).
\end{aligned}$$

Therefore, by the reasoning at the beginning of the proof, the lemma holds.

Case 3: $t_2 \in (t_e, t_n)$**.** In this case, for all t,

$$\begin{aligned}
g(t) = &\{\text{By the definition of } g(t) \text{ in (80)}\} \\
&\Delta(t_2) \cdot q^{\frac{t-t_2}{\rho}} \\
= &\{\text{By the definition of } \Delta(t_2) \text{ in (51) and the definition of } \Delta^e(t) \text{ in (47)}\} \\
&\Delta^\ell(t_e) \cdot q^{\frac{t_2-t_e}{\rho}} \cdot q^{\frac{t-t_2}{\rho}} \\
= &\{\text{Simplifying}\} \\
&\Delta^\ell(t_e) \cdot q^{\frac{t-t_e}{\rho}}.
\end{aligned} \tag{88}$$

We consider two subcases, depending on the value of $t_0 \leq t_2$ considered at the beginning of the lemma.

Case 3.1: $t_0 < t_e$**.** For arbitrary t, we have

$$\begin{aligned} g(t) &= \{\text{By (88)}\} \\ &\quad \Delta^{\ell}\left(t_e\right) \cdot q^{\frac{t-t_e}{\rho}} \\ &= \{\text{By Lemma 12}\} \\ &\quad \Delta\left(t_e\right) \cdot q^{\frac{t-t_e}{\rho}}. \end{aligned}$$

Therefore, by the reasoning in Case 1 (if $t_e = t_r$) or Case 2 (if $t_e > t_r$), $g(t_0) \geq \Delta\left(t_0\right)$, and the lemma holds by the reasoning at the beginning of the proof.

Case 3.2: $t_0 \geq t_e$**.** We have

$$\begin{aligned} g(t_0) &= \{\text{By (88) with } t = t_0\} \\ &\quad \Delta^{\ell}\left(t_e\right) \cdot q^{\frac{t_0-t_e}{\rho}} \\ &= \{\text{By the definition of } \Delta^{e}\left(t\right) \text{ in (47) and the definition of } \Delta\left(t\right) \text{ in (51)}\} \\ &\quad \Delta\left(t_0\right). \end{aligned}$$

Therefore, by the reasoning at the beginning of the proof, the lemma holds. □

We now bound the value of $x_i^s(s_r) + \Delta\left(t_{i,k}^b\right)$ using Lemma 46 when $t_{i,k}^b \in I_r$. Applying this result and Lemma 20 will allow us to prove Lemma 48, which is similar to Lemma 40 for the case that $t_{i,k}^b \in I_r$.

Lemma 47. *If* $t_{i,k}^b \in I_r$*, then*

$$x_i^s(s_r) + \Delta\left(t_{i,k}^b\right) \geq \frac{W_{i,k}^r + (m - u_{tot} - 1)C_i + O_{i,k} + L_i \cdot U_i^r \cdot x_i^{p_r}\left(t_{i,k}^b\right)}{u_{tot}}.$$

Proof. We have

$$\begin{aligned} &x_i^s(s_r) + \Delta\left(t_{i,k}^b\right) \\ &\geq \{\text{By Lemma 46}\} \end{aligned}$$

$$x_i^s(s_r) + \Delta\left(t_{i,k}^b - \rho\right) \cdot q$$

$= \{$By the definition of q in (49)$\}$

$$x_i^s(s_r) + \Delta\left(t_{i,k}^b - \rho\right) \cdot \frac{\sum_{m-1 \text{ largest}} U_j^r + \max_{\tau_j \in \tau} L_j \cdot U_j^v \cdot s_r}{u_{tot}}$$

$\geq \{$By the definition of "max"$\}$

$$x_i^s(s_r) + \Delta\left(t_{i,k}^b - \rho\right) \cdot \frac{\sum_{m-1 \text{ largest}} U_j^r + L_i \cdot U_i^v \cdot s_r}{u_{tot}}$$

$\geq \{$By the definition of $x_i^s(s_r)$ in Definition 13$\}$

$$\Bigg(\sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_r \cdot x_j^s(s_r) - S_j) + \sum_{\tau_j \in \tau} S_j + (m - u_{tot} - 1)C_i + O^{rn}$$
$$+ L_i \cdot U_i^v \cdot s_r \cdot x_i^s(s_r)\Bigg)/u_{tot} + \Delta\left(t_{i,k}^b - \rho\right) \cdot \frac{\sum_{m-1 \text{ largest}} U_j^r + L_i \cdot U_i^r \cdot s_r}{u_{tot}}$$

$= \{$Rearranging$\}$

$$\Bigg(\sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_r \cdot x_j^s(s_r) - S_j) + \sum_{m-1 \text{ largest}} U_j^r \cdot \Delta\left(t_{i,k}^b - \rho\right)$$
$$+ \sum_{\tau_j \in \tau} S_j + (m - u_{tot} - 1)C_i + O^{rn} + L_i \cdot U_i^v \cdot s_r \cdot (x_i^s(s_r) + \Delta\left(t_{i,k}^b - \rho\right))\Bigg)/u_{tot}. \quad (89)$$

For simplicity, we now consider part of this expression separately.

$$\sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_r \cdot x_j^s(s_r) - S_j) + \sum_{m-1 \text{ largest}} U_j^r \cdot \Delta\left(t_{i,k}^b - \rho\right) + \sum_{\tau_j \in \tau} S_j$$

$= \{$By the definition of U_j^r in (27)$\}$

$$\sum_{m-1 \text{ largest}} (C_j + U_j^r \cdot x_j^s(s_r) - S_j) + \sum_{m-1 \text{ largest}} U_j^r \cdot \Delta\left(t_{i,k}^b - \rho\right) + \sum_{\tau_j \in \tau} S_j$$

$\geq \{$Rearranging. Although the set of tasks in the new first sum may differ from either corresponding sum in the previous expression, that can only produce a smaller result.$\}$

$$\sum_{m-1 \text{ largest}} (C_j + U_j^r \cdot (x_j^s(s_r) + \Delta\left(t_{i,k}^b - \rho\right)) - S_j) + \sum_{\tau_j \in \tau} S_j$$

$= \{$By the definition of $x_j^{p_r}\left(t_{i,k}^b\right)$ in (63)$\}$

$$\sum_{m-1 \text{ largest}} (C_j + U_j^r \cdot x_j^{p_r}\left(t_{i,k}^b\right) - S_j) + \sum_{\tau_j \in \tau} S_j$$

$= \{$By the definition of $W_{i,k}^r$ in (79)$\}$

$$W^r_{i,k}. \tag{90}$$

Thus,

$$
\begin{aligned}
& x^s_i(s_r) + \Delta\left(t^b_{i,k}\right) \\
= & \{\text{By (89) and (90)}\} \\
& \frac{W^r_{i,k} + (m - u_{tot} - 1)C_i + O^{rn} + L_i \cdot U^v_i \cdot s_r \cdot \left(x^s_i(s_r) + \Delta\left(t^b_{i,k} - \rho\right)\right)}{u_{tot}} \\
\geq & \{\text{By the definition of } U^r_i \text{ in (27) and by Lemma 45}\} \\
& \frac{W^r_{i,k} + (m - u_{tot} - 1)C_i + O_{i,k} + L_i \cdot U^r_i \cdot \left(x^s_i(s_r) + \Delta\left(t^b_{i,k} - \rho\right)\right)}{u_{tot}} \\
= & \{\text{By the definition of } x^{p_r}_i\left(t^b_{i,k}\right) \text{ in (63)}\} \\
& \frac{W_{i,k} + (m - u_{tot} - 1)C_i + O_{i,k} + L_i \cdot U^r_i \cdot x^{p_r}_i\left(t^b_{i,k}\right)}{u_{tot}}.
\end{aligned}
$$

□

The next lemma is identical to Lemma 40, but for the case that $t^b_{i,k} \in I_o$.

Lemma 48. *If* $t_a \in I_r$, $t_a = y_{i,k}$ *for some* k, $\tau_{i,k}$ *is m-dominant for* L_i, *and* $t^b_{i,k} \in I_r$, *then* $x_i(t_a) = x^s_i(s_r) + \Delta(t_a)$ *is* x*-sufficient.*

Proof. We have

$$
\begin{aligned}
& x^s_i(s_r) + \Delta(y_{i,k}) \\
\geq & \{\text{By Lemma 20 with } t_0 = t^b_{i,k} \text{ and } t_1 = y_{i,k}\} \\
& x^s_i(s_r) + \Delta\left(t^b_{i,k}\right) + \phi \cdot (y_{i,k} - t^b_{i,k}) \\
\geq & \{\text{By Lemma 47}\} \\
& \frac{W^r_{i,k} + (m - u_{tot} - 1)C_i + O_{i,k} + L_i \cdot U^r_i \cdot x^{p_r}_i\left(t^b_{i,k}\right)}{u_{tot}} + \phi \cdot (y_{i,k} - t^b_{i,k}) \\
= & \{\text{Rearranging}\}
\end{aligned}
$$

$$\frac{W^r_{i,k} + u_{tot} \cdot \phi \cdot (y_{i,k} - t^b_{i,k}) + (m - u_{tot} - 1)C_i + O_{i,k} + L_i \cdot U^r_i \cdot x^{p_r}_i \left(t^b_{i,k}\right)}{u_{tot}}. \tag{91}$$

For simplicity, we now consider two parts of this expression separately. For the first,

$$\begin{aligned}
& W^r_{i,k} + u_{tot} \cdot \phi \cdot (y_{i,k} - t^b_{i,k}) - C_i \\
\geq & \{\text{By the definition of } \phi \text{ in (46)}\} \\
& W^r_{i,k} + \left(\sum_{\tau_j \in \tau} U^r_j - u_{tot}\right) \cdot (y_{i,k} - t^b_{i,k}) - C_i \\
= & \{\text{Rearranging}\} \\
& W^r_{i,k} + \sum_{\tau_j \in \tau} U^r_j \cdot (y_{i,k} - t^b_{i,k}) + u_{tot} \cdot (y_{i,k} - t^b_{i,k}) - C_i \\
\geq & \{\text{By Lemma 44 and the definition of } R_{i,k} \text{ in (20)}\} \\
& W_{i,k} - R_{i,k} - e_{i,k}.
\end{aligned} \tag{92}$$

And for the second,

$$\begin{aligned}
x^{p_r}_i \left(t^b_{i,k}\right) = & \{\text{By the definition of } x^{p_r}_i \left(t^b_{i,k}\right) \text{ in (63)}\} \\
& x^s_i(s_r) + \Delta \left(t^b_{i,k} - \rho\right) \\
\geq & \{\text{By Lemma 20 with } t_0 = t^b_{i,k} \text{ and } t_1 = y_{i,k}\} \\
& x^s_i(s_r) + \Delta \left(y_{i,k} - \rho\right) \\
= & \{\text{By the definition of } x^{p_r}_i \left(y_{i,k}\right) \text{ in (63)}\} \\
& x^{p_r}_i \left(y_{i,k}\right).
\end{aligned} \tag{93}$$

Putting it all together,

$$\begin{aligned}
& x^s_i(s_r) + \Delta \left(y_{i,k}\right) \\
\geq & \{\text{By (91)–(93)}\} \\
& \frac{W_{i,k} - R_{i,k} + (m - u_{tot})C_i - e_{i,k} + O_{i,k} + L_i \cdot U^r_j \cdot x^{p_r}_i \left(y_{i,k}\right)}{u_{tot}} \\
\geq & \{\text{By Property 8, because } t^c_{i,k} > y_{i,k} \geq t^b_{i,k} \geq t_r\}
\end{aligned}$$

$$\frac{W_{i,k} - R_{i,k} + (m - u_{tot} - 1)e_{i,k} + O_{i,k} + L_i \cdot U_j^r \cdot x_i^{p_r}(y_{i,k})}{u_{tot}}$$

$$\geq \{\text{By Lemma 33 with } x_i^p(y_{i,k}) = x_i^{p_r}(y_{i,k}),\ s_{ub} = s_r, \text{ and the definition of } U_i^r \text{ in (27)}\}$$

$$\frac{W_{i,k} - R_{i,k} + (m - u_{tot} - 1)e_{i,k} + O_{i,k} + L_i \cdot e_{i,k}^p}{u_{tot}}$$

$$\geq \{\text{By the definition of } x_{i,k}^m \text{ in (23)}\}$$

$$x_{i,k}^m.$$

By Theorem 5, $x_i(t_a) = x_{i,k}^m$ is x-sufficient. Therefore, by Property 3 with $c_0 = x_{i,k}^m$ and $c_1 = x_i^s(s_r) + \Delta(t_a)$, $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient. □

We now combine the results of Lemmas 40 and 48 into a single lemma that addresses Case E.

Lemma 49. *If $t_a \in I_r$, $t_a = y_{i,k}$ for some $\tau_{i,k}$, and $\tau_{i,k}$ is m-dominant for L_i, then $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient.*

Proof. If $t_{i,k}^b \in I_o$, then the lemma follows from Lemma 40. Otherwise, it follows from Lemma 48. □

We finally combine the lemmas previously proved in this section to show that $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient for arbitrary $t_a \in I_r$.

Theorem 6. *For arbitrary $t_a \in I_r$, $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ is x-sufficient.*

Proof. The lemmas referenced in Figure 9 exhaustively consider all possible cases for t_a. Furthermore, for each each lemma that requires that $x_i(t_0) = x_i^s(s_r) + \Delta(t_0)$ is x-sufficient, each considered t_0 is $y_{j,\ell}$ for some $\tau_{j,\ell}$ with $y_{j,\ell} \in [t_r, t_a)$. Therefore, the x-sufficiency of $x_i(t_a) = x_i^s(s_r) + \Delta(t_a)$ follows by strong induction over all times t_0 such that $t_0 = y_{j,\ell}$ for some $\tau_{j,\ell}$ and $y_{j,\ell} \in [t_r, t_a)$ or $t_0 = t_a$. □

4.4 Determining t_n

In this subsection, we provide a condition that the system can use to determine when to return the virtual time clock to normal speed, as our definition of t_n. We then provide a bound on when that condition must occur. Then, in Section 4.5, we will prove that $x_i(t) = x_i^s(1)$ is x-sufficient for $t_a \in I_n$.

Definition 17. t_d is the earliest time after t_r such that some CPU is idle and, for each $\tau_{i,k}$ pending and incomplete at t_d, $x_i(y_{i,k}) = x_i^s(1)$ is x-sufficient.

We will later show that such a time must exist. The results in this section are actually correct for *any* time that satisfies the stated condition. However, for the smallest dissipation bounds, the earliest should be selected.

Definition 18. If there are no pending jobs at t_d, then $t_n = t_d$. Otherwise, t_n is the last completion time of any job pending at t_d.

We will prove that, if the system continues to operate without new overload with $s(t) = s_r$, it will eventually achieve a state where $x_i(y_{i,k}) = x_i^s(1)$ is x-sufficient for all new $\tau_{i,k}$. We will then prove that, in this state, a CPU will eventually become idle. Such a point in time satisfies the conditions in the definition of t_d in Definition 17, unless an earlier time satisfying the same conditions exists. Therefore, by providing a bound on that time, we provide a dissipation bound.

We first provide analysis of a key time, which we will denote t_δ, such that $x_i(t_a) = x_i^s(1)$ is x-sufficient for $t \in [t_\delta, \infty)$. The following lemma considers the value of $\Delta(t_\delta)$.

Lemma 50. *$\Delta(t_\delta) = \delta$, where*

$$t_\delta \triangleq \begin{cases} t_r + \frac{\delta - \lambda}{\phi} & \text{If } \delta > \phi \cdot \frac{\rho}{\ln q} \\ t_e + \frac{\rho}{\ln q}(\ln(\delta) - \ln(\Delta^\ell(t_e))) & \text{Otherwise.} \end{cases} \tag{94}$$

Proof. We first note that, by the definition of λ in (40),

$$\lambda \geq \delta. \tag{95}$$

We consider two cases.

Case 1: $\delta > \phi \cdot \frac{\rho}{\ln q}$**.** We have

$$t_\delta = \{\text{By the definition of } t_\delta \text{ in (94)}\}$$

$$
\begin{aligned}
& t_r + \frac{\delta - \lambda}{\phi} \\
< & \left\{\text{Because } \delta > \phi \cdot \tfrac{\rho}{\ln q}\text{, and } \phi < 0 \text{ by Lemma 9}\right\} \\
& t_r + \frac{\phi \cdot \frac{\rho}{\ln q} - \lambda}{\phi} \\
= & \{\text{Simplifying}\} \\
& t_r + \frac{\rho}{\ln q} - \frac{\lambda}{\phi} \\
= & \left\{\text{By (48) and (95) and because } \delta > \phi \cdot \tfrac{\rho}{\ln q}\right\} \\
& t_e. \qquad (96)
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
t_\delta = & \{\text{By the definition of } t_\delta \text{ in (94)}\} \\
& t_r + \frac{\delta - \lambda}{\phi} \\
\geq & \{\text{By (95), and } \phi < 0\} \\
& t_r. \qquad (97)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta\left(t_\delta\right) = & \left\{\text{By the definition of } \Delta^\ell\left(t_\delta\right) \text{ in (45), the definition of } \Delta\left(t_\delta\right) \text{ in (51), (96), and (97)}\right\} \\
& \phi \cdot \left(t_\delta - t_r\right) + \lambda \\
= & \{\text{By the definition of } t_\delta \text{ in (94)}\} \\
& \phi \cdot \left(t_r + \frac{\delta - \lambda}{\phi} - t_r\right) + \lambda \\
= & \{\text{Simplifying}\} \\
& \delta.
\end{aligned}
$$

Case 2: $\delta \leq \phi \cdot \frac{\rho}{\ln q}$**.** In this case,

$$
t_\delta = \{\text{By the definition of } t_\delta \text{ in (94)}\}
$$

$$
\begin{aligned}
& t_e + \frac{\rho}{\ln q} \cdot (\ln \delta - \ln(\Delta^\ell(t_e))) \\
\geq\ & \{\text{Because } \Delta^\ell(t_e) \geq \delta \text{ by Lemma 12 and (95), and because } q < 1 \text{ by Lemma 10}\} \\
& t_e.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta(t_\delta) =\ & \{\text{By the definition of } \Delta(t_\delta) \text{ in (51) and the definition of} \\
& \Delta^e(t_\delta) \text{ in (47)}\} \\
& \Delta^\ell(t_e) \cdot q^{\frac{t_\delta - t_e}{\rho}} \\
=\ & \{\text{Rewriting}\} \\
& \Delta^\ell(t_e) \cdot e^{\frac{\ln q}{\rho}(t_\delta - t_e)} \\
=\ & \{\text{By (94)}\} \\
& \Delta^\ell(t_e) \cdot e^{\frac{\ln q}{\rho}(t_e + \frac{\rho}{\ln q}(\ln \delta - \ln(\Delta^\ell(t_e))) - t_e)} \\
=\ & \{\text{Simplifying}\} \\
& \Delta^\ell(t_e) \cdot e^{\ln \delta - \ln(\Delta^\ell(t_e))} \\
=\ & \{\text{Simplifying}\} \\
& \delta.
\end{aligned} \tag{98}
$$

□

We next provide a sufficient condition to ensure that $x_i(y_{i,k}) = x_i^s(1)$ is x-sufficient for any job $\tau_{i,k}$ pending at t_a.

Lemma 51. *If $t_a \geq t_n^{pre}$, where*

$$
t_n^{pre} \triangleq t_\delta + \rho, \tag{99}
$$

$\tau_{i,k}$ is pending at t_a, and $t_{i,k}^c \leq t_n$, then $x_i(y_{i,k}) = x_i^s(1)$ is x-sufficient.

Proof. We have

$$
t_a - \rho \geq \{\text{By the statement of the lemma}\}
$$

$$
\begin{aligned}
& t_n^{pre} - \rho \\
= & \{\text{By the definition of } t_n^{pre} \text{ in (99)}\} \\
& t_\delta. \qquad (100)
\end{aligned}
$$

By Lemma 30, $x_i^p(t_a) = x_i^{p_r}(t_a)$ is x^p-sufficient. Therefore,

$$
\begin{aligned}
y_{i,k} \geq & \{\text{By the definition of } x^p\text{-sufficient in Definition 16}\} \\
& t_a - (x_i^{p_r}(t_a) + e_{i,k}^c(t_a)) \\
= & \{\text{By the definition of } x_i^{p_r}(t_a) \text{ in (63)}\} \\
& t_a - (x_i^s(s_r) + \Delta(t_a - \rho) + e_{i,k}^c(t_a)) \\
\geq & \{\text{By Lemma 20 with } t_0 = t_r \text{ and } t_1 = t_a - \rho\text{, and by (100)}\} \\
& t_a - (x_i^s(s_r) + \Delta(t_r) + e_{i,k}^c(t_a)) \\
= & \{\text{By Lemma 16}\} \\
& t_a - (x_i^s(s_r) + \lambda + e_{i,k}^c(t_a)) \\
\geq & \{\text{By Property 8 because } \tau_{i,k} \text{ is pending at } t_a > t_r\text{, and by the definition of } e_{i,k}^c(t_a) \text{ in} \\
& \text{Definition 2}\} \\
& t_a - (x_i^s(s_r) + \lambda + C_i) \\
= & \{\text{By the definition of } \rho \text{ in (50)}\} \\
& t_a - \rho \\
\geq & \{\text{By the definition of } t_a \text{ in the statement of the lemma}\} \\
& t_n^{pre} - \rho \\
= & \{\text{By the definition of } t_n^{pre} \text{ in (99)}\} \\
& t_\delta. \qquad (101)
\end{aligned}
$$

Therefore,

$$
x_i^s(1) = \{\text{Rearranging}\}
$$

$$
\begin{aligned}
& x_i^s(s_r) + x_i^s(1) - x_i^s(s_r) \\
\geq & \{\text{By the definition of "min"}\} \\
& x_i^s(s_r) + \min_{\tau_j \in \tau}(x_j^s(1) - x_j^s(s_r)) \\
= & \{\text{By the definition of } \delta \text{ in (41)}\} \\
& x_i^s(s_r) + \delta \\
= & \{\text{By Lemma 50}\} \\
& x_i^s(s_r) + \Delta(t_\delta) \\
\geq & \{\text{By Lemma 20 with } t_0 = t_\delta \text{ and } t_1 = y_{i,k}, \text{ and by (101)}\} \\
& x_i^s(s_r) + \Delta(y_{i,k}).
\end{aligned}
$$

If $t_{i,k}^c \leq y_{i,k}$, then by Theorem 2, Lemma 13, and Property 3 with $c_0 = 0$ and $c_1 = x_i^s(s_r) + \Delta(y_{i,k})$, $x_i(y_{i,k}) = x_i^s(s_r) + \Delta(y_{i,k})$ is x-sufficient. Otherwise, by Theorem 6, $x_i(y_{i,k}) = x_i^s(s_r) + \Delta(y_{i,k})$ is x-sufficient. In either case, by Property 3 with $c_0 = x_i^s(s_r) + \Delta(y_{i,k})$ and $c_1 = x_i^s(1)$, $x_i(y_{i,k}) = x_i^s(1)$ is x-sufficient. □

We will now show that an idle instant must occur after t_δ. To do so, we will examine an interval over which more time is available to level C than is used by level-C tasks. We first bound in Lemma 52 the time available to level C in an interval starting at t_n^{pre}, with the ending point being arbitrary. Then, in Lemma 53, we bound the work executed at level C in an identically defined interval. In Lemma 54 we combine these results to show that idleness must occur in a sufficiently long interval.

Lemma 52. *If $t_1 \geq t_n^{pre}$, then at least*

$$u_{tot} \cdot (t_1 - t_n^{pre}) - \sum_{P_p \in P} \widehat{u_p} \sigma_p$$

units of processor time are available to level C over $[t_n^{pre}, t_1)$.

Proof. By the definition of $\beta_p(t_n^{pre}, t_1)$ in Definition 7, the total amount of processor time available to level C over $[t_n^{pre}, t_1)$ is

$$\begin{aligned}
&\sum_{P_p \in P} \beta_p(t_n^{pre}, t_1) \\
\geq &\{\text{By (9)}\} \\
&\sum_{P_p \in P} (\widehat{u_p} \cdot (t_1 - t_n^{pre}) - o_p(t_n^{pre}, t_1)) \\
\geq &\{\text{By Property 9, since } t_n^{pre} \geq t_r\} \\
&\sum_{P_p \in P} (\widehat{u_p} \cdot (t_1 - t_n^{pre}) - \widehat{u_p}\sigma_p) \\
= &\{\text{Rearranging}\} \\
&\sum_{P_p \in P} \widehat{u_p} \cdot (t_1 - t_n^{pre}) - \sum_{P_p \in P} \widehat{u_p}\sigma_p \\
= &\{\text{By the definition of } u_{tot} \text{ in (10)}\} \\
&u_{tot} \cdot (t_1 - t_n^{pre}) - \sum_{P_p \in P} \widehat{u_p}\sigma_p.
\end{aligned}$$

□

We now upper bound the amount of work completed by arbitrary τ_i over an identically-defined interval. By summing over all tasks, the total amount of work completed at level C over this interval can be derived.

Lemma 53. *If $t_1 \geq t_n^{pre}$, then at most*

$$2C_i - S_i + U_i^r \cdot x_i^s(1) + U_i^r \cdot (t_1 - t_n^{pre})$$

units of work execute from τ_i over $[t_n^{pre}, t_1)$.

Proof. If a job of τ_i executes in $[t_n^{pre}, t_1)$, then it must have $r_{i,k} < t_1$. Thus, we have

$$\begin{aligned}
v(y_{i,k}) = &\{\text{By the definition of } Y_i \text{ in (6)}\} \\
&v(r_{i,k}) + Y_i
\end{aligned}$$

$$
\begin{aligned}
&< \{\text{Because } r_{i,k} < t_1\} \\
&\quad v(t_1) + Y_i.
\end{aligned}
\tag{102}
$$

We therefore define $y_{\max}$ as the time such that $v(y_{\max}) = v(t_1) + Y_i$, so that $y_{i,k} < y_{\max}$ for all $\tau_{i,k}$ executing in $[t_n^{pre}, t_1)$.

We consider two cases.

Case 1: τ_i has no pending job at t_n^{pre}. In this case, all jobs of τ_i that run in $[t_n^{pre}, t_1)$ have $t_n^{pre} < r_{i,k} \leq y_{i,k} < y_{\max}$. Therefore, by the definition of $D_i^e(t_n^{pre}, y_{\max})$ in Definition 6, the total work from τ_i that runs in $[t_n^{pre}, t_1)$ is at most

$$
\begin{aligned}
&D_i^e(t_n^{pre}, y_{\max}) \\
\leq\ &\{\text{By Lemma 26}\} \\
&D_i^C(t_n^{pre}, y_{\max}) \\
\leq\ &\{\text{By Lemma 27}\} \\
&U_i^v \cdot (v(y_{\max}) - v(t_n^{pre})) + S_i \\
=\ &\{\text{By the definition of } y_{\max} \text{ above}\} \\
&U_i^v \cdot (v(t_1) + Y_i - v(t_n^{pre})) + S_i \\
=\ &\{\text{Rearranging}\} \\
&U_i^v \cdot Y_i + S_i + U_i^v \cdot (v(t_1) - v(t_n^{pre})) \\
=\ &\{\text{By the definition of } U_i^v \text{ in (26) and the definition of } S_i \text{ in (38)}\} \\
&C_i \cdot \frac{Y_i}{T_i} + C_i \cdot \left(1 - \frac{Y_i}{T_i}\right) + U_i^v \cdot (v(t_1) - v(t_n^{pre})) \\
=\ &\{\text{Simplifying}\} \\
&C_i + U_i^v \cdot (v(t_1) - v(t_n^{pre})) \\
=\ &\{\text{By Lemma 35 and the definition of } U_i^r \text{ in (27)}\} \\
&C_i + U_i^r \cdot (t_1 - t_n^{pre}) \\
\leq\ &\{\text{Because } S_i \leq C_i \text{ by the definition of } S_i \text{ in (38), and because } U_i^r > 0 \text{ and } x_i^s(1) \geq 0 \text{ by the}
\end{aligned}
$$

definition of $x_i^s(1)$ in (37)$\}$

$2C_i - S_i + U_i^r \cdot x_i^s(1) + U_i^r \cdot (t_1 - t_n^{pre})$.

Case 2: $\tau_{i,\ell}$ is the earliest pending job of τ_i at t_n^{pre}. We will use Lemma 32 with $t_2 = t_n^{pre}$, which requires a x^p-sufficient choice of $x_i^p(t_n^{pre})$. By Lemma 30, such a choice is

$$\begin{aligned} x_i^{p_r}(t_n^{pre}) &= \{\text{By the definition of } x_i^{p_r}(t_n^{pre}) \text{ in (63)}\} \\ &\quad x_i^s(s_r) + \Delta(t_n^{pre} - \rho) \\ &= \{\text{By the definition of } t_n^{pre} \text{ in (99)}\} \\ &\quad x_i^s(s_r) + \Delta(t_\delta) \\ &= \{\text{By Lemma 50}\} \\ &\quad x_i^s(s_r) + \delta. \end{aligned} \tag{103}$$

In this case, the work in $[t_n^{pre}, t_1)$ is at most $e_{i,\ell}^r(t_n^{pre})$ plus the work contributed by jobs $\tau_{i,k}$ with $b_{i,\ell} \le r_{i,k} \le y_{i,k} < y_{\max}$. By the definition of $D_i^e(b_{i,k}, y_{\max})$ in Definition 6, the total work from τ_i that runs in $[t_n^{pre}, t_1)$ is at most

$$\begin{aligned} & e_{i,\ell}^r(t_n^{pre}) + D_i^e(t_n^{pre}, y_{\max}) \\ \le & \{\text{By Lemma 32 with } j = i,\ t_2 = t_n^{pre},\ t_3 = y_{\max},\ s_{ub} = s_r, \text{ and } x_i^p(t_n^{pre}) = x_i^s(s_r) + \delta, \text{ by the} \\ & \text{definition of } U_i^r \text{ in (27)}\} \\ & C_i + U_i^r \cdot (x_i^s(s_r) + \delta) + U_i^v \cdot (v(y_{\max}) - v(t_n^{pre})) \\ = & \{\text{By the definition of } y_{\max} \text{ above}\} \\ & C_i + U_i^r \cdot (x_i^s(s_r) + \delta) + U_i^v \cdot (v(t_1) + Y_i - v(t_n^{pre})) \\ = & \{\text{Rearranging}\} \\ & C_i + U_i^v \cdot Y_i + U_i^r \cdot (x_i^s(s_r) + \delta) + U_i^v \cdot (v(t_1) - v(t_n^{pre})) \\ = & \{\text{By the definition of } U_i^v \text{ in (26)}\} \\ & C_i + C_i \cdot \frac{Y_i}{T_i} + U_i^r \cdot (x_i^s(s_r) + \delta) + U_i^v \cdot (v(t_1) - v(t_n^{pre})) \\ = & \{\text{Rearranging}\} \end{aligned}$$

$$2C_i - C_i \cdot \left(1 - \frac{Y_i}{T_i}\right) + U_i^r \cdot (x_i^s(s_r) + \delta) + U_i^v \cdot (v(t_1) - v(t_n^{pre}))$$

$= \{$By the definition of S_i in (38)$\}$

$$2C_i - S_i + U_i^r \cdot (x_i^s(s_r) + \delta) + U_i^v \cdot (v(t_1) - v(t_n^{pre}))$$

$= \{$By Lemma 35 and the definition of U_i^r in (27)$\}$

$$2C_i - S_i + U_i^r \cdot (x_i^s(s_r) + \delta) + U_i^r \cdot (t_1 - t_n^{pre})$$

$\leq \{$By the definition of δ in (41)$\}$

$$2C_i - S_i + U_i^r \cdot (x_i^s(s_r) + x_i^s(1) - x_i^s(s_r)) + U_i^r \cdot (t_1 - t_n^{pre})$$

$= \{$Simplifying$\}$

$$2C_i - S_i + U_i^r \cdot x_i^s(1) + U_i^r \cdot (t_1 - t_n^{pre}).$$

□

We now combine these results to show that idleness will happen in a sufficiently long interval starting at t_n^{pre}.

Lemma 54. *If $t_1 > t_n^{pre} + F$, where*

$$F \triangleq \frac{\sum_{P_p \in P} \widehat{u_p}\sigma_p + \sum_{\tau_i \in \tau}(2C_i - S_i + U_i^r \cdot (x_i^s(s_r) + \delta))}{u_{tot} - \sum_{\tau_i \in \tau} U_i^r} \tag{104}$$

(oFfset), then some CPU is idle for a nonzero period of time in $[t_n^{pre}, t_1)$.

Proof. We show that the difference between CPUs available to level C and level-C work that completes in $[t_n^{pre}, t_1)$ is positive. Using Lemmas 52 and 53, this difference is at least

$$u_{tot} \cdot (t_1 - t_n^{pre}) - \sum_{P_p \in P} \widehat{u_p}\sigma_p - \sum_{\tau_i \in \tau}(2C_i - S_i + U_i^r \cdot (x_i^s(s_r) + \delta) + U_i^r \cdot (t_1 - t_n^{pre}))$$

$= \{$Rearranging$\}$

$$\left(u_{tot} - \sum_{\tau_i \in \tau} U_i^r\right) \cdot (t_1 - t_n^{pre}) - \left(\sum_{P_p \in P} \widehat{u_p}\sigma_p + \sum_{\tau_i \in \tau}(2C_i - S_i + U_i^r \cdot (x_i^s(s_r) + \delta))\right)$$

$> \{$By the statement of the lemma$\}$

$$\left(u_{tot} - \sum_{\tau_i \in \tau} U_i^r\right) \cdot (t_n^{pre} + F - t_n^{pre}) - \left(\sum_{P_p \in P} \widehat{u_p}\sigma_p + \sum_{\tau_i \in \tau}(2C_i - S_i + U_i^r \cdot (x_i^s(s_r) + \delta))\right)$$

$= \{$By the definition of F in (104)$\}$

$0.$

$\square$

We now use this result, combined with Lemma 51 above, to bound t_d.

Lemma 55. $t_d \leq t_n^{pre} + F$.

Proof. By Lemma 54, there is a time in $[t_n^{pre}, t_n^{pre} + F]$ such that at least one CPU is idle. Furthermore, by Lemma 51, $x_i(y_{i,k}) = x_i^s(1)$ is x-sufficient for all $\tau_{i,k}$ pending and incomplete at this time. Therefore, the lemma follows by the definition of t_d in Definition 17. $\square$

Finally, we use the definition of t_d in Definition 17 and our bound on it in Lemma 55 in order to bound t_n.

Lemma 56. $t_n \leq t_n^{pre} + F + \max_{\tau_i \in \tau}(Y_i/s_r + x_i^s(1) + C_i)$.

Proof. If there are no jobs pending at t_d, then

$$
\begin{aligned}
t_n &= \{\text{By the definition of } t_n \text{ in Definition 18}\}\\
&\quad t_d\\
&\leq \{\text{By Lemma 55}\}\\
&\quad t_n^{pre} + F\\
&\leq \{\text{Because } Y_i \geq 0,\ s_r > 0,\ x_i^s(1) \geq 0 \text{ by the definition of } x_i^s(1) \text{ in (37), and } C_i > 0\}\\
&\quad t_n^{pre} + F + \max_{\tau_i \in \tau}(C_i/s_r + x_i^s(1) + C_i).
\end{aligned}
$$

Otherwise, let $\tau_{i,k}$ be the pending job at t_d with the latest completion time, so that $t_n = t_{i,k}^c$. We have

$$
\begin{aligned}
v(y_{i,k}) - v(t_d) &= \{\text{By the definition of } Y_i \text{ in (6)}\}\\
&\quad v(r_{i,k}) + Y_i - v(t_d)\\
&\leq \{\text{Because } \tau_{i,k} \text{ is pending at } t_d\}
\end{aligned}
$$

$$
\begin{aligned}
& v(t_d) + Y_i - v(t_d) \\
= & \{\text{Rearranging}\} \\
& Y_i.
\end{aligned} \tag{105}
$$

By Lemma 35 and (105), $s_r \cdot (y_{i,k} - t_d) \leq Y_i$. Rearranging,

$$y_{i,k} \leq t_d + Y_i/s_r. \tag{106}$$

Because $\tau_{i,k}$ is pending at t_d, by the definition of t_d in Definition 17, $x_i(y_{i,k}) = x_i^s(1)$ is x-sufficient. Thus, we have

$$
\begin{aligned}
t_{i,k}^c \leq & \{\text{By the definition of } x\text{-sufficient in Definition 8 and the definition of } t_d \text{ in Definition 17}\} \\
& y_{i,k} + x_i^s(1) + e_{i,k} \\
\leq & \{\text{By Property 8, because } \tau_{i,k} \text{ is pending at } t_d > t_r\} \\
& y_{i,k} + x_i^s(1) + C_i \\
\leq & \{\text{By (106)}\} \\
& t_d + Y_i/s_r + x_i^s(1) + C_i \\
\leq & \{\text{By Lemma 55}\} \\
& t_n^{pre} + F + Y_i/s_r + x_i^s(1) + C_i \\
\leq & \{\text{By the definition of “max”}\} \\
& t_n^{pre} + F + \max_{\tau_i \in \tau}(Y_i/s_r + x_i^s(1) + C_i).
\end{aligned}
$$

□

4.5 Proving that $x_i(t_a) = x_i^s(1)$ is x-sufficient for $t_a \in I_n$

In this subsection, we demonstrate that $x_i(t) = x_i^s(1)$ is x-sufficient for $t_a \in I_n$. Observe that unlike in Section 4.2, our choice of $x_i(t_a)$ does not depend on the specific value of t_a. We will consider the same cases that have been considered in previous sections, as now depicted in Figure 10.

We first consider Case A in Figure 10, in which $t_a < y_{i,0}$.

A. $t_a < y_{i,0}$ (Lemma 57).
B. $t_a = y_{i,k}$ for some k and $t^c_{i,k} \leq y_{i,k} + e_{i,k}$ (Lemma 58).
C. $t_a \in (y_{i,k}, y_{i,k+1})$ for some k (Lemma 60).
D. $t_a = y_{i,k}$ for some k, $t^c_{i,k} > y_{i,k} + e_{i,k}$, and $\tau_{i,k}$ is f-dominant for L_i (Lemma 61).
E. $t_a = y_{i,k}$ for some k, $t^c_{i,k} > y_{i,k} + e_{i,k}$, and $\tau_{i,k}$ is m-dominant for L_i (Lemma 63).

Figure 10: Cases considered when proving that $x_i(t_a) = x^s_i(1)$ is x-sufficient for $t_a \in I_n$

Lemma 57. *If $t_a < y_{i,0}$, then $x_i(t_a) = x^s_i(1)$ is x-sufficient.*

Proof. If $t_a < y_{i,0}$, then by Theorem 1, $x_i(t_a) = 0$ is x-sufficient. Furthermore, by the definition of $x^s_i(1)$ in (37), $x^s_i(1) \geq 0$. Therefore, by Property 3 with $c_0 = 0$ and $c_1 = x^s_i(1)$, $x_i(t_a) = x^s_i(1)$ is x-sufficient. □

The analysis of Case B in Figure 10 is simple, as was the case when analyzing the analogous case for $t_a \in I_r$.

Lemma 58. *If $t^c_{i,k} \leq y_{i,k} + e_{i,k}$, then $x_i(y_{i,k}) = x^s_i(1)$ is x-sufficient.*

Proof. By Theorem 2, $x_i(t_a) = 0$ is x-sufficient. Furthermore, by the definition of $x^s_i(1)$ in (37), $x^s_i(1) \geq 0$. Therefore, by Property 3 with $c_0 = 0$ and $c_1 = x^s_i(1)$, $x_i(t_a) = x^s_i(1)$ is x-sufficient. □

We will next consider Case C in Figure 10, in which $t_a \in (y_{i,k}, y_{i,k+1})$ for some k. For many of the results in this section, including our analysis of Case C, we will inductively assume that $x_i(y_{i,\ell}) = x^s_i(1)$ is x-sufficient for jobs with $y_{i,\ell} \in [t_d, t_a)$. Furthermore, the definition of t_d in Definition 17 allows us to make the same assumption about *any* job pending at t_a with $y_{i,\ell} \in (-\infty, t_a)$. This assumption will be used in many places in our proofs, so we state it as a separate lemma.

Lemma 59. *If $t_2 \in [t_d, \infty)$ and, for all $\tau_{i,\ell}$ with $y_{i,\ell} \in [t_d, t_2)$, $x_i(y_{i,\ell}) = x^s_i(1)$ is x-sufficient, then $x_i(y_{i,\ell}) = x^s_i(1)$ is x-sufficient for* all *jobs pending at t_2 with $y_{i,\ell} \in (-\infty, t_2)$.*

Proof. We consider an arbitrary $\tau_{i,\ell}$ with $y_{i,\ell} \in (-\infty, t_2)$. We consider two cases, depending on the value of $r_{i,\ell}$.

Case 1: $r_{i,\ell} \in (-\infty, t_d)$**.** In this case, because $\tau_{i,\ell}$ is pending at $t_2 \geq t_d$, $\tau_{i,\ell}$ is pending at t_d. Therefore, the lemma follows from the definition of t_d in Definition 17.

Case 2: $r_{i,\ell} \in [t_d, t_2)$**.** In this case, because $y_{i,\ell} \geq r_{i,\ell}$, $y_{i,\ell} \in [t_d, t_2)$. Thus, the lemma is true by assumption. □

We now address Case C directly.

Lemma 60. *If $t_a \in [t_d, \infty)$, $t_a \in (y_{i,k}, y_{i,k+1})$ for some k, and, for all $\tau_{i,\ell}$ with $y_{i,\ell} \in [t_d, t_a)$, $x_i(y_{i,\ell}) = x_i^s(1)$ is x-sufficient, then $x_i(t_a) = x_i^s(1)$ is x-sufficient.*

Proof. We consider two subcases, depending on whether $\tau_{i,k}$ is still pending at t_a.

Case 1: $\tau_{i,k}$ **is no longer pending at** t_a**.** In this case,

$$\begin{aligned} t_{i,k}^c &< t_a \\ &< \{\text{Because } x_i^s(1) \geq 0 \text{ by the definition of } x_i^s(1) \text{ in (37) and } e_{i,k} > 0\} \\ &\quad t_a + x_i^s(1) + e_{i,k} \end{aligned}$$

By the definition of x-sufficient in Definition 8, $x_i(t_a) = x_i^s(1)$ is x-sufficient.

Case 2: $\tau_{i,k}$ **is pending at** t_a**.** By Lemma 59 with $t_2 = t_a$, $x_i(y_{i,k}) = x_i^s(1)$ is x-sufficient. Furthermore, by Theorem 3 with $t = t_a$, $x_i(t_a) = \max\{0, x_i^s(1) - (t_a - y_{i,k})\}$ is x-sufficient. Additionally,

$$\begin{aligned} x_i^s(1) &\geq \{\text{Because } x_i^s(1) > 0 \text{ and } t_a > y_{i,k}\} \\ &\quad \max\{0, x_i^s(1) - (t_a - y_{i,k})\}. \end{aligned}$$

Therefore, by Property 3 with $c_0 = \max\{0, x_i^s(1) - (t_a - y_{i,k})\}$, and $c_1 = x_i^s(1)$, $x_i(t_a) = x_i^s(1)$ is x-sufficient. □

We now address Case D, in which $t_a = y_{i,k}$ for some k and $\tau_{i,k}$ is f-dominant for L_i.

Lemma 61. *If $t_a \in I_n$, $t_a = y_{i,k}$ for some $\tau_{i,k}$, $\tau_{i,k}$ is f-dominant for L_i, and $x_i(y_{i,\ell}) = x_i^s(1)$ is x-sufficient for all $\tau_{i,\ell}$ such that $y_{i,\ell} \in [t_d, y_{i,k})$, then $x_i(t_a) = x_i^s(1)$ is x-sufficient.*

Proof. Lemma 23 describes the relationship between $y_{i,k-1}$ and $y_{i,k}$ in virtual time. We now bound their difference in actual time.

$$\begin{aligned}
(y_{i,k} - y_{i,k-1}) &\geq \{\text{By Lemma 31 with } s_{ub} = 1\} \\
&\quad v(y_{i,k}) - v(y_{i,k-1}) \\
&\geq \{\text{By Lemma 23}\} \\
&\quad v(y_{i,k-1}) + T_i - v(y_{i,k-1}) \\
&= \{\text{Rearranging}\} \\
&\quad T_i.
\end{aligned}$$

Rearranging,

$$y_{i,k-1} \leq y_{i,k} - T_i. \tag{107}$$

We have

$$\begin{aligned}
x_i^s(1) &= \{\text{Rewriting}\} \\
&\quad y_{i,k-1} + x_i^s(1) + C_i - y_{i,k-1} - C_i \\
&\geq \{\text{By Property 8 and the definition of f-dominant for } L_i \text{ in Definition 10}\} \\
&\quad y_{i,k-1} + x_i^s(1) + e_{i,k-1} - y_{i,k-1} - C_i \\
&\geq \{\text{By Lemma 59, (107), and the definition of } x\text{-sufficient in Definition 8}\} \\
&\quad t^c_{i,k-1} - y_{i,k-1} - C_i \\
&\geq \{\text{By (107)}\} \\
&\quad t^c_{i,k-1} - y_{i,k} + T_i - C_i \\
&\geq \{\text{By the choice of } L_i \text{ in Definition 12}\} \\
&\quad t^c_{i,k-1} - y_{i,k} + A_i^{rn}(m - L_i - 1) - C_i \\
&\geq \{\text{By Lemma 8}\} \\
&\quad t^c_{i,k-1} - y_{i,k} + A_{i,k}(m - L_i - 1) - e_{i,k} \\
&= \{\text{By the definition of } x^f_{i,k} \text{ in (15)}\}
\end{aligned}$$

$x_{i,k}^f$.

Furthermore, by Theorem 4, because $\tau_{i,k}$ is f-dominant for L_i, $x_i(t_a) = x_i(y_{i,k}) = x_{i,k}^f$ is x-sufficient. Therefore, by Property 3 with $c_0 = x_{i,k}^f$ and $c_1 = x_i^s(1)$, $x_i(t_a) = x_i^s(1)$ is x-sufficient. □

We finally consider Case E in Figure 10, in which $t_a = y_{i,k}$ for some k and $\tau_{i,k}$ is m-dominant for L_i. We will reuse many of the lemmas from our analysis of the same case with $t_a \in I_r$. However, we will show in Lemma 62 below that in some circumstances we can use $x_i^p(t) = x_i^s(1)$ in place of $x_i^p(t) = x_i^{p_r}(t)$. The proof of Lemma 62, like the proof of Lemma 30, is based on Lemma 29 that considers $x_i(t)$ for certain values of t.

Lemma 62. *Let τ_j be arbitrary. If $t_2 \in [t_d, \infty)$ and for all $\tau_{j,\ell}$ such that $y_{j,\ell} \in [t_d, t_2)$, $x_j(y_{j,\ell}) = x_j^s(1)$ is x-sufficient, then $x_j^p(t_2) = x_j^s(1)$ is x^p-sufficient.*

Proof. We consider an arbitrary $\tau_{j,\ell}$ pending at t_2. We consider two cases, depending on the value of $y_{j,\ell}$.

Case 1: $y_{j,\ell} \in (-\infty, t_2)$**.** In this case, by Lemma 59 with $i = j$, $x_j(y_{j,\ell}) = x_j^s(1)$ is x-sufficient. Therefore, by Lemma 29, $y_{j,\ell} \geq t_2 - (x_j^s(1) + e_{j,\ell}^c(t_2))$.

Case 2: $y_{j,\ell} \in [t_2, \infty)$**.** In this case,

$$\begin{aligned} y_{j,\ell} &\geq t_2 \\ &\geq \{\text{Because } x_i^s(1) \geq 0 \text{ by the definition of } x_j^s(1) \text{ in (37), and because } e_{j,\ell}^c(t_2) \geq 0\} \\ &\quad t_2 - (x_j^s(1) + e_{j,\ell}^c(t_2)) \end{aligned}$$

Because $\tau_{j,\ell}$ was arbitrary, the lemma holds by the definition of x^p-sufficient in Definition 16. □

We now provide the analysis for Case E in Figure 10.

Lemma 63. *If $t_a = y_{i,k}$ for some k, $t_a \in [t_d, \infty)$, $x_j(y_{j,\ell}) = x_j^s(1)$ is x-sufficient for all $\tau_{j,\ell}$ with $y_{j,\ell} \in [t_d, t_a)$, and $\tau_{i,k}$ is m-dominant for L_i, then $x_i(t_a) = x_i^s(1)$ is x-sufficient.*

Proof. Because some processor is idle at t_d, by the definition of $t^b_{i,k}$, $t^b_{i,k} \geq t_d$. We have

$$x^s_i(1) \geq \{\text{By the definition of } x^s_i(1) \text{ in (37)}\}$$
$$\Bigg(\sum_{m-1 \text{ largest}} (C_j + U^v_j \cdot x^s_j(1) - S_j) + \sum_{\tau_j \in \tau} S_j + (m - u_{tot} - 1)C_i + O^{rn}$$
$$+ L_i \cdot U^v_i \cdot x^s_i(1)\Bigg)/u_{tot}. \tag{108}$$

For simplicity, we separately consider a subset of this expression.

$$\sum_{m-1 \text{ largest}} (C_j + U^v_j \cdot x^s_j(1) - S_j) + \sum_{\tau_j \in \tau} S_j - C_i$$
$$= \{\text{Rewriting}\}$$
$$\sum_{m-1 \text{ largest}} (C_j + U^v_j \cdot x^s_j(1) - S_j) + \sum_{\tau_j \in \tau} S_j + \sum_{\tau_j \in \tau} U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k}))$$
$$- \sum_{\tau_j \in \tau} U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) + e_{i,k} - C_i - e_{i,k}$$
$$\geq \{\text{By Lemma 43 with } s_{ub} = 1 \text{ and by Lemma 62}\}$$
$$W_{i,k} - \sum_{\tau_j \in \tau} U^v_j \cdot (v(y_{i,k}) - v(t^b_{i,k})) - e_{i,k}$$
$$\geq \{\text{By Property 11}\}$$
$$W_{i,k} - u_{tot} \cdot (v(y_{i,k}) - v(t^b_{i,k})) - e_{i,k}$$
$$\geq \{\text{By Lemma 31 with } s_{ub} = 1\}$$
$$W_{i,k} - u_{tot} \cdot (y_{i,k} - t^b_{i,k}) - e_{i,k}$$
$$= \{\text{By the definition of } R_{i,k} \text{ in (20)}\}$$
$$W_{i,k} - R_{i,k} - e_{i,k}. \tag{109}$$

Putting it all together,

$$x^s_i(1) \geq \{\text{By (108) and (109)}\}$$
$$\frac{W_{i,k} - R_{i,k} + (m - u_{tot})C_i - e_{i,k} + O^{rn} + L_i \cdot U^v_i \cdot x^s_i(1)}{u_{tot}}$$
$$\geq \{\text{By Lemma 33 with } s_{ub} = 1 \text{ and Lemma 62}\}$$

$$
\begin{aligned}
&\frac{W_{i,k} - R_{i,k} + (m - u_{tot})C_i - e_{i,k} + O^{rn} + L_i \cdot e^p_{i,k}}{u_{tot}} \\
\geq\ &\{\text{By Property 8, because } t^c_{i,k} > y_{i,k} \geq t^b_{i,k} > t_r\} \\
&\frac{W_{i,k} - R_{i,k} + (m - u_{tot} - 1)e_{i,k} + O^{rn} + L_i \cdot e^p_{i,k}}{u_{tot}} \\
\geq\ &\{\text{By Lemma 45}\} \\
&\frac{W_{i,k} - R_{i,k} + (m - u_{tot} - 1)e_{i,k} + O_{i,k} + L_i \cdot e^p_{i,k}}{u_{tot}} \\
=\ &\{\text{By the definition of } x^m_{i,k} \text{ in (23)}\} \\
&x^m_{i,k}. \qquad (110)
\end{aligned}
$$

Because $\tau_{i,k}$ is m-dominant for L_i, by Theorem 5, $x_i(t_a) = x^m_{i,k}$ is x-sufficient. Thus, by Property 3 with $c_0 = x^m_{i,k}$ and $c_1 = x^s_i(1)$, $x_i(t_a) = x^s_i(1)$ is x-sufficient. □

We finally combine the lemmas previously proved in this subsection to show that $x_i(t_a) = x^s_i(1)$ is x-sufficient for arbitrary $t_a \in I_n$.

Theorem 7. *For arbitrary $t_a \in I_n$, $x_i(t_a) = x^s_i(1)$ is x-sufficient.*

Proof. The lemmas referenced in Figure 9 exhaustively consider all possible cases for t_a. Furthermore, for each lemma requires that $x_i(t_0) = x^s_i(1)$ is x-sufficient for some t_0, the considered t_0 is $y_{j,\ell}$ for some $\tau_{j,\ell}$ with $y_{j,\ell} \in [t_d, t_a)$. Therefore, the x-sufficiency of $x_i(t_a) = x^s_i(1)$ follows by strong induction over all times t_0 such that $t_0 = y_{j,\ell}$ for some $\tau_{j,\ell}$ and $y_{j,\ell} \in [t_d, t_a)$ or $t_0 = t_a$. □

BIBLIOGRAPHY

References

Baruah, S., Koren, G., Mishra, B., Raghunathan, A., Rosier, L., and Shasha, D. (1991). On-line scheduling in the presence of overload. In *Proceedings of the 32nd Annual Symposium On Foundations of Computer Science*, pages 100–110.

Beccari, G., Caselli, S., Reggiani, M., and Zanichelli, F. (1999). Rate modulation of soft real-time tasks in autonomous robot control systems. In *Proceedings of the 11th Euromicro Conference on Real-Time Systems*, pages 21–28.

Block, A. (2008). *Adaptive Multiprocessor Real-Time Systems*. PhD thesis, The University of North Carolina at Chapel Hill.

Burns, A. and Davis, R. (2013). Mixed criticality systems - a review. `http://www-users.cs.york.ac.uk/~burns/review.pdf`.

Buttazzo, G. and Stankovic, J. (1993). RED: Robust earliest deadline scheduling. In *Proceedings of the 3rd International Workshop on Responsive Computing Systems*, pages 100–111.

Chakraborty, S., Kunzli, S., and Thiele, L. (2003). A general framework for analysing system properties in platform-based embedded system designs. In *Proceedings of the 2003 Design, Automation and Test in Europe Conference and Exhibition*, pages 190–195.

Erickson, J. P., Anderson, J. H., and Ward, B. C. (2014). Fair lateness scheduling: reducing maximum lateness in G-EDF-like scheduling. *Real-Time Systems*, 50(1):5–47.

Garyali, P. (2010). On best-effort utility accrual real-time scheduling on multiprocessors. Master's thesis, The Virginia Polytechnic Institute and State University.

Herman, J., Kenna, C., Mollison, M., Anderson, J., and Johnson, D. (2012). Rtos support for multicore mixed-criticality systems. In *Proceedings of the 18th IEEE Real-Time and Embedded Technology and Applications Symposium*, pages 197–208.

Jan, M., Zaourar, L., and Pitel, M. (2013). Maximizing the execution rate of low-criticality tasks in mixed criticality systems. In *Proceedings of the 1st Workshop on Mixed Criticality Systems*, pages 43–48.

Koren, G. and Shasha, D. (1992). d^{over}: an optimal on-line scheduling algorithm for overloaded real-time systems. In *Proceedings of the 13th Real-Time Systems Symposium*, pages 290–299.

Leontyev, H. and Anderson, J. H. (2010). Generalized tardiness bounds for global multiprocessor scheduling. *Real-Time Systems*, 44(1-3):26–71.

Leontyev, H., Chakraborty, S., and Anderson, J. H. (2011). Multiprocessor extensions to real-time calculus. *Real-Time Systems*, 47(6):562–617.

Locke, C. (1986). *Best-Effort Decision Making for Real-Time Scheduling*. PhD thesis, Carnegie Mellon University.

Mollison, M. S., Erickson, J. P., Anderson, J. H., Baruah, S. K., and Scoredos, J. A. (2010). Mixed-criticality real-time scheduling for multicore systems. In *Proceedings of the IEEE International Conference on Embedded Software and Systems*, pages 1864–1871, Washington, DC, USA. IEEE Computer Society.

Santy, F., George, L., Thierry, P., and Goossens, J. (2012). Relaxing mixed-criticality scheduling strictness for task sets scheduled with fp. In *Proceedings of the 24th Euromicro Conference on Real-Time Systems*, pages 155–165.

Santy, F., Raravi, G., Nelissen, G., Nelis, V., Kumar, P., Goossens, J., and Tovar, E. (2013). Two protocols to reduce the criticality level of multiprocessor mixed-criticality systems. In *Proceedings of the 21st International Conference on Real-Time Networks and Systems*, pages 183–192, New York, NY, USA. ACM.

Stoica, I., Abdel-Wahab, H., Jeffay, K., Baruah, S., Gehrke, J., and Plaxton, C. (1996). A proportional share resource allocation algorithm for real-time, time-shared systems. In *Proceedings of the 17th IEEE Real-Time Systems Symposium*, pages 288–299.

Su, H. and Zhu, D. (2013). An elastic mixed-criticality task model and its scheduling algorithm. In *Proceedings of the 2013 Design, Automation Test in Europe Conference Exhibition (DATE)*, pages 147–152.

Su, H., Zhu, D., and Mosse, D. (2013). Scheduling algorithms for elastic mixed-criticality tasks in multicore systems. In *Proceedings of the 19th IEEE International Conference on Embedded and Real-Time Computing Systems and Applications*, pages 352–357.

Zhang, L. (1990). Virtual clock: A new traffic control algorithm for packet switching networks. In *Proceedings of the 5th ACM Symposium on Communications Architectures & Protocols*, pages 19–29, New York, NY, USA. ACM.

A Notation

$A_{i,k}(v)$	Function used to account for completion time in the Few Tasks Case (See (13))
$A_i^{rn}(v)$	Upper bound on $A_{i,k}(v)$ for $I_r \cup I_n$ (see (31))
$b_{i,k}$	Earliest time such that $v(b_{i,k}) = v(r_{i,k}) + T_i$ (see Definition 5)
C_i	Worst-case execution time of τ_i in the absence of overload
$D_i^C(t_0, t_1)$	Upper bound on $D_i^e(t_0, t_1)$ when $t_0 > t_r$ (see Definition 6)
$D_i^e(t_0, t_1)$	Total execution cost from jobs $\tau_{i,k}$ with $t_0 \leq r_{i,k} \leq y_{i,k} \leq t_1$ (see Definition 6)
$e_{i,k}$	Actual execution of $\tau_{i,k}$
$e_{i,k}^c(t)$	Work completed by $\tau_{i,k}$ before actual time t
$e_{i,k}^p$	Work after $y_{i,k}$ for jobs of τ_i prior to $\tau_{i,k}$ (See Definition 9)
$e_{i,k}^r(t)$	Work remaining for $\tau_{i,k}$ after actual time t
F	Constant that guarantees idleness in $[t_n^{pre}, t_n^{pre} + F]$ (see (104))
I_n	Interval after the system has recovered from an overload and is operating at normal speed
I_o	Interval during which overload occurs, $(-\infty, t_r)$ (see (28))
I_r	Interval during which the system is recovering from overload
L	Arbitrary integer parameter with $0 \leq L < m$
L_i	Selection of L for τ_i in Section 4 (see Definition 12)
m	Number of CPUs in the system
n	Number of level-C tasks in the system
$O_{i,k}$	Term to account for supply restriction overload (See (19))
$O_{i,k}^o$	Term used to bound the contribution to $O_{i,k}$ from I_o
O^{rn}	Term used to bound the contribution to $O_{i,k}$ from $I_r \cup I_n$
$o_p(t_0, t_1)$	Supply restriction overload over $[t_0, t_1)$ (see (8))
P	Set of all processors

P_p	Processor p
q	Base of the exponential function in $\Delta^e(t)$ (see (49))
$r_{i,k}$	Release time of $\tau_{i,k}$
$R_{i,k}$	Term to account for work completed in $[t^b_{i,k}, y_{i,k})$ (See (20))
$R^o_{i,k}$	Term to account for the contribution of I_o to $R_{i,k}$
S_i	Term used to account for $Y_i < T_i$ (see (38))
$s(t)$	Speed of virtual time at actual time t
s_r	Speed of virtual time during recovery interval
T_i	Minimum separation time between jobs of τ_i in virtual time
t_a	Actual time under immediate analysis
$t^c_{i,k}$	Completion time of $\tau_{i,k}$
t_d	Time at which some processor is idle and $x_j(y_{j,\ell}) = x^s_j(1)$ for all pending $\tau_{j,\ell}$ (see Definition 17)
t_e	Time at which $\Delta(t)$ switches from linear to exponential (see (48))
t_n	Time at which the virtual clock can return to normal speed, because all jobs pending at t_d are complete (see Definition 18)
t^{pre}_n	Time after which $x_i(y_{i,k}) = x^s_i(1)$ is x-sufficient for all pending jobs (see (99))
t_r	Time at start of recovery interval I_r
t_s	Time at which virtual clock actually slows to stable value
t_δ	Time at which $\Delta(t_\delta) = \delta$ (see (94))
$\widehat{u_p}$	Nominal utilization (of availability) of P_p
u_{tot}	Sum of nominal utilizations over all processors (see (10))
U^r_i	Utilization of τ_i in I_n (see (27))
U^v_i	Utilization of τ_i in virtual time (see (26)
$v(t)$	Virtual time corresponding to actual time t (see (4))
$W_{i,k}$	Term to account for work (See (16))
$W^o_{i,k}$	Term to account for the contribution of I_o to $W_{i,k}$
$W^r_{i,k}$	Term to account for part of $W_{i,k}$ when $t^b_{i,k} \in I_r$

$x_i(t)$	Upper bound described in Definition 8
$\dot{x}_i(t)$	Upper bound described in Definition 14
$x_i^p(t)$	Bound described in Definition 16
$x_i^{p_r}(t)$	Particular choice of $x_i^p(t)$ defined in (63)
$x_i^s(s_I)$	Asymptotic bound on $x_i(t)$ in the absence of overload when $s(t) = s_I$
$x_{i,k}^f$	x-sufficient choice of $x_i(y_{i,k})$ in the Few Tasks Case
$x_{i,k}^m$	x-sufficient choice of $x_i(y_{i,k})$ in the Many Tasks Case
$y_{i,k}$	PP of $\tau_{i,k}$
Y_i	Relative PP of τ_i in virtual time
$\beta_p(t_0, t_1)$	Total time in $[t_0, t_1)$ when P_p is available to level C (see Definition 7)
δ	$\min_{\tau_i \in \tau} x_i^s(1) - x_i^s(s_r)$ (defined in (41))
$\Delta(t)$	Function such that $x_i(t) = x_i^s(s_r) + \Delta(t)$ is x-sufficient for t in I_r (see (51))
$\Delta^e(t)$	Exponential component of $\Delta(t)$ (see (47))
$\Delta^\ell(t)$	Linear component of $\Delta(t)$ (see (45))
$\theta_{i,k}$	A set of jobs pending at $t_{i,k}^b$ (see Lemma 4)
$\overline{\theta_{i,k}}$	Set of tasks without jobs in $\theta_{i,k}$ (see Lemma 4)
κ	Jobs with $y_{i,k} \in I_o$ and $t_{i,k}^c \in I_o \cup I_r$
λ	Constant value of $\Delta(t)$ in $(-\infty, t_r)$ (see (40))
ρ	Upper bound on amount of time that a job is pending after t_r or its PP in I_r (see (50))
σ_p	Constant used to characterize supply restriction (see Property 9)
τ	Set of all level-C tasks
τ_i	Task i
$\tau_{i,k}$	Job k of τ_i
ϕ	Slope of $\Delta^\ell(t)$ (see (46))
ψ	Jobs with $t_{i,k}^b \in I_o$ and $y_{i,k} \in I_n \cup I_r$
$\Omega_{i,k}(j)$	Indicator variable to account for pessimism in $W_{i,k}$ (see (76))

B Computing and Analyzing $x_i^s(s_I)$

Recall that $x_i^s(s_I)$ is defined in (37). As promised in Section 4, we now describe how to use linear programming to compute $x_i^s(s_I)$ and prove several results mentioned in that section.

Our technique for computing $x_i^s(s_I)$ involves formulating a linear program very similar to that described by Erickson et al. (2014) in the absence of restricted supply.

The formulation of our LP is based on the following theorem, which corresponds to Corrollary 1 in (Erickson et al., 2014).

Theorem 8. *If* $\forall i, u_{tot} - L_i \cdot U_i^v \cdot s_I > 0$ *and*

$$\forall i, x_i^c \triangleq \max\left\{0, \frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v \cdot s_I}\right\}, \tag{111}$$

*(*choice*), where*

$$s \triangleq \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j) + \sum_{\tau_j \in \tau} S_j, \tag{112}$$

then $x_i^s(s_I) = x_i^c$ *satisifies the definition of* $x_i^s(s_I)$ *in (37).*

Observe that s *is independent of the task index* i*.*

Proof. Let $\tau_i \in \tau$ be arbitrary. We consider two cases, one for each term of the $\max$ in the definition of x_i^c in the statement of the lemma.

Case 1: $x_i^c = 0$**.** In this case, by the definition of x_i^c in the statement of the lemma and the condition that $u_{tot} - L_i \cdot U_i^v \cdot s_I > 0$,

$$s + (m - u_{tot} - 1)C_i + O^{rn} \leq 0. \tag{113}$$

Therefore,

$$\Bigg(\sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j) + \sum_{\tau_j \in \tau} S_j + (m - u_{tot} - 1)C_i + O^{rn}$$
$$+ L_i \cdot U_j^v \cdot s_I \cdot x_i^c\Bigg)/u_{tot}$$
$$= \{\text{By the definition of } s \text{ in (112)}\}$$

$$
\begin{aligned}
&\frac{s + (m - u_{tot} - 1)C_i + O^{rn} + L_i \cdot U_i^v \cdot s_I \cdot x_i^c}{u_{tot}} \\
&= \{\text{By the case we are considering}\} \\
&\frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot}} \\
&\leq \{\text{By (113)}\} \\
&0.
\end{aligned}
$$

Thus, $x_i^s(s_I) = x_i^c$ satisfies the definition of $x_i^s(s_I)$ in (37) for τ_i.

Case 2: $x_i^c > 0$. In this case, by the definition of x_i^c in the statement of the lemma we have

$$x_i^c = \frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v \cdot s_I}.$$

We manipulate this expression to the form in the definition of $x_i^s(s_I)$ in (37). Multiplying both sides by $\frac{u_{tot} - L_i \cdot U_i^v \cdot s_I}{u_{tot}}$ yields

$$\frac{u_{tot} - L_i \cdot U_i^v \cdot s_I}{u_{tot}} \cdot x_i^c = \frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot}}.$$

Adding $\frac{L_i \cdot U_i^v \cdot s_I}{u_{tot}} \cdot x_i^c$ to both sides yields

$$x_i^c = \frac{s + (m - u_{tot} - 1)C_i + O^{rn} + L_i \cdot U_i^v \cdot s_I \cdot x_i^c}{u_{tot}}.$$

Finally, substituting the expression for s in (112),

$$
\begin{aligned}
x_i^c = \Bigg(&\sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j) + \sum_{\tau_j \in \tau} S_j + (m - u_{tot} - 1)C_i + O^{rn} \\
&+ L_i \cdot U_i^v \cdot s_I \cdot x_i^c \Bigg) / u_{tot}.
\end{aligned}
$$

By the case we are considering, both sides of this expression must be greater than zero. Thus, $x_i^s(s_I) = x_i^c$ satisfies the definition of $x_i^s(s_I)$ in (37) for τ_i. □

Our LP has, for each τ_i, a variable x_i^c as in Theorem 8, corresponding to $x_i^s(s_I)$ and an auxilliary variable z_i. Our LP also has task-independent variables s (as in Theorem 8), G (corresponding to the first sum in the definition of s in (112)), S_{sum} (corresponding to the second sum in the definition of s in (112)) and auxilliary variable b. All other quantities that appear are constants.

We will present constraint sets for determining x_i^c in the same order as the constraint sets in Section 4 in (Erickson et al., 2014). The first constraint set ensures that

$$x_i^c \geq \max\left\{0, \frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v \cdot s_I}\right\}. \tag{114}$$

It corresponds to Constraint Set 1 in Erickson et al. (2014) that defined x_i, but provides only an inequality. Although Theorem 8 requires equality, this discrepancy will be handled in Lemma 66 below.

Constraint Set 1.

$$\forall i : x_i^c \geq \frac{s + (m - u_{tot} - 1) \cdot C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v \cdot s_I},$$

$$\forall i : x_i^c \geq 0.$$

Because we consider S_i to be a constant, we do not require a constraint that corresponds with Constraint Set 2 in (Erickson et al., 2014).

The next constraint set is used to determine the value of the first sum in the definition of s in (112). This sum corresponds with $G(\vec{x}, \vec{Y})$ in (Erickson et al., 2014), so the constraint is almost identical to Constraint Set 3 in (Erickson et al., 2014). As discussed there, this constraint actually ensures that G provides an upper bound on that sum, rather than an exact value. In other words, it actually guarantees that

$$G \geq \sum_{m-1 \text{ largest}} (C_i + U_i^v \cdot s_I \cdot x_i^c - S_i). \tag{115}$$

Although Theorem 8 requires equality, this discrepancy will be handled in Lemma 65 below.

Constraint Set 2.

$$G = b(m - 1) + \sum_{\tau_i \in \tau} z_i,$$

$$\forall i : z_i \geq 0,$$

$$z_i \geq C_i + U_i^v \cdot s_I \cdot x_i^c - S_i - b.$$

Rather than having a constraint that corresponds to Constraint Set 4 in (Erickson et al., 2014), we define a constant

$$S_{\text{sum}} = \sum_{\tau_j \in \tau} S_j. \tag{116}$$

The next constraint provides a bound on the value of s. This constraint differs from the definition of s in (112) because it is an inequality. However, we will show in Lemma 65 below that for an optimal solution to the LP, it reduces to an equality. This constraint corresponds to Constraint Set 5 in (Erickson et al., 2014).

Constraint Set 3.

$$s \geq G + S_{\text{sum}}.$$

We must show that, for some appropriate optimization function, an optimal solution to Constraint Sets 1–3 can be used to compute the values of s and x_i^c described in Theorem 8. We will show below that if we minimize s and an optimal solution is found, we can use the resulting s in the definition of x_i^c in (111), and (112) must be satisfied as well with that choice of x_i^c.

To do so, we first characterize the value of s for any feasible solution, providing a lower bound in Lemma 64 and a characterization relating to an optimal value in Lemma 65. Observe that the expression in Lemma 64 is identical to the definition of s in (112), except that it replaces the equality with an inequality.

Lemma 64. *For any feasible assignment of variables satisfying Constraint Sets 1–3,*

$$s \geq \sum_{m-1 \text{ largest}} (C_i + U_i^v \cdot s_I \cdot x_i^c - S_i) + \sum_{\tau_j \in \tau} S_i.$$

Proof. We have

$$\begin{aligned} s \geq\ & \{\text{By Constraint Set 3}\} \\ & G + S_{\text{sum}} \\ \geq\ & \{\text{By (115)}\} \end{aligned}$$

$$\sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j) + S_{\text{sum}}$$

$$= \{\text{By the definition of } S_{\text{sum}} \text{ in (116)}\}$$

$$\sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j) + \sum_{\tau_j \in \tau} S_j.$$

□

The next lemma will be used to characterize the optimal value of s with an appropriate optimization function.

Lemma 65. *If V is a feasible assignment of variables that satisfies Constraint Sets 1–3 and*

$$s > \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j) + \sum_{\tau_j \in \tau} S_i,$$

then there is also a feasible assignment of variables V' (with variable assignments denoted with primes) such that $s > s'$. In other words, s has not taken its optimal value upon minimization.

Proof. We use the following assignment for V':

$$s' = \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j) + S_{\text{sum}}, \tag{117}$$

$$\forall i, x_i^{c\prime} = x_i^c, \tag{118}$$

$$G' = \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j), \tag{119}$$

$$b' = (m-1)^{th} \text{ largest value of } C_j + U_j^v \cdot s_I \cdot x_j^c - S_j, \tag{120}$$

$$\forall i, z_i' = \max\{0, C_j + U_j^v \cdot s_I \cdot x_j^c - S_j - b'\}. \tag{121}$$

By the statement of the lemma, the definition of S_{sum} in (116), and by (117),

$$s > s'. \tag{122}$$

We will first show that Constraint Set 1 holds by considering arbitrary τ_i.

$$
\begin{aligned}
x_i^{c\prime} &= \{\text{By (118)}\} \\
&\quad x_i^c \\
&\geq \{\text{By (114)}\} \\
&\quad \max\left\{0, \frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v \cdot s_I}\right\} \\
&\geq \{\text{By (122)}\} \\
&\quad \max\left\{0, \frac{s' + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v \cdot s_I}\right\}.
\end{aligned} \tag{123}
$$

Constraint Set 2 holds by (118)–(121).

To show Constraint Set 3 holds,

$$
\begin{aligned}
s' &= \{\text{By (117)}\} \\
&\quad \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j) + S_{\text{sum}} \\
&= \{\text{By (119)}\} \\
&\quad G + S_{\text{sum}}.
\end{aligned}
$$

□

By Lemmas 64–66, we can minimize s as our optimization objective, and the definition of s in Theorem 8 must be satisfied by the resulting solution. However, the definition of x_i^c in (111) is not guaranteed to hold, because Constraint Set 1 also guaranteed only an inequality. Fortunately, we can use the resulting value of s in (111) to compute correct values of x_i^c, as shown in the following lemma.

Lemma 66. *If V is a feasible assignment of variables satisfying Constraint Sets 1–3, then there is also an assignment V' (with variables denoted with primes, as before) such that $s' = s$ and $\forall i, x_i^c = \max\left\{0, \frac{s+(m-u_{tot}-1)C_i+O^{rn}}{u_{tot}-L_i \cdot U_i^v \cdot s_I}\right\}$.*

Proof. We use the following assignment for V':

$$s' = s, \tag{124}$$

$$\forall i, x_i^{c\prime} = \max\left\{0, \frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v \cdot s_I}\right\}, \tag{125}$$

$$G' = \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^{c\prime} - S_j), \tag{126}$$

$$b' = (m-1)^{th} \text{ largest value of } C_j + U_j^v \cdot s_I \cdot x_j^{c\prime} - S_j, \tag{127}$$

$$\forall i, z_i' = \max\{0, C_j + U_j^v \cdot s_I \cdot x_j^{c\prime} - S_j - b'\}. \tag{128}$$

Constraint Set 1 holds by (125).

Constraint Set 2 holds by (126)–(128).

To show that Constraint Set 3 holds, we first show that for arbitrary j,

$$\begin{aligned} x_j^c \geq & \{\text{By (114)}\} \\ & \max\left\{0, \frac{s + (m - u_{tot} - 1)C_j + O^{rn}}{u_{tot} - L_j \cdot U_j^v \cdot s_I}\right\} \\ = & \{\text{By (125)}\} \\ & x_j^{c\prime}. \end{aligned} \tag{129}$$

Then, we have

$$\begin{aligned} s' = & \{\text{By (124)}\} \\ & s \\ \geq & \{\text{By Lemma 64}\} \\ & \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j) + \sum_{\tau_j \in \tau} S_j \\ \geq & \{\text{By (129)}\} \\ & \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^{c\prime} - S_j) + \sum_{\tau_j \in \tau} S_j \\ = & \{\text{By (126)}\} \end{aligned}$$

$$
\begin{aligned}
&G' + \sum_{\tau_j \in \tau} S_j \\
= &\{\text{By the definition of } S_{\text{sum}} \text{ in (116)}\} \\
&G' + S_{\text{sum}}.
\end{aligned}
$$

□

We will next show that, if Property 10 holds, a minimum feasible s does in fact exist. While proving this result, we will several times exploit the fact that S_j is nonnegative, as shown now.

Lemma 67. *For all* j, $S_j \geq 0$.

Proof. We have

$$
\begin{aligned}
S_j = &\{\text{By the definition of } S_j \text{ in (38)}\} \\
&C_j \cdot \left(1 - \frac{Y_i}{T_i}\right) \\
\geq &\{\text{Because } Y_i \leq T_i\} \\
&C_j \cdot \left(1 - \frac{T_j}{T_j}\right) \\
= &\{\text{Cancelling}\} \\
&0.
\end{aligned}
$$

□

We now show that a lower bound on s exists for feasible assignments. We will later show that feasible assignments do exist. Together, these results are sufficient to show that an optimal value of s exists.

Lemma 68. *For any feasible assignment of variables* V *satisfying Constraint Sets 1–3,*

$$
s \geq \sum_{m-1 \text{ largest}} C_i.
$$

Proof. Now, we show that

$$
\begin{aligned}
s \geq & \{\text{By Lemma 64}\} \\
& \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_i^c - S_j) + \sum_{\tau_j \in \tau} S_j \\
\geq & \{\text{By Lemma 67; observe that each } S_j \text{ appearing in the first summation} \\
& \text{also appears in the second}\} \\
& \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_i^c) \\
\geq & \{\text{By Constraint Set 1}\} \\
& \sum_{m-1 \text{ largest}} C_j.
\end{aligned}
$$

□

We now show that, given Property 10, the LP is feasible.

Lemma 69. *If Property 10 holds, then a feasible assignment of variables V for Constraint Sets 1–3 exists.*

Proof. Let $C_{\max}$ be the largest C_i in the system. For notational convenience, let

$$U_{\text{sum}} \triangleq \sum_{m-1 \text{ largest}} U_j^v, \tag{130}$$

$$L_{\max}^t \triangleq \max_{\tau_j \in \tau} L_j U_j^v \tag{131}$$

(*t*erm). We use the following assignment for V:

$$s = \max\left\{C_{\max}, \frac{U_{\text{sum}} \cdot (mC_{\max} + O^{rn}) + (u_{tot} - L_{\max}^t) \cdot ((m-1)C_{\max} + S_{\text{sum}})}{u_{tot} - L_{\max}^t - U_{\text{sum}}}\right\}, \tag{132}$$

$$\forall i, x_i^c = \frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v \cdot s_I}, \tag{133}$$

$$G = \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s_I \cdot x_j^c - S_j), \tag{134}$$

$$b = (m-1)^{th} \text{ largest value of } C_j + U_j^v \cdot s_I \cdot x_j^c - S_j, \tag{135}$$

$$\forall i, z_i = \max\{0, C_j + U_j^v \cdot s_I \cdot x_j^c - S_j - b\}. \tag{136}$$

To demonstrate that Constraint Set 1 holds, we first bound the expression that appears in the denominator of the definition of x_i^c in (133).

$$\begin{aligned}
u_{tot} - L_j \cdot U_j^v \cdot s_I &\geq \{\text{Because } s_I \leq 1\} \\
&\quad u_{tot} - L_j \cdot U_j^v \\
&\geq \{\text{By the definition of ``}\max\text{''}\} \\
&\quad u_{tot} - \max_{\tau_j \in \tau} L_j \cdot U_j^v \\
&> \{\text{By Property 10}\} \\
&\quad \sum_{m-1 \text{ largest}} U_j^v \\
&\geq \{\text{Because each } U_j^v > 0\} \\
&\quad 0.
\end{aligned} \tag{137}$$

The first constraints in Constraint Set 1 hold by (133). We now show that the second constraint also holds.

$$\begin{aligned}
x_i^c &= \{\text{By (133)}\} \\
&\quad \frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v \cdot s_I} \\
&\geq \{\text{By (132) and (137)}\} \\
&\quad \frac{C_{\max} + (m - u_{tot} - 1)C_i + O^{rn}}{L_i \cdot U_i^v \cdot s_I} \\
&\geq \{\text{By the definition of } C_{\max} \text{ and by (137)}\} \\
&\quad \frac{(m - u_{tot})C_i + O^{rn}}{L_i \cdot U_i^v \cdot s_I} \\
&\geq \{\text{By (137), because } C_i \text{ and } O^{rn} \text{ are nonnegative}\} \\
&\quad 0.
\end{aligned} \tag{138}$$

Constraint Set 2 holds by (134)–(136).

By (132),

$$s \geq \frac{U_{\text{sum}} \cdot (mC_{\max} + O^{rn}) + (u_{tot} - L^t_{\max}) \cdot ((m-1)C_{\max} + S_{\text{sum}})}{u_{tot} - L^t_{\max} - U_{\text{sum}}}. \tag{139}$$

To show that Constraint Set 3 holds, we start by multiplying both sides of (139) by $\frac{u_{tot} - L^t_{\max} - U_{\text{sum}}}{u_{tot} - L^t_{\max}}$, resulting in

$$\frac{u_{tot} - L^t_{\max} - U_{\text{sum}}}{u_{tot} - L^t_{\max}} \cdot s \geq \frac{U_{\text{sum}} \cdot (mC_{\max} + O^{rn})}{u_{tot} - L^t_{\max}} + (m-1)C_{\max} + S_{\text{sum}}.$$

Adding $\frac{U_{\text{sum}}}{u_{tot} - L^t_{\max}} \cdot s$ to both sides yields

$$s \geq \frac{U_{\text{sum}} \cdot (s + mC_{\max} + O^{rn})}{u_{tot} - L^t_{\max}} + (m-1)C_{\max} + S_{\text{sum}}. \tag{140}$$

Thus, we have

$$\begin{aligned}
s \geq\ & \{\text{Rewriting (140)}\} \\
& \sum_{m-1 \text{ largest}} \left(C_{\max} + U^v_j \cdot \frac{s + mC_{\max} + O^{rn}}{u_{tot} - L^t_{\max}} \right) + S_{\text{sum}} \\
\geq\ & \{\text{By the definition of } L^t_{\max} \text{ in (131), and because } s_I \leq 1\} \\
& \sum_{m-1 \text{ largest}} \left(C_{\max} + U^v_j \cdot \frac{s + mC_{\max} + O^{rn}}{u_{tot} - L_j \cdot U^v_j \cdot s_I} \right) + S_{\text{sum}} \\
\geq\ & \{\text{By (137) and the definition of } C_{\max}\} \\
& \sum_{m-1 \text{ largest}} \left(C_j + U^v_j \cdot \frac{s + mC_j + O^{rn}}{u_{tot} - L_j \cdot U^v_j \cdot s_I} \right) + S_{\text{sum}} \\
\geq\ & \{\text{Because } C_j > 0 \text{ and by (137)}\} \\
& \sum_{m-1 \text{ largest}} \left(C_j + U^v_j \cdot \frac{s + (m - u_{tot} - 1)C_j + O^{rn}}{u_{tot} - L_j \cdot U^v_j \cdot s_I} \right) + S_{\text{sum}} \\
=\ & \{\text{By (133)}\} \\
& \sum_{m-1 \text{ largest}} \left(C_j + U^v_j \cdot x^c_i \right) + S_{\text{sum}}
\end{aligned}$$

$\geq$ {Because $0 < s_I \leq 1$ and by (138)}

$$\sum_{m-1 \text{ largest}} \left(C_j + U_j^v \cdot s_I \cdot x_i^c\right) + S_{\text{sum}}$$

$\geq$ {By Lemma 67}

$$\sum_{m-1 \text{ largest}} \left(C_j + U_j^v \cdot s_I \cdot x_i^c - S_j\right) + S_{\text{sum}}$$

$=$ {By (134)}

$$G + S_{\text{sum}}..$$

□

The one promised result that remains to be shown is that a feasible solution exists for any $s'_I \leq 1$ as long as one exists for $s_I = 1$. We show that result now.

Lemma 70. *If a feasible assignment of variables V satisfies Constraint Sets 1–3 with $s_I = 1$ and $u_{tot} - L_i \cdot U_i^v > 0$, then a feasible assignment of variables V' satisfies Constraint Sets 1–3 for arbitary $s'_I \leq 1$.*

Proof. We use the following assignment for V':

$$s' = s, \tag{141}$$

$$\forall i, x_i^{c\prime} = x_i^c, \tag{142}$$

$$G' = \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s'_I \cdot x_j^c - S_j), \tag{143}$$

$$b' = (m-1)^{th} \text{ largest value of } C_j + U_j^v \cdot s'_I \cdot x_j^c - S_j, \tag{144}$$

$$\forall i, z'_i = \max\{0, C_j + U_j^v \cdot s'_I \cdot x_j^c - S_j - b'\}. \tag{145}$$

Because $u_{tot} - L_i \cdot U_i^v > 0$ and $0 < s_I \leq 1$,

$$u_{tot} - L_i \cdot s_I \cdot U_i^v > 0. \tag{146}$$

To show that Constraint Set 1 holds, we consider two cases for each τ_i.

Case 1: $s + (m - u_{tot} - 1)C_i + O^{rn} \leq 0$. In this case,

$$
\begin{aligned}
x_i^{c\prime} &= \{\text{By (142)}\} \\
&\quad x_i^c \\
&\geq \{\text{By Constraint Set 1}\} \\
&\quad 0 \\
&= \{\text{By (146) and the case we are considering}\} \\
&\quad \max\left\{0, \frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot s_I \cdot U_i^v}\right\}.
\end{aligned}
$$

Case 2: $s + (m - u_{tot} - 1)C_i + O_{i,k} > 0$. In this case,

$$
\begin{aligned}
x_i^{c\prime} &= \{\text{By (142)}\} \\
&\quad x_i^c \\
&\geq \{\text{By (114) with } s_I = 1\} \\
&\quad \max\left\{0, \frac{s + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v}\right\} \\
&= \{\text{By (141)}\} \\
&\quad \max\left\{0, \frac{s' + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot U_i^v}\right\} \\
&\geq \{\text{By (146) and the case we are considering, because } 0 < s_I < 1\} \\
&\quad \max\left\{0, \frac{s' + (m - u_{tot} - 1)C_i + O^{rn}}{u_{tot} - L_i \cdot s_I \cdot U_i^v}\right\}.
\end{aligned}
$$

Constraint Set 2 holds by (142) and (119)–(145).

To show Constraint Set 3 holds,

$$
\begin{aligned}
s' &= \{\text{By (141)}\} \\
&\quad s \\
&\geq \{\text{By Lemma 64 with } s_I = 1\} \\
&\quad \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot x_j^c - S_j) + S_{\text{sum}}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \{\text{Because } 0 < s'_I \leq 1\} \\
&\quad \sum_{m-1 \text{ largest}} (C_j + U_j^v \cdot s'_I \cdot x_j^c - S_j) + S_{\text{sum}} \\
&= \{\text{By (142) and (143)}\} \\
&\quad G + S_{\text{sum}}.
\end{aligned}
$$

□

Finally, we briefly discuss the use of LP to determine a choice of Y_i. The value of Y_i in our analysis is assumed to remain constant when the virtual clock speed changes. Furthermore, during the typical behavior of a system, $s_I = 1$ should be used. Therefore, if using linear programming to determine the best choice of Y_i, it should be done using $s_I = 1$.

Up to this point, Y_i has been assumed to be a constant. Similarly, S_i, which depends on Y_i by the definition of S_i in (38), and S_{sum}, which depends on S_i by the definition of S_{sum} in (116), have also been considered to be constants. These can be changed to variables as long as the following constraint sets are added. The first constrains the choice of Y_i itself to match the assumptions used in our analysis.

Constraint Set 4.

$$
\begin{aligned}
&\forall i, Y_i \geq 0, \\
&\forall i, T_i \geq Y_i.
\end{aligned}
$$

The next constraint set simply specifies the value of S_i according to the definition of S_i in (38).

Constraint Set 5.

$$\forall i, S_i = C_i \cdot \left(1 - \frac{Y_i}{T_i}\right).$$

The final constraint set specifies the value of S_{sum} according to the definition of S_{sum} in (116).

Constraint Set 6.

$$S_{\text{sum}} = \sum_{\tau_j \in \tau} S_j.$$

Any optimization function can be used that ensures, under an optimal solution, the minimal value of s with respect to the chosen values of Y_i. If the optimization function has such a property, then all of the reasoning in this appendix will continue to hold. Because each x_i^c cannot increase as a result of an increase in s, such a property is easy to achieve for a reasonable optimization function.