Some Conditions for Accurate Surface Registration

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ABSTRACT

In this technical report, we derive some conditions that allow the ICP (Iterative Closest Point) surface registration algorithm to attain some absolute accuracy when aligning two surfaces.

1 INTRODUCTION

3D shape alignment is an important part of many applications. It is used for object recognition in which newly acquired shapes in the environment are fitted to model shapes in the database. For reverse engineering and building real-world models for virtual reality, it is used to align multiple partial range scans to form models that are more complete. For autonomous range acquisition, 3D surface registration is used to accurately localize the range scanner, and to align data from multiple scans so that they can be merged to allow planning of the next views.

Since its introduction by Besl and McKay [Besl1992], the ICP (Iterative Closest Point) algorithm has become the most widely used method for aligning three-dimensional shapes, especially 3D surfaces obtained by range scanners. A similar algorithm was also introduced by Chen and Medioni [Chen1992]. Rusinkiewicz and Levoy [Rusinkiewicz2001] provide a recent survey of the many ICP variants based on the original ICP concept.

During autonomous range acquisition, each newly acquired range image must be registered to the current model of the scene before it can be merged with the latter. Therefore, when planning for a new view to make the next scan, before the scan is actually made, the view planner must ensure that the new range image acquired from the planned view can be successfully registered with the current model, and to within certain error tolerance. The success of the registration depends on a few factors, for example, the registration algorithm used, the initial relative pose between the two surfaces, the "frequencies" on the surfaces, the amount of overlap between the two surfaces, the amount of measurement errors, and the shape constraint on the 3D rigid-body transformation between the two surfaces. In this technical report, the registration method we consider is the ICP algorithm [Besl1992], and among the many other factors, we consider only the amount of measurement errors and the shape constraint on the 3D rigid-body transformation.

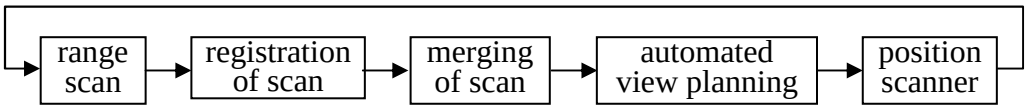

Figure 1.1: The autonomous acquisition cycle with automated view planning.

In his doctoral thesis [Simon1996], Simon proposed a means to measure the relative amount of shape constraint on a surface, which is proportional to how well the surface can be registered using the ICP algorithm. More specifically, his method is able to compute, for a set of surface

points, a value that represents the amount of constraint on the 3D rigid-body transformation when the surface is being registered with itself. This value is only good for comparing with the values of other point sets, so as to determine which point set has the best constraint on the transformation. The value does not tell the absolute accuracy that can be achieved with each point set during the registration. In this technical report, we extend Simon's results to derive conditions and constraints required to achieve some given absolute accuracies.

In order to understand the derivations in this technical report, the reader has to be familiar with the material in 3.3.1 and Section 3.3.3 of [Simon1996]. We adopt Simon's notation.

Unlike Simon's derivation, in the following derivations, we consider translation and rotation separately. This makes sense because translation and rotation are parameterized by different entity types (translation is parameterized by distances and rotation by angles). By doing so, as we will see, the result of the rotation error analysis becomes independent of the scale of the object.

2 CONDITIONS FOR TRANSLATIONAL ACCURACY

Following the derivation in [Simon1996], the translational component of $\boldsymbol{V}(\boldsymbol{x}_s)$ can be written as

$$\boldsymbol{V}_{\boldsymbol{\tau}}(\boldsymbol{x}_s) = \frac{\partial}{\partial \boldsymbol{\tau}} D(T_{\boldsymbol{\tau}}(\boldsymbol{x}_s)) = \boldsymbol{n}_{\boldsymbol{x}_s} \tag{A1}$$

where $\boldsymbol{\tau} = \left[t_x, t_y, t_z\right]^{\mathrm{T}}$, $\partial\boldsymbol{\tau} = \left[\partial t_x, \partial t_y, \partial t_z\right]^{\mathrm{T}}$ and $\boldsymbol{n}_{\boldsymbol{x}_s}$ is the unit surface normal at point $\boldsymbol{x}_s$. Then

$$D(T_{\boldsymbol{\tau}}(\boldsymbol{x}_s)) = \boldsymbol{V}_{\boldsymbol{\tau}}^{\mathrm{T}}(\boldsymbol{x}_s)\, d\boldsymbol{\tau} \tag{A2}$$

Squaring this equation results in

$$D^2(T_{\boldsymbol{\tau}}(\boldsymbol{x}_s)) = d\boldsymbol{\tau}^{\mathrm{T}} \boldsymbol{V}_{\boldsymbol{\tau}}(\boldsymbol{x}_s)\, \boldsymbol{V}_{\boldsymbol{\tau}}^{\mathrm{T}}(\boldsymbol{x}_s)\, d\boldsymbol{\tau} = d\boldsymbol{\tau}^{\mathrm{T}} \boldsymbol{M}_{\boldsymbol{\tau}}(\boldsymbol{x}_s)\, d\boldsymbol{\tau} \tag{A3}$$

where $\boldsymbol{M}_{\boldsymbol{\tau}}(\boldsymbol{x}_s) = \boldsymbol{V}_{\boldsymbol{\tau}}(\boldsymbol{x}_s)\boldsymbol{V}_{\boldsymbol{\tau}}^{\mathrm{T}}(\boldsymbol{x}_s)$ is a symmetric, positive semi-definite 3x3 matrix. Summing the quantity in Eq. A3 over a set *P* of discrete surface points results in

$$E_P(T_{\boldsymbol{\tau}}(\boldsymbol{x}_s)) = \sum_{\boldsymbol{x}_s \in P} D^2(T_{\boldsymbol{\tau}}(\boldsymbol{x}_s)) = d\boldsymbol{\tau}^{\mathrm{T}} \left(\sum_{\boldsymbol{x}_s \in P} \boldsymbol{M}_{\boldsymbol{\tau}}(\boldsymbol{x}_s) \right) d\boldsymbol{\tau} = d\boldsymbol{\tau}^{\mathrm{T}} \Psi_{\boldsymbol{\tau}} d\boldsymbol{\tau} \tag{A4}$$

The symmetric 3x3 matrix $\Psi_{\boldsymbol{\tau}}$ is the sum of the $\boldsymbol{M}_{\boldsymbol{\tau}}(\boldsymbol{x}_s)$ matrices evaluated at each point in *P*. $\Psi_{\boldsymbol{\tau}}$ is a scatter matrix that contains information about the distribution of the original $\boldsymbol{V}_{\boldsymbol{\tau}}(\boldsymbol{x}_s)$ vectors over all points in the set *P*. By decomposing $\Psi_{\boldsymbol{\tau}}$ using principal component analysis, we get

$$\Psi_{\boldsymbol{\tau}} = Q_{\boldsymbol{\tau}} \Lambda_{\boldsymbol{\tau}} Q_{\boldsymbol{\tau}}^{\mathrm{T}} = Q_{\boldsymbol{\tau}} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} Q_{\boldsymbol{\tau}}^{\mathrm{T}} \tag{A5}$$

where λ_1, λ_2 and λ_3 are the eigenvalues of $\Psi_{\boldsymbol{\tau}}$, with $\lambda_1 \geq \lambda_2 \geq \lambda_3$, and the columns of $Q_{\boldsymbol{\tau}}$ are the corresponding unit eigenvectors. Since the sum of the eigenvalues is equal to the trace of the original matrix [Strang1986],

$$\begin{aligned}\lambda_1 + \lambda_2 + \lambda_3 &= \text{trace}\left(\Psi_{\boldsymbol{\tau}}\right) \\ &= \text{trace}\left(\sum_{\boldsymbol{x}_s \in P} \boldsymbol{M}_{\boldsymbol{\tau}}(\boldsymbol{x}_s)\right) = \text{trace}\left(\sum_{\boldsymbol{x}_s \in P} \boldsymbol{V}_{\boldsymbol{\tau}}(\boldsymbol{x}_s)\boldsymbol{V}_{\boldsymbol{\tau}}^{\mathrm{T}}(\boldsymbol{x}_s)\right) \\ &= \sum_{\boldsymbol{x}_s \in P} \left|\boldsymbol{V}_{\boldsymbol{\tau}}(\boldsymbol{x}_s)\right|^2 = \sum_{\boldsymbol{x}_s \in P} \left|\boldsymbol{n}_{\boldsymbol{x}_s}\right|^2 = \sum_{\boldsymbol{x}_s \in P} 1 = M\end{aligned} \tag{A6}$$

where M is the number of points in P.

For the alignment of the surface to be successful, it must be that $\lambda_i > 0$ for $i = 1, 2, \text{and } 3$. Ideally, we wish to select the set of points for P such that $\lambda_1 = \lambda_2 = \lambda_3 = M/3$, which means that the surface alignment is constrained equally in all three orthogonal translation directions. The minimum requirement for alignment is $M = 3$, and the three surface normals must span the 3D space. In practice, because of errors in the range measurements, it is necessary to have $M \geq M_{\min} > 3$, such that the translational alignment can be performed to a certain desired accuracy, assuming the surface is already correctly oriented.

The translation that resulted from the alignment can be decomposed into three orthogonal directions. In the following, we investigate the relationship between $M_{\min}$ and translational alignment accuracy by looking at the alignment errors in the three orthogonal directions. Without loss of generality, we choose the ***x***, ***y***, and ***z*** directions.

Let $\left[x_i, y_i, z_i\right]^{\mathrm{T}}$ be the 3D coordinates of the *i*th true surface point, where $i = 1, 2, \ldots, M$. Suppose there are two sets of measurements of the surface points, producing the point set A with coordinates $\left[x_i + u_{Ai}, y_i + v_{Ai}, z_i + w_{Ai}\right]^{\mathrm{T}}$ and the point set B with coordinates $\left[x_i + u_{Bi}, y_i + v_{Bi}, z_i + w_{Bi}\right]^{\mathrm{T}}$, where $\{u_{Ai}, u_{Bi}\}$, $\{v_{Ai}, v_{Bi}\}$, and $\{w_{Ai}, w_{Bi}\}$ are measurement errors in the ***x***, ***y***, and ***z*** directions, respectively.

Suppose point set B is to be translated so that it is aligned with point set A. For the alignment, the correspondences between points in point sets A and B are known. The alignment uses the least-squares (least-sum-of-squares) error metric, where we want to find the translation vector $\boldsymbol{\tau} = \left[t_x, t_y, t_z\right]^{\mathrm{T}}$ to translate point set B to minimize

$$\begin{aligned}SSE &= \sum_{i=1}^{M} \left| \begin{bmatrix} x_i + u_{Ai} \\ y_i + v_{Ai} \\ z_i + w_{Ai} \end{bmatrix} - \begin{bmatrix} x_i + u_{Bi} \\ y_i + v_{Bi} \\ z_i + w_{Bi} \end{bmatrix} - \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \right|^2 \\ &= \sum_{i=1}^{M} \left(u_{Ai} - u_{Bi} - t_x\right)^2 + \sum_{i=1}^{M} \left(v_{Ai} - v_{Bi} - t_y\right)^2 + \sum_{i=1}^{M} \left(w_{Ai} - w_{Bi} - t_z\right)^2\end{aligned} \tag{A7}$$

In the following, we determine the conditions such that the alignment error is within a specified threshold.

Differentiating *SSE* with respect to t_x, t_y, and t_z, we have

$$\begin{aligned}\frac{\partial(SSE)}{\partial t_x} &= -2\sum_{i=1}^{M}\left(u_{Ai} - u_{Bi} - t_x\right)\\ \frac{\partial(SSE)}{\partial t_y} &= -2\sum_{i=1}^{M}\left(v_{Ai} - v_{Bi} - t_y\right)\\ \frac{\partial(SSE)}{\partial t_z} &= -2\sum_{i=1}^{M}\left(w_{Ai} - w_{Bi} - t_z\right)\end{aligned} \tag{A8}$$

SSE is minimum when $\frac{\partial(SSE)}{\partial t_x} = \frac{\partial(SSE)}{\partial t_y} = \frac{\partial(SSE)}{\partial t_z} = 0$, and they are true when

$$\begin{aligned}&-2\sum_{i=1}^{M}\left(u_{Ai} - u_{Bi} - t_x\right) = 0\\ &\Rightarrow Mt_x = \sum_{i=1}^{M}\left(u_{Ai} - u_{Bi}\right)\\ &\Rightarrow t_x = \frac{1}{M}\sum_{i=1}^{M}u_{Ai} - \frac{1}{M}\sum_{i=1}^{M}u_{Bi} = \overline{u_A} - \overline{u_B}\end{aligned} \tag{A9}$$

and similarly

$$\begin{aligned}t_y &= \frac{1}{M}\sum_{i=1}^{M}v_{Ai} - \frac{1}{M}\sum_{i=1}^{M}v_{Bi} = \overline{v_A} - \overline{v_B}\\ t_z &= \frac{1}{M}\sum_{i=1}^{M}w_{Ai} - \frac{1}{M}\sum_{i=1}^{M}w_{Bi} = \overline{w_A} - \overline{w_B}\end{aligned} \tag{A10}$$

We now first consider only the translation in the ***x*** direction. Let u_{Ai} and u_{Bi} be the values of the random variables U_{Ai} and U_{Bi}, respectively, and $\overline{u_A}$ and $\overline{u_B}$ be the values of the random variables $\overline{U_A}$ and $\overline{U_B}$, respectively. Suppose each U_{Ai} has normal distribution with mean μ_{U_A} and standard deviation σ_{U_A}, and each U_{Bi} has normal distribution with mean μ_{U_B} and standard deviation σ_{U_B}, i.e.

$$U_{Ai} \sim N\left(\mu_{U_A}, \sigma_{U_A}\right) \quad \text{and} \quad U_{Bi} \sim N\left(\mu_{U_B}, \sigma_{U_B}\right), \tag{A11}$$

then the sampling distributions [Walpole1993] of $\overline{U_A}$ and $\overline{U_B}$ are

$$\overline{U_A} \sim N\left(\mu_{U_A}, \frac{\sigma_{U_A}}{\sqrt{M}}\right) \quad \text{and} \quad \overline{U_B} \sim N\left(\mu_{U_B}, \frac{\sigma_{U_B}}{\sqrt{M}}\right) \tag{A12}$$

Since $t_x = \overline{u_A} - \overline{u_B}$, we have

$$T_x \sim N\left(\mu_{U_A} - \mu_{U_B}, \sqrt{\frac{\sigma_{U_A}^2}{M} + \frac{\sigma_{U_B}^2}{M}}\right) \tag{A13}$$

where t_x is the value of the random variable T_x.

In practice, we can assume $\mu_{U_A} = \mu_{U_B} = 0$. With the above assumptions, we can be $(1 - \alpha)$ 100% confident that the translational alignment error in the $\boldsymbol{x}$ direction will not exceed $\varepsilon > 0$ when the following condition is true [Walpole1993]:

$$\varepsilon^2 M \geq \left(z_{\alpha/2}\sqrt{\sigma_{U_A}^2 + \sigma_{U_B}^2}\right)^2 \tag{A14}$$

where

$$P\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) = 1 - \alpha \tag{A15}$$

and Z is a random variable that has the standard normal distribution.

If U_{Ai} and U_{Bi} are not normal distributions, then the condition in Eq. (A14) is still a good approximations as long as M is greater than 30 [Walpole1993].

Normally, σ_{U_A}, σ_{U_B}, σ_{V_A}, σ_{V_B}, σ_{W_A}, and σ_{W_B} are not constant and they vary depending on factors such as the choice of the coordinate system with respect to the surface's orientation, the incident angle of the laser to the surface point, the surface reflectance properties, and the distance between the sensor and the surface point. To simplify the analysis, and the fact that we can be more conservative in this case, we can assume all the above standard deviations are less than or equal to the worst possible RMS error in range measurement, e_{RMS}. Then from Eq. (A14), we obtain the more conservative condition

$$\varepsilon^2 M \geq 2\left(z_{\alpha/2} e_{RMS}\right)^2 \tag{A16}$$

However, the condition in Eq. (A16) is only true in a special case. When a point $\boldsymbol{p}_i = [x_i, y_i, z_i]^{\mathrm{T}}$ is translated by a small distance t_x in the $\boldsymbol{x}$ direction, its contribution to the energy function (the constraint), $E_P(T_{\boldsymbol{\tau}}(\boldsymbol{p}_i))$ in Eq. (A4), is

$$D^2(T_{\boldsymbol{\tau}}(\boldsymbol{p}_i)) = d\boldsymbol{\tau}^{\mathrm{T}} \boldsymbol{V}_{\boldsymbol{\tau}}(\boldsymbol{p}_i) \boldsymbol{V}_{\boldsymbol{\tau}}^{\mathrm{T}}(\boldsymbol{p}_i)\, d\boldsymbol{\tau} = \left(\boldsymbol{n}_i \cdot \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix}\right)^2 = \left(\begin{bmatrix} n_{i,x} \\ n_{i,y} \\ n_{i,z} \end{bmatrix} \cdot \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix}\right)^2 = t_x^2 n_{i,x}^2 \tag{A17}$$

where $\boldsymbol{n}_i$ is the unit surface normal at $\boldsymbol{p}_i$. We can easily see that $D^2(T_{\boldsymbol{\tau}}(\boldsymbol{p}_i))$ is maximum only if $\boldsymbol{n}_i = [\pm 1, 0, 0]^{\mathrm{T}}$, and in this case we get

$$E_P(T_{\boldsymbol{\tau}}(\boldsymbol{p}_i)) = \sum_{\boldsymbol{x}_s \in P} D^2(T_{\boldsymbol{\tau}}(\boldsymbol{p}_i)) = t_x^2 \sum_{i=1}^{M} n_{i,x}^2 = t_x^2 \sum_{i=1}^{M} (\pm 1)^2 = t_x^2 M \tag{A18}$$

The R.H.S. of Eq. (A18) is basically the same as the L.H.S. of Eq. (A16). This makes sense because Eq. (A16) is derived on the assumption that each point in point set *B* is matched correctly with the corresponding point in point set *A*. The vector between each pair of corresponding points is parallel to the direction of the translation. When the translation is $[t_x,0,0]^{\mathrm{T}}$, this is equivalent to having a surface normal $\boldsymbol{n}_i = [\pm 1,0,0]^{\mathrm{T}}$ at every point of one of the point sets. Therefore, we can say that the condition in Eq. (A16) is true only in the special case when each point $\boldsymbol{p}_i$ is at its maximum constraint on the translation in the $\boldsymbol{x}$ direction, i.e. when $\boldsymbol{n}_i = [\pm 1,0,0]^{\mathrm{T}}$. However, in the general case when each $\boldsymbol{n}_i$ is a true surface normal on the surface, and it can be any unit vector, we have

$$E_P(T_{\boldsymbol{\tau}}(\boldsymbol{p}_i)) = \sum_{\boldsymbol{x}_s \in P} D^2(T_{\boldsymbol{\tau}}(\boldsymbol{p}_i)) = t_x^2 \sum_{i=1}^{M} n_{i,x}^2 \tag{A19}$$

Now, for the general case, we can be $(1-\alpha)$ 100% confident that the translational alignment error in the $\boldsymbol{x}$ direction will not exceed $\varepsilon > 0$ when the following condition is true:

$$\varepsilon^2 \sum_{i=1}^{M} n_{i,x}^2 \geq 2\left(z_{\alpha/2} e_{RMS}\right)^2 \tag{A20}$$

Similarly, we can obtain the following conditions for translation in the $\boldsymbol{y}$ and $\boldsymbol{z}$ directions, respectively:

$$\varepsilon^2 \sum_{i=1}^{M} n_{i,y}^2 \geq 2\left(z_{\alpha/2} e_{RMS}\right)^2 \quad \text{and} \quad \varepsilon^2 \sum_{i=1}^{M} n_{i,z}^2 \geq 2\left(z_{\alpha/2} e_{RMS}\right)^2 \tag{A21}$$

By combining the three inequalities in Eq. (A20) and (A21), and Eq. (A6), we get

$$\begin{aligned} & \varepsilon^2 \sum_{i=1}^{M} \left(n_{i,x}^2 + n_{i,y}^2 + n_{i,z}^2\right) \geq 3 \cdot 2\left(z_{\alpha/2} e_{RMS}\right)^2 \\ \Rightarrow \quad & \varepsilon^2 M \geq 6\left(z_{\alpha/2} e_{RMS}\right)^2 \quad \Rightarrow \quad M \geq M_{\min} = 6\left(\frac{z_{\alpha/2} e_{RMS}}{\varepsilon}\right)^2 \\ \Rightarrow \quad & \lambda_1 + \lambda_2 + \lambda_3 \geq M_{\min} = 6\left(\frac{z_{\alpha/2} e_{RMS}}{\varepsilon}\right)^2 \end{aligned} \tag{A22}$$

With the results in Eq. (A20), (A21) and (A22), in the general case when $\boldsymbol{n}_i$ can be any unit vector,

> we can be $(1-\alpha)^3$ 100% confident that the translational alignment error in *any* direction will not exceed $\sqrt{\varepsilon^2 + \varepsilon^2 + \varepsilon^2} = \sqrt{3}\,\varepsilon$ when $\lambda_1 \geq M_{\min}/3$, $\lambda_2 \geq M_{\min}/3$ and $\lambda_3 \geq M_{\min}/3$, where $M_{\min} = 6\left(z_{\alpha/2} e_{RMS}/\varepsilon\right)^2$.

3 CONDITIONS FOR ROTATIONAL ACCURACY

The rotational component of $\boldsymbol{V}(\boldsymbol{x}_s)$ can be written as

$$\boldsymbol{V}_\theta(\boldsymbol{x}_s) = \frac{\partial}{\partial \boldsymbol{\theta}} D(T_\theta(\boldsymbol{x}_s)) = \boldsymbol{x}_s \times \boldsymbol{n}_{\boldsymbol{x}_s} \tag{B1}$$

where $\boldsymbol{\theta} = \left[\omega_x, \omega_y, \omega_z\right]^{\mathrm{T}}$, $\partial\boldsymbol{\theta} = \left[\partial\omega_x, \partial\omega_y, \partial\omega_z\right]^{\mathrm{T}}$ and $\boldsymbol{n}_{\boldsymbol{x}_s}$ is the unit surface normal at point $\boldsymbol{x}_s$. Then

$$D(T_\theta(\boldsymbol{x}_s)) = \boldsymbol{V}_\theta^{\mathrm{T}}(\boldsymbol{x}_s)\, d\boldsymbol{\theta} \tag{B2}$$

Squaring this equation results in

$$D^2(T_\theta(\boldsymbol{x}_s)) = d\boldsymbol{\theta}^{\mathrm{T}} \boldsymbol{V}_\theta(\boldsymbol{x}_s)\, \boldsymbol{V}_\theta^{\mathrm{T}}(\boldsymbol{x}_s)\, d\boldsymbol{\theta} = d\boldsymbol{\theta}^{\mathrm{T}} \boldsymbol{M}_\theta(\boldsymbol{x}_s)\, d\boldsymbol{\theta} \tag{B3}$$

where $\boldsymbol{M}_\theta(\boldsymbol{x}_s) = \boldsymbol{V}_\theta(\boldsymbol{x}_s)\boldsymbol{V}_\theta^{\mathrm{T}}(\boldsymbol{x}_s)$ is a symmetric, positive semi-definite 3x3 matrix. Summing the quantity in Eq. B3 over a set *P* of discrete surface points results in

$$E_P(T_\theta(\boldsymbol{x}_s)) = \sum_{\boldsymbol{x}_s \in P} D^2(T_\theta(\boldsymbol{x}_s)) = d\boldsymbol{\theta}^{\mathrm{T}} \left(\sum_{\boldsymbol{x}_s \in P} \boldsymbol{M}_\theta(\boldsymbol{x}_s) \right) d\boldsymbol{\theta} = d\boldsymbol{\theta}^{\mathrm{T}} \Psi_\theta d\boldsymbol{\theta} \tag{B4}$$

The symmetric 3x3 matrix Ψ_θ is the sum of the $\boldsymbol{M}_\theta(\boldsymbol{x}_s)$ matrices evaluated at each point in *P*. Ψ_θ is a scatter matrix that contains information about the distribution of the original $\boldsymbol{V}_\theta(\boldsymbol{x}_s)$ vectors over all points in the set *P*. By decomposing Ψ_θ using principal component analysis, we get

$$\Psi_\theta = Q_\theta \Lambda_\theta Q_\theta^{\mathrm{T}} = Q_\theta \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix} Q_\theta^{\mathrm{T}} \tag{B5}$$

where γ_1, γ_2 and γ_3 are the eigenvalues of Ψ_θ, with $\gamma_1 \geq \gamma_2 \geq \gamma_3$, and the columns of Q_θ are the corresponding unit eigenvectors. Since the sum of the eigenvalues is equal to the trace of the original matrix [Strang1986],

$$\begin{aligned} S &= \gamma_1 + \gamma_2 + \gamma_3 \\ &= \mathrm{trace}\left(\Psi_\theta\right) = \mathrm{trace}\left(\sum_{\boldsymbol{x}_s \in P} \boldsymbol{M}_\theta(\boldsymbol{x}_s) \right) = \mathrm{trace}\left(\sum_{\boldsymbol{x}_s \in P} \boldsymbol{V}_\theta(\boldsymbol{x}_s)\boldsymbol{V}_\theta^{\mathrm{T}}(\boldsymbol{x}_s) \right) \\ &= \sum_{\boldsymbol{x}_s \in P} \left|\boldsymbol{V}_\theta(\boldsymbol{x}_s)\right|^2 = \sum_{\boldsymbol{x}_s \in P} \left|\boldsymbol{x}_s \times \boldsymbol{n}_{\boldsymbol{x}_s}\right|^2 = \sum_{\boldsymbol{x}_s \in P} \left|\boldsymbol{x}_s\right|^2 sin^2\, \phi_{\boldsymbol{x}_s} \end{aligned} \tag{B6}$$

where $\phi_{\boldsymbol{x}_s}$ is the angle between the vector $\boldsymbol{x}_s$ and $\boldsymbol{n}_{\boldsymbol{x}_s}$.

For the alignment of the surface to be successful, it must be that $\gamma_i > 0$ for $i = 1, 2, \text{and } 3$. Ideally, we wish to select the set of points for *P* such that $\gamma_1 = \gamma_2 = \gamma_3 = S/3$, which means that the surface alignment is constrained equally in the rotations about all three orthogonal axes. The

minimum requirement for alignment is $M = 3$, where M is the number of points in P, and the three vectors in the set $\left\{ \boldsymbol{x}_s \times \boldsymbol{n}_{\boldsymbol{x}_s} : \boldsymbol{x}_s \in P \right\}$ must span the 3D space. In practice, because of errors in the range measurements, it is necessary to have $M \geq M_{\min} > 3$, such that the rotational alignment can be performed to a certain desired accuracy, assuming the surface is already correctly translated. In the following, we investigate the conditions necessary to attain a specified accuracy in rotational alignment.

Let $[x_i, y_i, z_i]^{\mathrm{T}}$ be the 3D coordinates of the ith true surface point, where $i = 1, 2, \ldots, M$. Suppose there are two sets of measurements of the surface points, producing the point set A with coordinates $[x_i + u_{Ai}, y_i + v_{Ai}, z_i + w_{Ai}]^{\mathrm{T}}$ and the point set B with coordinates $[x_i + u_{Bi}, y_i + v_{Bi}, z_i + w_{Bi}]^{\mathrm{T}}$, where $\{u_{Ai}, u_{Bi}\}$, $\{v_{Ai}, v_{Bi}\}$, and $\{w_{Ai}, w_{Bi}\}$ are measurement errors in the ***x***, ***y***, and ***z*** directions, respectively.

Suppose point set B is to be rotated about the origin so that it is aligned with point set A. For the alignment, the correspondences between points in point sets A and B are known. The alignment uses the least-squares (least-sum-of-squares) error metric, where we want to find the rotation vector $\boldsymbol{\theta} = [\omega_x, \omega_y, \omega_z]^{\mathrm{T}}$ to rotate point set B to minimize

$$\begin{aligned} SSE &= \sum_{i=1}^{M} \left| \begin{bmatrix} x_i + u_{Ai} \\ y_i + v_{Ai} \\ z_i + w_{Ai} \end{bmatrix} - \boldsymbol{R}(\omega_z) \cdot \boldsymbol{R}(\omega_y) \cdot \boldsymbol{R}(\omega_x) \cdot \begin{bmatrix} x_i + u_{Bi} \\ y_i + v_{Bi} \\ z_i + w_{Bi} \end{bmatrix} \right|^2 \\ &= \sum_{i=1}^{M} \left| \begin{bmatrix} x_i + u_{Ai} \\ y_i + v_{Ai} \\ z_i + w_{Ai} \end{bmatrix} - \boldsymbol{R}(\omega_x, \omega_y, \omega_z) \cdot \begin{bmatrix} x_i + u_{Bi} \\ y_i + v_{Bi} \\ z_i + w_{Bi} \end{bmatrix} \right|^2 \end{aligned} \tag{B7}$$

where $\boldsymbol{R}(\omega_x)$, $\boldsymbol{R}(\omega_y)$ and $\boldsymbol{R}(\omega_z)$ are the rotation matrices for rotations of ω_x, ω_y and ω_z radians about the x-axis, y-axis and z-axis, respectively. Since the rotations will be small, we can approximate the matrix $\boldsymbol{R}(\omega_x, \omega_y, \omega_z)$ by using the approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ when $\theta \approx 0$. With this approximation, we get

$$\begin{aligned} SSE &\approx \sum_{i=1}^{M} \left| \begin{bmatrix} x_i + u_{Ai} \\ y_i + v_{Ai} \\ z_i + w_{Ai} \end{bmatrix} - \begin{bmatrix} 1 & \omega_x\omega_y - \omega_z & \omega_x\omega_z + \omega_y \\ \omega_z & \omega_x\omega_y\omega_z + 1 & \omega_y\omega_z - \omega_x \\ -\omega_y & \omega_x & 1 \end{bmatrix} \cdot \begin{bmatrix} x_i + u_{Bi} \\ y_i + v_{Bi} \\ z_i + w_{Bi} \end{bmatrix} \right|^2 \\ &\approx \sum_{i=1}^{M} \left| \begin{bmatrix} x_i + u_{Ai} \\ y_i + v_{Ai} \\ z_i + w_{Ai} \end{bmatrix} - \begin{bmatrix} 1 & -\omega_z & \omega_y \\ \omega_z & 1 & -\omega_x \\ -\omega_y & \omega_x & 1 \end{bmatrix} \cdot \begin{bmatrix} x_i + u_{Bi} \\ y_i + v_{Bi} \\ z_i + w_{Bi} \end{bmatrix} \right|^2 \end{aligned} \tag{B8}$$

To simplify the analysis, we first consider only the rotation about the x-axis. This gives us

$$SSE_x \approx \sum_{i=1}^{M} \left| \begin{bmatrix} x_i + u_{Ai} \\ y_i + v_{Ai} \\ z_i + w_{Ai} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\omega_x \\ 0 & \omega_x & 1 \end{bmatrix} \cdot \begin{bmatrix} x_i + u_{Bi} \\ y_i + v_{Bi} \\ z_i + w_{Bi} \end{bmatrix} \right|^2 \tag{B9}$$
$$= \sum_{i=1}^{M} \left((u_{Ai} - u_{Bi})^2 + (v_{Ai} - v_{Bi} + \omega_x (z_i + w_{Bi}))^2 + (w_{Ai} - w_{Bi} - \omega_x (y_i + v_{Bi}))^2 \right)$$

Differentiating SSE_x with respect to ω_x, we have

$$\frac{\partial (SSE_x)}{\partial \omega_x} \approx 2 \sum_{i=1}^{M} ((z_i + w_{Bi})(v_{Ai} - v_{Bi} + \omega_x (z_i + w_{Bi})) - (y_i + v_{Bi})(w_{Ai} - w_{Bi} - \omega_x (y_i + v_{Bi}))) \tag{B10}$$

Normally $|z_i| >> |w_{Bi}|$ and $|y_i| >> |v_{Bi}|$, so we can further simplify the above by

$$\frac{\partial (SSE_x)}{\partial \omega_x} \approx 2 \sum_{i=1}^{M} (z_i (v_{Ai} - v_{Bi} + \omega_x z_i) - y_i (w_{Ai} - w_{Bi} - \omega_x y_i)) \tag{B11}$$

SSE_x is minimum when $\frac{\partial (SSE_x)}{\partial \omega_x} = 0$, and it is true when

$$2 \sum_{i=1}^{M} (z_i (v_{Ai} - v_{Bi} + \omega_x z_i) - y_i (w_{Ai} - w_{Bi} - \omega_x y_i)) = 0$$
$$\Rightarrow \omega_x \sum_{i=1}^{M} (y_i^2 + z_i^2) = \sum_{i=1}^{M} (y_i (w_{Ai} - w_{Bi}) - z_i (v_{Ai} - v_{Bi})) \tag{B12}$$

Let v_{Ai}, v_{Bi}, w_{Ai} and w_{Bi} be the values of the random variables V_{Ai}, V_{Bi}, W_{Ai} and W_{Bi}, respectively. We also let $\omega_x \sum_{i=1}^{M} (y_i^2 + z_i^2)$ be the value of the random variable Q_x. Suppose each V_{Ai}, V_{Bi}, W_{Ai} and W_{Bi} has normal distribution with mean μ_{V_A}, μ_{V_B}, μ_{W_A} and μ_{W_B}, and standard deviation σ_{V_A}, σ_{V_B}, σ_{W_A} and σ_{W_B}, respectively, i.e.

$$\begin{aligned} V_{Ai} &\sim N(\mu_{V_A}, \sigma_{V_A}) \quad \text{and} \quad V_{Bi} \sim N(\mu_{V_B}, \sigma_{V_B}), \\ W_{Ai} &\sim N(\mu_{W_A}, \sigma_{W_A}) \quad \text{and} \quad W_{Bi} \sim N(\mu_{W_B}, \sigma_{W_B}), \end{aligned} \tag{B13}$$

then the mean [Walpole1993] of Q_x is

$$\mu_{Q_x} = \sum_{i=1}^{M} (y_i (\mu_{W_A} - \mu_{W_B}) - z_i (\mu_{V_A} - \mu_{V_B})) \tag{B14}$$

and the standard deviation [Walpole1993] of Q_x is

$$\sigma_{Q_x} = \sqrt{\sum_{i=1}^{M} (y_i^2 \sigma_{W_A}^2 + y_i^2 \sigma_{W_B}^2 + z_i^2 \sigma_{V_A}^2 + z_i^2 \sigma_{V_B}^2)} \tag{B15}$$

Since $\sigma_{Q_x} \to \infty$ as $M \to \infty$, Q_x satisfies the Lindeberg condition, and by Lindeberg's Theorem [Ash1972, pp. 336–337], Q_x converges to a normal distribution, i.e.

$$Q_x \sim N\left(\mu_{Q_x}, \sigma_{Q_x}\right)$$

$$Q_x \sim N\left(\sum_{i=1}^{M}\left(y_i\left(\mu_{W_A} - \mu_{W_B}\right) - z_i\left(\mu_{V_A} - \mu_{V_B}\right)\right),\ \sqrt{\sum_{i=1}^{M}\left(y_i^2\sigma_{W_A}^2 + y_i^2\sigma_{W_B}^2 + z_i^2\sigma_{V_A}^2 + z_i^2\sigma_{V_B}^2\right)}\right) \tag{B16}$$

In practice, we can assume $\mu_{W_A} = \mu_{W_B} = \mu_{V_A} = \mu_{V_B} = 0$. With the above assumptions, we can be $(1 - \alpha)$ 100% confident that the rotational alignment error about the *x*-axis will not exceed $\varepsilon > 0$ radians when the following condition is true:

$$\varepsilon\sum_{i=1}^{M}\left(y_i^2 + z_i^2\right) \geq z_{\alpha/2}\sqrt{\sum_{i=1}^{M}\left(y_i^2\sigma_{W_A}^2 + y_i^2\sigma_{W_B}^2 + z_i^2\sigma_{V_A}^2 + z_i^2\sigma_{V_B}^2\right)} \tag{B17}$$

where

$$P\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) = 1 - \alpha \tag{B18}$$

and *Z* is a random variable that has the standard normal distribution.

Normally, σ_{U_A}, σ_{U_B}, σ_{V_A}, σ_{V_B}, σ_{W_A} and σ_{W_B} are not constant and they vary depending on factors such as the choice of the coordinate system with respect to the surface's orientation, the incident angle of the laser to the surface point, the surface reflectance properties, and the distance between the sensor and the surface point. To simplify the analysis, and the fact that we can be more conservative in this case, we can assume all the above standard deviations are less than or equal to the worst possible RMS error in range measurement, e_{RMS}. Then from Eq. (B17), we obtain the more conservative condition

$$\varepsilon\sum_{i=1}^{M}\left(y_i^2 + z_i^2\right) \geq z_{\alpha/2}\sqrt{\sum_{i=1}^{M}\left(y_i^2 e_{RMS}^2 + y_i^2 e_{RMS}^2 + z_i^2 e_{RMS}^2 + z_i^2 e_{RMS}^2\right)}$$

$$\Rightarrow\ \varepsilon^2\sum_{i=1}^{M}\left(y_i^2 + z_i^2\right) \geq 2\left(z_{\alpha/2} e_{RMS}\right)^2 \tag{B19}$$

However, the condition in Eq. (B19) is only valid for a special case. When a point $\boldsymbol{p}_i = \left[x_i, y_i, z_i\right]^{\mathrm{T}}$ is rotated a small angle ω_x about the *x*-axis, its contribution to the energy function (the constraint), $E_P(T_\theta(\boldsymbol{p}_i))$ in Eq. (B4), is

$$D^2(T_\theta(\boldsymbol{p}_i)) = d\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{V}_\theta(\boldsymbol{p}_i)\boldsymbol{V}_\theta^{\mathrm{T}}(\boldsymbol{p}_i)\,d\boldsymbol{\theta}$$

$$= \left(\boldsymbol{p}_i \times \boldsymbol{n}_i \cdot \begin{bmatrix}\omega_x \\ 0 \\ 0\end{bmatrix}\right)^2 = \left(\begin{bmatrix}x_i \\ y_i \\ z_i\end{bmatrix} \times \begin{bmatrix}n_{i,x} \\ n_{i,y} \\ n_{i,z}\end{bmatrix} \cdot \begin{bmatrix}\omega_x \\ 0 \\ 0\end{bmatrix}\right)^2 = \omega_x^2\left(y_i n_{i,z} - z_i n_{i,y}\right)^2 \tag{B20}$$

where $\boldsymbol{n}_i$ is the unit surface normal at $\boldsymbol{p}_i$. $D^2(T_\theta(\boldsymbol{p}_i))$ is maximum if $\boldsymbol{n}_i$ is perpendicular to both the vector $\boldsymbol{p}_i$ and the *x*-axis, i.e.

$$\boldsymbol{n}_i = normalize\left(\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \times \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}\right) = normalize\left(\begin{bmatrix} 0 \\ \pm z_i \\ \mp y_i \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \pm z_i \Big/ \sqrt{y_i^2 + z_i^2} \\ \mp y_i \Big/ \sqrt{y_i^2 + z_i^2} \end{bmatrix} \tag{B21}$$

Then,

$$\begin{aligned} D^2(T_\theta(\boldsymbol{p}_i)) &= \omega_x^2 \left(y_i n_{i,z} - z_i n_{i,y}\right)^2 \\ &= \omega_x^2 \left(y_i \left(\mp y_i \Big/ \sqrt{y_i^2 + z_i^2} \right) - z_i \left(\pm z_i \Big/ \sqrt{y_i^2 + z_i^2} \right) \right)^2 = \omega_x^2 \left(y_i^2 + z_i^2\right) \end{aligned} \tag{B22}$$

and

$$E_P(T_\theta(\boldsymbol{p}_i)) = \sum_{\boldsymbol{x}_s \in P} D^2(T_\theta(\boldsymbol{p}_i)) = \omega_x^2 \sum_{i=1}^{M} \left(y_i^2 + z_i^2\right) \tag{B23}$$

The R.H.S. of Eq. (B23) is basically the same as the L.H.S. of Eq. (B19). This makes sense because Eq. (B19) is derived on the assumption that each point in point set *B* is matched correctly with the corresponding point in point set *A*. The vector between each pair of corresponding points is parallel to the direction of the rotation. When the rotation is about the *x*-axis, this vector is perpendicular to both the vector $\boldsymbol{p}_i$ (coordinates of the point *i* in point set *B*) and the *x*-axis. This vector can be treated as the normal $\boldsymbol{n}_i$. Therefore, we can say that the condition in Eq. (B19) is true only in the special case when each point $\boldsymbol{p}_i$ is at its maximum constraint on the rotation about the *x*-axis, i.e. $\boldsymbol{n}_i$ is perpendicular to both the vector $\boldsymbol{p}_i$ and the *x*-axis. However, in the general case when each $\boldsymbol{n}_i$ is a true surface normal on the surface, and it can be any unit vector, we have

$$\begin{aligned} & D^2(T_\theta(\boldsymbol{p}_i)) = \omega_x^2 \left(y_i n_{i,z} - z_i n_{i,y}\right)^2 \\ & \Rightarrow E_P(T_\theta(\boldsymbol{p}_i)) = \sum_{\boldsymbol{x}_s \in P} D^2(T_\theta(\boldsymbol{p}_i)) = \omega_x^2 \sum_{i=1}^{M} \left(y_i n_{i,z} - z_i n_{i,y}\right)^2 \end{aligned} \tag{B24}$$

Now, for the general case, we can be $(1 - \alpha)$ 100% confident that the rotational alignment error about the *x*-axis will not exceed $\varepsilon > 0$ radians when the following condition is true:

$$\varepsilon^2 \sum_{i=1}^{M} \left(y_i n_{i,z} - z_i n_{i,y}\right)^2 \;\geq\; 2\left(z_{\alpha/2} e_{RMS}\right)^2 \tag{B25}$$

The conditions for rotations about the *y*-axis and *z*-axis can be similarly derived respectively as

$$\varepsilon^2 \sum_{i=1}^{M} \left(z_i n_{i,x} - x_i n_{i,z}\right)^2 \;\geq\; 2\left(z_{\alpha/2} e_{RMS}\right)^2 \quad \text{and} \quad \varepsilon^2 \sum_{i=1}^{M} \left(x_i n_{i,y} - y_i n_{i,x}\right)^2 \;\geq\; 2\left(z_{\alpha/2} e_{RMS}\right)^2 \tag{B26}$$

By combining the three inequalities in Eq. (B25) and (B26), and Eq. (B6), we get

$$\varepsilon^2 \sum_{i=1}^{M} \left(\left(y_i n_{i,z} - z_i n_{i,y} \right)^2 + \left(z_i n_{i,x} - x_i n_{i,z} \right)^2 + \left(x_i n_{i,y} - y_i n_{i,x} \right)^2 \right) \geq 3 \cdot 2 \left(z_{\alpha/2} e_{RMS} \right)^2$$

$$\Rightarrow \quad \varepsilon^2 \sum_{\boldsymbol{x}_s \in P} \left| \boldsymbol{p}_i \times \boldsymbol{n}_i \right|^2 \geq 6 \left(z_{\alpha/2} e_{RMS} \right)^2 \quad \Rightarrow \quad \gamma_1 + \gamma_2 + \gamma_3 \geq S_{\min} = 6 \left(\frac{z_{\alpha/2} e_{RMS}}{\varepsilon} \right)^2 \tag{B27}$$

With the results in Eq. (B25), (B26) and (B27), in the general case when $\boldsymbol{n}_i$ can be any unit vector,

> we can be $(1 - \alpha)^3$ 100% confident that the rotational alignment errors about the *x*-axis, *y*-axis and *z*-axis, respectively, will not exceed $\varepsilon > 0$ radians when $\gamma_1 \geq S_{\min}/3$, $\gamma_2 \geq S_{\min}/3$ and $\gamma_3 \geq S_{\min}/3$, where $S_{\min} = 6 \left(z_{\alpha/2} e_{RMS} / \varepsilon \right)^2$.

4 NOTES AND REMARKS

- The above analyses and their results assume the two surfaces to be registered are already very well aligned.

- For the evaluation of the rotational constraint in Section 3, the values of the eigenvalues γ_1, γ_2 and γ_3 are dependent on the scale of the surface. Simon [Simon1996] suggested rescaling the surface so that the average distance of the selected surface points from the origin is 1. However, in our error analysis, this normalization is not necessary because we do not compare the eigenvalues γ_1, γ_2 and γ_3 of the rotational constraint with those, λ_1, λ_2 and λ_3, of the translational constraint.

- Both of the above analyses do not consider errors in the surface normals at the measured points. The surface normal at a point can be estimated by fitting a plane to the points in the neighborhood of the candidate point. The estimation is more accurate if larger neighborhood is used. In addition, we would like to select only points in relatively smoother regions on the surface, i.e. smaller error when plane-fitting a large neighborhood.

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REFERENCES

[Ash1972] Robert B. Ash. *Real Analysis and Probability*. Academic Press, 1972.

[Besl1992] Paul J. Besl, and Neil D. McKay. *A Method for Registration of 3-D Shapes*. IEEE Transactions on Pattern Analysis and Machine Intelligence (PAMI), 14(2), pp. 239–256, 1992.

[Chen1992] Yang Chen, and Gerard Medioni. *Object Modeling by Registration of Multiple Range Images*. International Journal of Image and Vision Computing, 10(3), pp. 145–155, 1992.

[Rusinkiewicz2001] Szymon Rusinkiewicz, and Marc Levoy. *Efficient Variants of the ICP Algorithm*. Proceedings of the International Conference on 3-D Digital Imaging and Modeling (3DIM), pp. 145–152, 2001.

[Simon1996] David A. Simon. *Fast and Accurate Shape-Based Registration*. Ph. D. Dissertation, Carnegie Mellon University, CMU-RI-TR-96-45, 1996. Also available at http://www.ri.cmu.edu/pub_files/pub1/simon_david_1996_1/simon_david_1996_1.pdf

[Strang1988] Gibert Strang. *Linear Algebra and Its Applications, Third Edition*. Harcourt College Publishers, 1988.

[Walpole1993] Ronald E. Walpole, and Raymond H. Myers. *Probability and Statistics for Engineers and Scientists, Fifth Edition*. Prentice Hall, 1993.