Calibrating The HiBall Wand

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1 INTRODUCTION

The *HiBall wand* is an elongated hard plastic that can be attached rigidly to the bottom of a HiBall [1, 2, 3] (see Figure 1). The other end of the wand is a moderately-sharp metal tip. In the HMD Lab at UNC-Chapel Hill, it is a convenient device for measuring the 3D position of any point in a space where the HiBall can be tracked by an *optical ceiling tracker*.

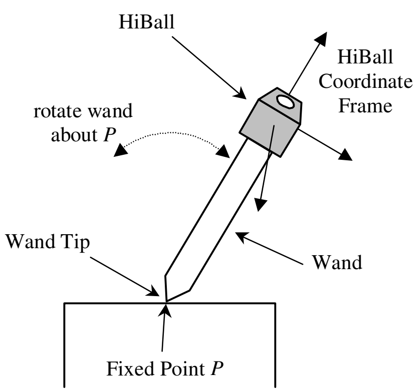

Figure 1: *Calibrating the HiBall wand.*

When a HiBall is being tracked by a ceiling tracker, its poses (orientations and positions) within the fixed *ceiling coordinate frame* are known, and can be obtained from the ceiling tracker system. Each pose tells how the *HiBall's local coordinate frame* is positioned relative to the ceiling coordinate frame.

To use the tip of the wand to measure the 3D position of a point with respect to the ceiling coordinate frame, we first need to know the tip's 3D position in the HiBall's local coordinate frame. If the tip's position in the HiBall coordinate frame is known, we can always use the HiBall's current pose in the ceiling coordinate frame to express the tip's position with respect to the ceiling coordinate frame.

This article explains a method to calibrate the wand—to find the tip's position in the HiBall coordinate frame.

2 WAND CALIBRATION

The calibration consists of two steps. The first step collects the necessary data, and the second step computes the result using the collected data.

2.1 Data Collection

With the HiBall attached to the wand, we place the tip of the wand at a fixed point, P, on a firm and stable surface within the tracking range of the ceiling tracker. While always keeping the tip fixed at P, we slowly rotate the wand about its tip (see Figure 1). While rotating the wand, the pose of the HiBall is continuously acquired from the tracker system, and recorded.

We need to obtain at least two different poses of the HiBall (the next section explains why). Moreover, it is desirable to have poses that are very different, which can be obtained by rotating the wand by a large angle.

Each recorded pose of the HiBall is actually made up of a rotation $\mathbf{R}_i$ and a translation $\mathbf{T}_i$ that transform the ceiling coordinate frame into the HiBall coordinate frame. In other words, $\mathbf{R}_i$ and $\mathbf{T}_i$ transform a point's position in the HiBall coordinate frame into a position in the ceiling coordinate frame.

2.2 Solution Computation

The crucial observation necessary to solve for the tip's position in the HiBall coordinate frame is that when the wand is being rotated about its tip fixed at point P, the point P always has a constant 3D position in the HiBall coordinate frame and also has another (possibly the same) constant 3D position in the ceiling coordinate frame.

Let $(x_H, y_H, z_H)^T$ and $(x_C, y_C, z_C)^T$ be the position of P in the HiBall and ceiling coordinate frames, respectively. Both $(x_H, y_H, z_H)^T$ and $(x_C, y_C, z_C)^T$ are unknown, but $(x_H, y_H, z_H)^T$ is the one we are interested in solving. Next, for each pose $\mathbf{R}_i$ and $\mathbf{T}_i$, we can write the following equation:

$$\begin{pmatrix} x_C \\ y_C \\ z_C \end{pmatrix} = \left(\mathbf{R}_i \mid \mathbf{T}_i\right) \begin{pmatrix} x_H \\ y_H \\ z_H \\ 1 \end{pmatrix} \qquad \text{(EQ-1)}$$

where $\mathbf{R}_i$ is a 3×3 rotation matrix and $\mathbf{T}_i$ is a 3×1 column vector. We can expand EQ-1 to get

$$\begin{pmatrix} x_C \\ y_C \\ z_C \end{pmatrix} = \begin{pmatrix} r_{i11} & r_{i12} & r_{i13} & t_{i1} \\ r_{i21} & r_{i22} & r_{i23} & t_{i2} \\ r_{i31} & r_{i32} & r_{i33} & t_{i3} \end{pmatrix} \begin{pmatrix} x_H \\ y_H \\ z_H \\ 1 \end{pmatrix} \qquad \text{(EQ-2)}$$

By doing the matrix multiplication, we get the following three equations:

$$
\begin{aligned}
x_C &= r_{i11}x_H + r_{i12}y_H + r_{i13}z_H + t_{i1} \\
y_C &= r_{i21}x_H + r_{i22}y_H + r_{i23}z_H + t_{i2} \\
z_C &= r_{i31}x_H + r_{i32}y_H + r_{i33}z_H + t_{i3}
\end{aligned} \tag{EQ-3}
$$

After rearranging, we have

$$
\begin{aligned}
x_C - r_{i11}x_H - r_{i12}y_H - r_{i13}z_H &= t_{i1} \\
y_C - r_{i21}x_H - r_{i22}y_H - r_{i23}z_H &= t_{i2} \\
z_C - r_{i31}x_H - r_{i32}y_H - r_{i33}z_H &= t_{i3}
\end{aligned} \tag{EQ-4}
$$

Next, we rewrite EQ-4 as matrix multiplication

$$
\begin{pmatrix} 1 & 0 & 0 & -r_{i11} & -r_{i12} & -r_{i13} \\ 0 & 1 & 0 & -r_{i21} & -r_{i22} & -r_{i23} \\ 0 & 0 & 1 & -r_{i31} & -r_{i32} & -r_{i33} \end{pmatrix} \begin{pmatrix} x_C \\ y_C \\ z_C \\ x_H \\ y_H \\ z_H \end{pmatrix} = \begin{pmatrix} t_{i1} \\ t_{i2} \\ t_{i3} \end{pmatrix} \tag{EQ-5}
$$

EQ-5 is in the form **Ax** = **b**, where **A** and **b** are known. Our task now is to solve for **x**. However, since **x** has 6 unknowns and **A** has only 3 rows (equations), we do not have enough constraints to solve for **x** yet. We need to have at least 6 rows in **A** (provided they are all independent). Since each pose of the HiBall provides 3 rows in **A**, we need at least 2 poses to get at least 6 rows in **A**.

Since there are errors in the poses obtained from the tracker system, we would like to use as many poses as possible to minimize the error in our solution of **x**. Let $n \geq 2$ be the number of poses collected. Then, by writing them in the form of EQ-5, and combining them, we have

$$
\begin{pmatrix} 1 & 0 & 0 & -r_{111} & -r_{112} & -r_{113} \\ 0 & 1 & 0 & -r_{121} & -r_{122} & -r_{123} \\ 0 & 0 & 1 & -r_{131} & -r_{132} & -r_{133} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 0 & 0 & -r_{n11} & -r_{n12} & -r_{n13} \\ 0 & 1 & 0 & -r_{n21} & -r_{n22} & -r_{n23} \\ 0 & 0 & 1 & -r_{n31} & -r_{n32} & -r_{n33} \end{pmatrix} \begin{pmatrix} x_C \\ y_C \\ z_C \\ x_H \\ y_H \\ z_H \end{pmatrix} = \begin{pmatrix} t_{11} \\ t_{12} \\ t_{13} \\ . \\ . \\ . \\ t_{n1} \\ t_{n2} \\ t_{n3} \end{pmatrix} \tag{EQ-6}
$$

which we will refer to as

$$
\mathbf{Ax} = \mathbf{b} \tag{EQ-7}
$$

Because of errors in the measurements, **Ax** = **b** is almost always an inconsistent system, i.e. **b** is not in the *range* of **A**. Generally, the columns of **A** are independent, therefore **A** has a *rank* of 6. So, the *least squares solution* (see Appendix) of **Ax** = **b** is just

$$
\bar{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}.
$$

$\mathbf{A}^T\mathbf{A}$ is invertible because it is a 6×6 matrix and it has the same rank as **A**.

The last step of the calibration is just to extract x_H, y_H, and z_H from $\bar{\mathbf{x}}$. $(x_H, y_H, z_H)^T$ is the 3D position of the tip in the HiBall coordinate frame.

Using MATLAB [5], the least squares solution $\bar{\mathbf{x}}$ can be computed as follows:

```
x = A \ b
```

provided `A` and `b` have already been set up as in EQ-6.

REFERENCES

[1] http://www.3rdtech.com/HiBall.htm

[2] http://www.cs.unc.edu/~tracker/

[3] Greg Welch, Gary Bishop, Leandra Vicci, Stephen Brumback, Kurtis Keller, D'nardo Colucci. *The HiBall Tracker: High-Performance Wide-Area Tracking for Virtual and Augmented Environments.* Proceedings of the ACM Symposium on Virtual Reality Software and Technology 1999 (VRST '99).

[4] Gilbert Strang. *Linear Algebra and Its Applications, Third Edition* (1988). International Thomson Publishing.

[5] MATLAB. See http://www.mathworks.com/

APPENDIX

Least Squares Solution

The column vectors in **A** span a 6-dimensional subspace *S*, embedded in a 3*n*-dimensional space. In order for **Ax** = **b** to have a unique solution, the 3*n*-dimensional vector **b** must lie in the subspace *S*. However, errors in measurements usually cause **b** to lie outside *S*. In this case, we call **Ax** = **b** an inconsistent system, and there is no solution **x** such that **Ax** = **b**. In spite of this, we do not just give up here. Inconsistent equations arise in practice and have to be solved in some way.

One way we can do in the case when **b** lies outside *S* is to find a point **p**, such that it is in *S* and it is the point in *S* that is closest to **b**. To find the point **p**, we can project **b** onto *S*. To help you understand, you can imagine that *S* is a 2-dimensional plane embedded in a 3-dimensional space, and **b** is a 3-dimensional point not on the plane. Then if you project **b** onto the plane *S*, you get **p**, which is the point on the plane closest to **b**.

For the system in EQ-7, **p** can be computed from **b** as in

$$
\mathbf{p} = \mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}. \tag{EQ-A}
$$

However, instead of **p**, we are more interested in solving for $\bar{\mathbf{x}}$ such that $\mathbf{A}\bar{\mathbf{x}} = \mathbf{p}$. By comparing $\mathbf{A}\bar{\mathbf{x}} = \mathbf{p}$ with EQ-A, we can see that

$$
\bar{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b},
$$

which is what we call the least squares solution of **Ax** = **b**. It is called the least squares solution because $|\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}|^2$ is the minimum of all the $\mathbf{x} \in \mathbf{R}^6$.

You can refer to [4] for more details.